MULTIFRACTAL ANALYSIS OF BERNOULLI CONVOLUTIONS ASSOCIATED WITH SALEM NUMBERS

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Abstract. We consider the multifractal structure of the Bernoulli convolution ν_λ, where λ^{-1} is a Salem number in $(1,2)$. Let $\tau(q)$ denote the L^q spectrum of ν_λ. We show that if $\alpha \in [\tau'(+\infty), \tau'(0+)]$, then the level set

$$E(\alpha) := \left\{ x \in \mathbb{R} : \lim_{r\to 0} \frac{\log \nu_\lambda([x-r, x+r])}{\log r} = \alpha \right\}$$

is non-empty and $\dim_H E(\alpha) = \tau^*(\alpha)$, where τ^* denotes the Legendre transform of τ. This result extends to all self-conformal measures satisfying the asymptotically weak separation condition. We point out that the interval $[\tau'(+\infty), \tau'(0+)]$ is not a singleton when λ^{-1} is the largest real root of the polynomial $x^n - x^{n-1} - \cdots - x + 1$, $n \geq 4$.

1. Introduction

For any $\lambda \in (0,1)$, let ν_λ denote the distribution of $\sum_{n=0}^{\infty} \epsilon_n \lambda^n$ where the coefficients ϵ_n are either -1 or 1, chosen independently with probability $\frac{1}{2}$ for each. It is the infinite convolution product of the distributions $\frac{1}{2}(\delta_{-\lambda^n} + \delta_{\lambda^n})$, giving rise to the term "infinite Bernoulli convolution" or simply "Bernoulli convolution". The Bernoulli convolution can be expressed as a self-similar measure ν_λ satisfying the equation

$$\nu_\lambda = \frac{1}{2}\nu_\lambda \circ S_1^{-1} + \frac{1}{2}\nu_\lambda \circ S_2^{-1}, \tag{1.1}$$

where $S_1(x) = \lambda x - 1$ and $S_2(x) = \lambda x + 1$. These measures have been studied since the 1930's, revealing surprising connections with a number of areas in mathematics, such as harmonic analysis, fractal geometry, number theory, dynamical systems, and others, see [28].

The fundamental question about ν_λ is to decide for which $\lambda \in (\frac{1}{2}, 1)$ this measure is absolutely continuous and for which λ it is singular. It is well known that for each $\lambda \in (1/2, 1)$, ν_λ is continuous, and it is either purely absolutely continuous or purely singular. Solomyak [35] proved that ν_λ is absolutely continuous for a.e. $\lambda \in (1/2, 1)$. In the other direction, Erdös [4] proved that if λ^{-1} is a Pisot number,

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i.e. an algebraic integer whose algebraic conjugates are all inside the unit disk, then ν_λ is singular. It is an open problem whether the Pisot reciprocals are the only class of λ's in $(\frac{1}{2}, 1)$ for which ν_λ is singular. This question is far from being answered. There appears to be a general belief that the best candidates for counter-examples are the reciprocals of *Salem numbers*. Recall that a positive number β is called a Salem number if it is an algebraic integer whose algebraic conjugates all have modulus no greater than 1, with at least one of which on the unit circle. Indeed, as Kahane observed, when λ^{-1} is a Salem number, the Fourier transform of ν_λ has no uniform decay at infinity (cf. [28, Lemma 5.2]). A well-known class of Salem numbers are the largest real roots β_n of the polynomials $x^n - x^{n-1} - \cdots - x + 1$; where $n \geq 4$. It was shown by Wang and the author in [15] that for any $\epsilon > 0$, the density of ν_{1/β_n}, if it exists, is not in $L^{3+\epsilon}(\mathbb{R})$ when n is large enough.

In this paper, we study the local dimensions and the multifractal structure of ν_λ when λ^{-1} is a Salem number in $(1, 2)$. Few results along this direction have been known in the literature. Before formulating our results, we first recall some basic notation used in the multifractal analysis. The reader is referred to [6] for details.

Let μ be a finite Borel measure in $\mathbb{R}^d$ with compact support. For $x \in \mathbb{R}^d$ and $r > 0$, let $B_r(x)$ denote the closed ball centered at x of radius r. For $q \in \mathbb{R}$, the L^q *spectrum* of μ is defined as

$$\tau_\mu(q) = \liminf_{r \to 0} \frac{\log \Theta_\mu(q; r)}{\log r},$$

where

$$\Theta_\mu(q; r) = \sup \sum_i \mu(B_r(x_i))^q, \qquad r > 0,\ q \in \mathbb{R}, \tag{1.2}$$

and the supremum is taken over all families of disjoint balls $\{B_r(x_i)\}_i$ with $x_i \in \operatorname{supp}(\mu)$. It is easily checked that $\tau_\mu(q)$ is a concave function of q over $\mathbb{R}$. For $x \in \mathbb{R}^d$, the *local dimension* of μ at x is defined as

$$d_\mu(x) = \lim_{r \to 0} \frac{\log \mu(B_r(x))}{\log r},$$

provided that the limit exists. For $\alpha \in \mathbb{R}$, denote

$$E_\mu(\alpha) = \{x \in \mathbb{R} : \ d_\mu(x) = \alpha\},$$

which is called the *level set* of μ.

One of the main objectives of multifractal analysis is to study the *dimension spectrum* $\dim_H E_\mu(\alpha)$ and its relation with the L^q spectrum $\tau_\mu(q)$, here $\dim_H$ denotes the Hausdorff dimension. The celebrated heuristic principle known as the *multifractal formalism* which was first introduced by some physicists [16], states that for "good" measures μ, the dimension spectrum $\dim_H E_\mu(\alpha)$ can be recovered by the L^q-spectrum $\tau_\mu(q)$ through the Legendre transform:

$$\dim_H E_\mu(\alpha) = \tau_\mu^*(\alpha) := \inf\{\alpha q - \tau_\mu(q) : \ q \in \mathbb{R}\}. \tag{1.3}$$

For more backgrounds of the multifractal formalism, we refer to the books [6, 31]. The multifractal formalism has been verified to hold for many natural measures including for example, self-similar measures satisfying the well-known *open set condition* [3, 26, 27]. In the recent decade, there have been a lot of interest in studying the validity of the multifractal formalism for self-similar measures with *overlaps* (see, e.g., [12] and the references therein).

The main result of the paper is the following.

Theorem 1.1. *Let $\lambda \in (1/2, 1)$ so that λ^{-1} is a Salem number. Then*

(i) $E_{\nu_\lambda}(\alpha) \neq \emptyset$ *if* $\alpha \in [\tau'_{\nu_\lambda}(+\infty), \tau'_{\nu_\lambda}(0+)]$, *where* $\tau'_{\nu_\lambda}(+\infty) := \lim_{q\to+\infty} \tau_{\nu_\lambda}(q)/q$, *and* $\tau'_{\nu_\lambda}(0+)$ *denotes the right derivative of* τ_{ν_λ} *at* 0.

(ii) *For any* $\alpha \in [\tau'_{\nu_\lambda}(+\infty), \tau'_{\nu_\lambda}(0+)]$,

$$\dim_H E_{\nu_\lambda}(\alpha) = \tau_{\nu_\lambda}^*(\alpha) := \inf\{\alpha q - \tau_{\nu_\lambda}(q) : \ q \in \mathbb{R}\}. \tag{1.4}$$

In short, the above theorem says that the Bernoulli convolution ν_λ fulfils the multifractal formalism over $q > 0$, when λ^{-1} is a Salem number. As an application, we obtain the following information about the range of local dimensions of ν_λ associated with certain Salem numbers.

Theorem 1.2. *For $n \geq 4$, let β_n be the largest real root of the polynomials $x^n - x^{n-1} - \cdots - x + 1$, and let $\lambda_n = \beta_n^{-1}$. Then for $\lambda = \lambda_n$, $\tau'_{\nu_\lambda}(+\infty) < 1 \leq \tau'_{\nu_\lambda}(0+)$; and hence the range of local dimensions of ν_λ contains a non-degenerate interval.*

The above results shed somewhat new light on the study of Bernoulli convolutions. In [36] Solomyak asked whether the multifractal analysis can provide some information about the range of local dimensions of Bernoulli convolutions associated with non-Pisot numbers. Theorem 1.2 provides a positive answer.

Theorem 1.2 also provides a hint that ν_{λ_n} might be singular for all $n \geq 4$. Nevertheless, this hint is not direct, since there exists a self-similar measure μ on $\mathbb{R}$ such that μ is absolutely continuous and the range of local dimensions of μ contains a non-degenerate interval (see Proposition 5.1).

Let us give some historic remarks. In the literature there have been a lot of works considering the multifractal structure of Bernoulli convolutions associated with Pisot numbers (see, e.g., [24, 17, 20, 32, 19, 21, 22, 8, 14, 9, 11, 12]). Here we give a brief summary. Assume that λ^{-1} is a Pisot number in $(1, 2)$. In this case, the local distribution of ν_λ can be characterized via matrix products, and as a result, the local dimensions of ν_λ can be described as the Lyapunov exponents of the associated random matrices, whilst the L^q-spectrum corresponds to the pressure function of matrix products [19, 9, 8]. It was shown by Lau and Ngai [21] that ν_λ satisfies the weak separation condition, and (1.4) holds for those $\alpha = \tau'_{\nu_\lambda}(q)$, $q > 0$, provided that $\tau'_{\nu_\lambda}(q)$ exists. Later in [8] we proved that, indeed, τ_{ν_λ} is differentiable on $(0, +\infty)$. Recently in [11], it was shown that there exists an interval I in the support of ν_λ so that, for the restriction of ν_λ on I, the multifractal formalism is valid on the whole range of the local dimensions, regardless of whether there are phase transitions at $q < 0$. This result is extended to self-similar measures satisfying the weak separation condition [12]. The L^q spectra and the dimension spectra can be computed explicitly in some concrete cases. For $\lambda = \frac{\sqrt{5}-1}{2}$ (the golden ratio case), an explicit formula of $\tau_{\nu_\lambda}(q)$ on $q > 0$ was obtained in [22] and was extended to $q \in \mathbb{R}$ in [9]; it was showed in [9] that τ_{ν_λ} has a non-differentiable point in $(-\infty, 0)$ (the so-called *phase transition* behavior); nevertheless, (1.4) still holds for all those $\alpha \in [\tau'_{\nu_\lambda}(+\infty), \tau'_{\nu_\lambda}(-\infty)]$ [14]. The phase transition behaviors and exceptional multifractal phenomena were further found and considered in [23, 34, 37] for other self-similar measures. Rather than the golden ratio case, the explicit formulas of the L^q spectra and the dimension spectra of ν_λ were obtained in [9, 25] when λ is the unique positive root of $x^n + x^{n-1} + \cdots + x - 1$, $n \geq 3$; in this case, τ_{ν_λ} is differentiable over $\mathbb{R}$.

When λ is an arbitrary number in $(1/2, 1)$, the only known result so far is that $E_{\nu_\lambda}(\alpha) \neq \emptyset$ and (1.4) holds for those $\alpha = \tau'_{\nu_\lambda}(q)$, $q > 1$, provided that $\tau'_{\nu_\lambda}(q)$ exists at q [10]. This result extends to all self-conformal measures. In the case that λ^{-1} is a Salem number, the condition $q > 1$ can be relaxed to $q > 0$ [10]. However, it still remains open whether τ_{ν_λ} is differentiable over $(0, \infty)$ for each λ. Although by concavity τ_{ν_λ} has at most countably many non-differentiable points, no much information can be provided for the range $\{\alpha : \ \alpha = \tau'_{\nu_\lambda}(q) \text{ for some } q > 0\}$.

Let us illustrate the main idea in our proof of Theorem 1.1. Assume that λ^{-1} is a Salem number in $(1, 2)$. The IFS $\{\lambda x - 1, \lambda x + 1\}$ may not satisfy the weak separation condition (see Remark 3.3), hence the previous approaches via matrix products and the thermodynamic formalism in [11, 12] are not efficient in this new setting. For $n \in \mathbb{N}$, denote

$$t_n = \sup_{x \in \mathbb{R}} \#\{S_{i_1 \dots i_n} : \ i_1 \dots i_n \in \{1, 2\}^n, \ S_{i_1 \dots i_n}(K) \cap [x - \lambda^n, x + \lambda^n] \neq \emptyset\},$$

where S_1, S_2 are given as in (1.1), $S_{i_1 \dots i_n} := S_{i_1} \circ \cdots \circ S_{i_n}$ and $K := [-\frac{1}{1-\lambda}, \frac{1}{1-\lambda}]$ is the attractor of $\{S_1, S_2\}$. The following simple property is our starting point (see, e.g. [10] for a proof):

$$\lim_{n\to\infty} \frac{\log t_n}{n} = 0. \tag{1.5}$$

Due to this property, we can manage to setup the following local box-counting principle. Let $n \in \mathbb{N}$, $x \in \mathbb{R}$ with $\nu_\lambda(B_{2^{-n-1}}(x)) > 0$. Let $q > 0$ so that $\alpha = \tau'_{\nu_\lambda}(q)$ exists and let $k \in \mathbb{N}$. Then when m is suitably large (which can be controlled delicately by n, q, k and $\nu_\lambda(B_{2^{-n}}(x))/\nu_\lambda(B_{2^{-n-1}}(x))$), there exist $N \geq 2^{m(\tau^*_{\nu_\lambda}(\alpha)-1/k)}$ many disjoint balls $B_{2^{-n-m}}(x_i)$, $i = 1, \ldots, N$, contained in $B_{2^{-n}}(x)$ such that

$$\frac{\nu_\lambda(B_{2^{-n-m}}(x_i))}{\nu_\lambda(B_{2^{-n}}(x))} \in \left(2^{-m(\alpha+1/k)},\ 2^{-m(\alpha-1/k)}\right),$$

and $\nu_\lambda(B_{2^{-n-m+1}}(x_i))/\nu_\lambda(B_{2^{-n-m-1}}(x_i))$ is bounded from above by a constant independent of n, m. This local box-counting principle is much stronger than the standard box-counting principle originated in [16] (see also, Proposition 3.3 in [12]). According to this principle, for any $\alpha \in [\tau'_{\nu_\lambda}(+\infty), \tau'_{\nu_\lambda}(0+)]$, we can give a delicate construction of a Cantor-type subset of $E_{\nu_\lambda}(\alpha)$ with Moran structure such that its Hausdorff dimension is greater or equal to $\tau^*_{\nu_\lambda}(\alpha)$; this shows that $\dim_H E_{\nu_\lambda}(\alpha) = \tau^*_{\nu_\lambda}(\alpha)$, since the upper bound $\dim_H E_{\nu_\lambda}(\alpha) \leq \tau^*_{\nu_\lambda}(\alpha)$ always holds (see, e.g., Theorem 4.1 in [21]).

Using the similar idea, we can extend the result of Theorem 1.1 to any self-conformal measure which satisfies the asymptotically weak separation condition (see Def. 3.2). That is,

Theorem 1.3. *Let ν be a self-conformal measure on $\mathbb{R}^d$ satisfying the asymptotically weak separation condition. Then for $\alpha \in [\tau'_\nu(+\infty), \tau'_\nu(0+)]$, $E_\nu(\alpha) \neq \emptyset$ and $\dim_H E_\nu(\alpha) = \tau^*_\nu(\alpha)$.*

We remark that the asymptotically weak separation condition is strictly weaker than the weak separation condition introduced in [21] (see Remark 3.3).

The paper is arranged in the following manner: in Sect. 2, we show that for a general measure μ in $\mathbb{R}^d$, the multifractal formalism is valid if certain local box-counting principle holds for μ; we prove Theorem 1.3 in Sect. 3 by showing that this local box-counting principle holds for self-conformal measures on $\mathbb{R}^d$ satisfying the asymptotically weak separation condition; in Sect. 4, we prove Theorem 1.2; in Sect. 5, we construct an example of absolutely continuous self-similar measure on $\mathbb{R}$ with non-trivial range of local dimensions.

2. A general scheme for the validity of the multifractal formalism

Let μ be a finite Borel measure μ in $\mathbb{R}^d$ with compact support. Let $\tau(q) := \tau_\mu(q)$ be the L^q-spectrum of μ, and let $E(\alpha) := E_\mu(\alpha)$ denote the level set of μ. (See Sect.1 for the definitions.) Assume that $\tau(q) \in \mathbb{R}$ for each $q \in \mathbb{R}$. In this section we show that the multifractal formalism is valid for μ if certain local box-counting principle holds for μ.

Define

(2.1) $$\Omega = \{q \in \mathbb{R} : \text{ the derivative } \tau'(q) \text{ exists}\} \quad \text{and} \quad \Omega_+ = \Omega \cap (0, \infty).$$

Since τ is concave on $\mathbb{R}$, Ω is dense in $\mathbb{R}$ and Ω_+ is dense in $(0, \infty)$.

Definition 2.1. We say that μ has an *asymptotically good multifractal structure over* $\mathbb{R}$ (resp., $\mathbb{R}_+$) if there is a dense subset Λ of Ω (resp. Ω_+) such that for each $q \in \Lambda$ and $k \in \mathbb{N}$, there exist positive numbers $a(q,k)$, $b(q,k)$, $f_n(q,k)$, $n = 0, 1, 2, \cdots$, such that the following properties hold:

(i)

(2.2) $$\lim_{k\to\infty} b(q,k) = 0, \qquad \lim_{n\to\infty} f_n(q,k)/n = 0.$$

(ii) Let $n \geq 0$ and $x \in \mathbb{R}$ so that $\mu(B_{2^{-n-1}}(x)) > 0$. Then for any integer m with

(2.3) $$m \geq f_n(q,k) + a(q,k) \log \frac{\mu(B_{2^{-n}}(x))}{\mu(B_{2^{-n-1}}(x))},$$

there are disjoint balls $B_{2^{-n-m}}(x_i) \subset B_{2^{-n}}(x)$, $i = 1, \ldots, N$, such that

$$N \geq 2^{m(\tau'(q)q - \tau(q) - b(q,k))},$$

$$2^{-m(\tau'(q)+1/k)} \leq \frac{\mu(B_{2^{-n-m}}(x_i))}{\mu(B_{2^{-n}}(x))} \leq 2^{-m(\tau'(q)-1/k)},$$

and

$$\frac{\mu(B_{2^{-n-m+1}}(x_i))}{\mu(B_{2^{-n-m-1}}(x_i))} \leq f_{n+m}(q,k).$$

The main result in this section is the following.

Theorem 2.2. (a) *Assume that μ has an asymptotically good multifractal structure over $\mathbb{R}$. Let $\alpha_{\min} = \lim_{q\to\infty} \tau(q)/q$ and $\alpha_{\max} = \lim_{q\to-\infty} \tau(q)/q$. Then $E(\alpha) \neq \emptyset$ if and only if $\alpha \in [\alpha_{\min}, \alpha_{\max}] \cap \mathbb{R}$.*[1] *Furthermore, for any $\alpha \in [\alpha_{\min}, \alpha_{\max}] \cap \mathbb{R}$,*

$$\dim_H E(\alpha) = \tau^*(\alpha) = \inf\{\alpha q - \tau(q) : \ q \in \mathbb{R}\}.$$

(b) *Assume that μ has an asymptotical multifractal structure over $\mathbb{R}^+$. Then for $\alpha \in [\alpha_{\min}, \tau'(0+)]$, we have $E(\alpha) \neq \emptyset$ and $\dim_H E(\alpha) = \tau^*(\alpha)$.*

[1] $\alpha_{\min}$ is always non-negative and finite. It is possible that $\alpha_{\max} = +\infty$.

A key idea in the proof of the above theorem is to construct Cantor-type subsets of $E(\alpha)$ with a special Moran construction.

Definition 2.3. Let $B \subset \mathbb{R}^d$ be a closed ball. Let $\{N_\ell\}_{\ell\geq 1}$ be a sequence of positive integers. Let $D = \bigcup_{\ell\geq 0} D_k$ with $D_0 = \{\emptyset\}$ and $D_\ell = \{\omega = (i_1 i_2 \cdots i_\ell) : 1 \leq i_j \leq N_j, 1 \leq j \leq \ell\}$. Suppose that $\mathcal{G} = \{B_\omega : \omega \in D\}$ is a collection of closed balls of radius r_ω in $\mathbb{R}^d$. We say that $\mathcal{G}$ fulfills the *Moran structure*, provided it satisfies the following conditions:

(1) $B_\emptyset = B$, $B_{\omega j} \subset B_\omega$ for any $\omega \in D_{\ell-1}, 1 \leq j \leq N_\ell$;
(2) $B_\omega \cap B_{\omega'} = \emptyset$ for $\omega, \omega' \in D_\ell$ with $\omega \neq \omega'$.
(3) $\lim_{k\to\infty} \max_{\omega\in D_\ell} r_\omega = 0$;
(4) For all $\omega\eta \neq \omega'\eta,\ \omega, \omega' \in D_m,\ \omega\eta, \omega'\eta \in D_n, m \leq n$,

$$\frac{r_{\omega\eta}}{r_\omega} = \frac{r_{\omega'\eta}}{r_{\omega'}}.$$

If $\mathcal{G}$ fulfills the above Moran structure, we call

$$F = \bigcap_{\ell=1}^{\infty} \bigcup_{\omega\in D_\ell} B_\omega$$

the *Moran set* associated with $\mathcal{G}$.

For $\ell \in \mathbb{N}$, let

$$c_\ell = \min_{(i_1\cdots i_\ell)\in D_\ell} \frac{r_{i_1\cdots i_\ell}}{r_{i_1\cdots i_{\ell-1}}}, \quad M_\ell = \max_{(i_1\cdots i_\ell)\in D_\ell} r_{i_1\cdots i_\ell}.$$

Proposition 2.4. [13, Proposition 3.1]. *For a Moran set F defined as above, suppose furthermore*

$$\lim_{k\to\infty} \frac{\log c_\ell}{\log M_\ell} = 0. \tag{2.4}$$

Then we have

$$\dim_H F = \liminf_{\ell\to\infty} s_\ell,$$

where s_ℓ satisfies the equation $\sum_{\omega\in D_\ell} r_\omega^{s_\ell} = 1$ for each k.

Proof of Theorem 2.2. We only prove part (a) of the theorem, since the proof of part (b) is essentially identical. We divide the proof into several steps.

Step 1. If $\alpha \in \overline{\{\tau'(q) : q \in \Omega\}}$, then $E(\alpha) \neq \emptyset$ and $\dim_H E(\alpha) \geq \tau^*(\alpha)$.

Let Λ and $a(q,k), b(q,k), f_n(q,k)$ ($q \in \Lambda,\ k, n \in \mathbb{N}$) be given as in Def. 2.1. We can assume that $\lim_{n\to\infty} f_n(q,k) = \infty$, since in Def. 2.1, we can change $f_n(q,k)$ to $\max\{f_n(q,k),\ \log n\}$ with no harm.

Fix $\alpha \in \overline{\{\tau'(q):\ q \in \Omega\}}$. Since τ is a concave function on $\mathbb{R}$ and Λ is dense in Ω, there exists a sequence $(q_j)_{j=1}^{\infty} \subset \Lambda$ such that $\lim_{j\to\infty} \tau'(q_j) = \alpha$. Note that τ^* is also a concave function (and hence lower semi-continuous) on $[\alpha_{\min}, \alpha_{\max}] \cap \mathbb{R}$ (see [33]). Hence

$$\tau^*(\alpha) \le \liminf_{j\to\infty} \tau^*(\tau'(q_j)) = \liminf_{j\to\infty} (\tau'(q_j)q_j - \tau(q_j)). \tag{2.5}$$

Take a sequence $(k_j)_{j=1}^{\infty}$ of positive integers such that $\lim_{j\to\infty} k_j = \infty$ and

$$b_j := b(q_j, k_j) \to 0, \text{ as } j \to \infty. \tag{2.6}$$

Pick $x_0 \in \mathbb{R}$ such that $\mu(B_{1/2}(x_0)) > 0$. Set

$$A_0 = \frac{\mu(B_1(x_0))}{\mu(B_{1/2}(x_0))}.$$

Clearly $1 \le A_0 < \infty$. Then due to (2.2), we can define a sequence $(L_j)_{j=1}^{\infty}$ of positive integers recursively such that $L_1 \ge 2$ and

$$n \ge f_0(q_1, k_1) + a(q_1, k_1)\log A_0 \quad \text{ if } n \ge L_1 \tag{2.7}$$

and

$$n \ge f_n(q_1, k_1) + a(q_1, k_1)\log f_n(q_1, k_1) \quad \text{ if } n \ge L_1 \tag{2.8}$$

and

$$\frac{n}{j+1} \ge f_n(q_{j+1}, k_{j+1}) + a(q_{j+1}, k_{j+1})\log(f_n(q_{j+1}, k_{j+1}) + f_n(q_j, k_j)) \tag{2.9}$$

if $n \ge L_j$, $j = 1, 2, \ldots$.

Construct a sequence of positive integers $(n_\ell)_{\ell=1}^{\infty}$ recursively by setting $n_1 = L_1$ and for $\ell \ge 2$,

$$n_\ell = \text{ the smallest integer greater than } (n_1 + \cdots + n_{\ell-1})/\theta(\ell), \tag{2.10}$$

where $\theta(\ell)$ denotes the unique positive integer j satisfying

$$L_0 + \cdots + L_{j-1} \le \ell < L_0 + \cdots + L_j.$$

Here we take the convention $L_0 = 0$. Clearly,

$$0 \le \theta(\ell+1) - \theta(\ell) \le 1, \quad \lim_{\ell\to\infty} \theta(\ell) = \infty, \quad \lim_{\ell\to\infty} \frac{\theta(\ell+1)}{\theta(\ell)} = 1. \tag{2.11}$$

Moreover,

$$\lim_{\ell\to\infty} \frac{n_1 + \cdots + n_{\ell-1}}{n_1 + \cdots + n_\ell} = \lim_{\ell\to\infty} \frac{n_1 + \cdots + n_{\ell-1}}{(n_1 + \cdots + n_{\ell-1})(1 + 1/\theta(\ell))} = 1. \tag{2.12}$$

Combining (2.11), (2.12) and (2.10), we have

$$\lim_{\ell\to\infty} \frac{n_\ell}{n_{\ell-1}} = 1, \qquad \lim_{\ell\to\infty} \frac{n_\ell}{n_1+\cdots+n_{\ell-1}} = 0. \tag{2.13}$$

By (2.7), we have

$$n_1 = L_1 \geq f_0(q_1,k_1) + a(q_1,k_1)\log \frac{\mu(B_1(x_0))}{\mu(B_{1/2}(x_0))}. \tag{2.14}$$

We claim that for any $\ell \geq 1$,

$$n_{\ell+1} \geq f_{n_1+\cdots+n_\ell}(q_{\theta(\ell+1)}+k_{\theta(\ell+1)}) + a(q_{\theta(\ell+1)},k_{\theta(\ell+1)})\log f_{n_1+\cdots+n_\ell}(q_{\theta(\ell)},k_{\theta(\ell)}). \tag{2.15}$$

To prove (2.15), fix ℓ and set $j=\theta(\ell+1)$. First we consider the case that $j=1$. In this case, by (2.10), $n_{\ell+1}\geq n_1+\cdots+n_\ell$. Note that in this case $\theta(\ell)=1$, hence (2.15) follows from (2.8). Next we assume $j\geq 2$. Then $\theta(\ell)=j$ or $j-1$. By the definition of θ,

$$L_{j-1} \leq \ell+1 \leq n_1+\cdots+n_\ell.$$

Since $n_{\ell+1}\geq (n_1+\cdots+n_\ell)/j$, (2.15) follows from (2.9).

Denote $\lambda_j = \tau'(q_j)q_j - \tau(q_j) - b_j$ for $j\in\mathbb{N}$. Then by (2.5)-(2.6), we have

$$\liminf_{j\to\infty} \lambda_j \geq \tau^*(\alpha). \tag{2.16}$$

Define a sequence $(N_\ell)_{\ell=1}^\infty$ by

$$N_\ell = \max\left\{1,\ \left[2^{n_\ell \lambda_{\theta(\ell)}}\right]\right\},$$

where $[x]$ denotes the integer part of x.

Let $D=\bigcup_{\ell\geq 0} D_\ell$ with $D_0=\{\emptyset\}$ and $D_\ell = \{\omega=(i_1i_2\cdots i_\ell) : 1\leq i_j\leq N_j, 1\leq j\leq \ell\}$. We will construct a collection $\mathcal{G}=\{B_\omega : \omega\in D\}$ of closed balls of radius r_ω in $\mathbb{R}^d$ recursively, which has Moran structure and satisfies the following properties:

- (p1) $B_\emptyset = B_1(x_0)$;
- (p2) $r_\omega = 2^{-(n_1+\cdots+n_\ell)}$ for each $\omega\in D_\ell$;
- (p3) For each $\ell\geq 1$, $\omega\in D_{\ell-1}$ and $1\leq i\leq N_\ell$,

$$2^{-n_\ell(\tau'(q_{\theta(\ell)})+1/k_{\theta(\ell)})} \leq \frac{\mu(B_{\omega i})}{\mu(B_\omega)} \leq 2^{-n_\ell(\tau'(q_{\theta(\ell)})-1/k_{\theta(\ell)})}.$$

  and

$$\mu(2B_{\omega i})/\mu(\frac{1}{2}B_{\omega i}) \leq f_{n_1+\cdots+n_\ell}(q_{\theta(\ell)},k_{\theta(\ell)}) \leq n_1+\cdots+n_\ell,$$

  here and afterwards, cB denotes $B_{cr}(x)$ when $B=B_r(x)$.

The construction is done by induction. We first set $B_\emptyset = B_1(x_0)$. Since μ has an asymptotical multifractal structure, by (2.14) and Def. 2.1, there exist N_1 disjoint closed balls $\{B_i\}_{i=1}^{N_1}$ of radius 2^{-n_1}, contained in $B_\emptyset$, such that

$$2^{-n_1(\tau'(q_1)+1/k_1)} \le \frac{\mu(B_i)}{\mu(B_\emptyset)} \le 2^{-n_1(\tau'(q_1)-1/k_1)}$$

and

$$\frac{\mu(2B_i)}{\mu(\frac{1}{2}B_i)} \le f_{n_1}(q_1, k_1) \le n_1.$$

Relabel this family of N_1 balls by $\{B_\omega : \omega \in D_1\}$. Then (p3) holds in the case $\ell = 1$ (noting that $\theta(1) = 1$).

Assume we have constructed well the family of disjoint balls $\{B_\omega : \omega \in D_\ell\}$ for some $\ell \ge 1$ so that each ball in this family has radius $2^{-n_1-\cdots-n_\ell}$, and (p3) holds for ℓ. Next we construct $\{B_{\omega'} : \omega' \in D_{\ell+1}\}$. Fix $\omega \in D_\ell$. Since (p3) holds for ℓ, we have

$$\mu(B_\omega)/\mu\big(\frac{1}{2}B_\omega\big) \le f_{n_1+\cdots+n_\ell}(q_{\theta(\ell)}, k_{\theta(\ell)}).$$

Combining the above inequality with (2.15) yields

$$n_{\ell+1} \ge f_{n_1+\cdots+n_\ell}(q_{\theta(\ell+1)}, k_{\theta(\ell+1)}) + a(q_{\theta(\ell+1)}, k_{\theta(\ell+1)}) \log \frac{\mu(B_\omega)}{\mu\big(\frac{1}{2}B_\omega\big)}.$$

By Def. 2.1, there exist $N_{\ell+1}$ disjoint balls of radius $2^{-n_1-\cdots-n_{\ell+1}}$, which we denote as $B_{\omega i}$, $i = 1, \ldots, N_{\ell+1}$, such that $B_{\omega i} \subset B_\omega$ and

$$2^{-n_{\ell+1}(\tau'(q_{\theta(\ell+1)})+1/k_{\theta(\ell+1)})} \le \frac{\mu(B_{\omega i})}{\mu(B_\omega)} \le 2^{-n_{\ell+1}(\tau'(q_{\theta(\ell+1)})-1/k_{\theta(\ell+1)})}$$

and

$$\frac{\mu(2B_{\omega i})}{\mu(\frac{1}{2}B_{\omega i})} \le f_{n_1+\cdots+n_{\ell+1}}(q_{\theta(\ell+1)}, k_{\theta(\ell+1)}).$$

Now letting ω vary over D_ℓ, we get the family $\{B_{\omega i} :\ \omega \in D_\ell,\ 1 \le i \le N_{\ell+1}\} := \{B_{\omega'} : \omega' \in D_{\ell+1}\}$. Clearly, (p3) holds for $\ell + 1$.

Hence by induction, we can construct well $\mathcal{G} := \{B_\omega :\ \omega \in D\}$ which has the Moran structure and satisfies (p1)-(p3). Clearly, by (p3), for each $\ell \ge 1$ and $\omega \in D_\ell$ we have

$$\prod_{i=1}^{\ell} 2^{-n_i(\tau'(q_{\theta(i)})+1/k_{\theta(i)})} \le \frac{\mu(B_\omega)}{\mu(B_\emptyset)} \le \prod_{i=1}^{\ell} 2^{-n_i(\tau'(q_{\theta(i)})-1/k_{\theta(i)})}. \tag{2.17}$$

Let $F = \bigcap_{\ell=1}^\infty \bigcup_{\omega\in D_\ell} B_\omega$ be the Moran set associated with $\mathcal{G}$. We can use Proposition 2.4 to determine the Hausdorff dimension of F. Indeed in our case, $c_\ell = 2^{-n_\ell}$

and $M_\ell = 2^{-n_1-\cdots-n_\ell}$, hence by (2.13), the assumption (2.4) fulfills. Thus by Proposition 2.4 and (2.16),

$$\dim_H F = \liminf_{\ell\to\infty} \frac{\log(N_1\cdots N_\ell)}{\log(2^{n_1+\cdots+n_\ell})} \geq \liminf_{\ell\to\infty} \lambda_{\theta(\ell)} \geq \tau^*(\alpha).$$

In the end of this step, we show that $F \subset E(\alpha)$ and hence $\dim_H E(\alpha) \geq \dim_H F \geq \tau^*(\alpha)$. To see this, let $x \in F$. Let $r > 0$ be a small number. Then there exists $\ell \geq 1$ such that

$$2^{-n_1-\cdots-n_{\ell+1}} \leq r < 2^{-n_1-\cdots-n_\ell}. \tag{2.18}$$

Clearly, $B_r(x)$ contains a ball, say $B_{\omega'}$, for some $\omega' \in D_{\ell+2}$. On the other hand, $B_r(x)$ intersects at least one ball, say B_ω, for some $\omega \in D_\ell$, which implies $B_r(x) \subseteq 2B_\omega$. Hence we have

$$\mu(B_r(x)) \geq \mu(B_{\omega'}) \quad \text{and} \quad \mu(B_r(x)) \leq \mu(2B_\omega) \leq (n_1+\cdots+n_\ell)\mu(B_\omega). \tag{2.19}$$

Combining (2.19) with (2.17), (2.18) and (2.13) yields

$$\lim_{r\to 0} \frac{\log\mu(B_r(x))}{\log r} = \lim_{i\to\infty} \tau'(q_{\theta(i)}) + 1/k_{\theta(i)} = \alpha.$$

That is, $x \in E(\alpha)$. Hence we have $F \subset E(\alpha)$. This finishes the proof of Step 1.

Step 2. If $\alpha = p\tau'(q_1) + (1-p)\tau'(q_2)$ for some $0 < p < 1$ and $q_1, q_2 \in \Omega$, then $E(\alpha) \neq \emptyset$ and $\dim_H E(\alpha) \geq p\tau^*(\alpha_1) + (1-p)\tau^*(\alpha_2)$, where $\alpha_1 := \tau'(q_1)$ and $\alpha_2 = \tau'(q_2)$.

The proof of this step is quite similar to that in Step 1. We only list the main different points.

Fix $q_1, q_2 \in \Omega$ and $0 < p < 1$. Since Λ is dense in Ω, there exist two sequences $(q_{1,j})_{j=1}^\infty$, $(q_{2,j})_{j=1}^\infty \subset \Lambda$ such that $\lim_{j\to\infty} q_{i,j} = q_i$, $i = 1,2$. Since τ is concave, we have $\lim_{j\to\infty} \tau'(q_{i,j}) = \tau'(q_i) = \alpha_i$, $i = 1,2$. By (2.2), there exists a sequence of integers $(k_j) \uparrow \infty$ such that $\lim_{j\to\infty} b(q_{i,j}, k_j) = 0$.

By (2.2), we can define a sequence $(L_j)_{j=0}^\infty$ of integers such that $L_0 = 0$ and for $j \geq 1$,

$$n \geq f_0(q_{i,1}, k_1) + a(q_{i,1}, k_1)\log A_0 \quad \text{if } n \geq L_1,\ i = 1,2,$$

$$n \geq f_n(q_{i,1}, k_1) + a(q_{i,1}, k_1)\log f_n(q_{i,1}, k_1) \quad \text{if } n \geq L_1,\ i = 1,2,$$

and

$$\frac{n}{j+1} \geq f_n(q_{i,j+1}, k_{j+1}) + a(q_{i,j+1}, k_{j+1})\log(f_n(q_{i,j+1}, k_{j+1}) + f_n(q_{i,j}, k_j))$$

if $n \geq L_j$, $j = 1, 2, \ldots$, $i = 1,2$. Note that the sequence $(L_j)_{j=0}^\infty$ may be different from what we constructed in Step 1.

Construct $(n_\ell)_{\ell=1}^\infty$ from $(L_j)_{j=0}^\infty$ in the same way as in Step 1. Again, we use $\theta(\ell)$ denote the unique positive integer j satisfying $\sum_{s=0}^{j-1} L_s \leq \ell < \sum_{s=0}^{j} L_s$.

For $\ell \geq 1$, set

$$t_\ell = \begin{cases} 1 & \text{if } \{\ell\sqrt{2}\} \in [0,p), \\ 2 & \text{if } \{\ell\sqrt{2}\} \in [p,1), \end{cases} \tag{2.20}$$

where $\{x\}$ denotes the fractional part of x, and define

$$u_\ell = \tau'(q_{t_\ell,\theta(\ell)})q_{t_\ell,\theta(\ell)} - \tau(q_{t_\ell,\theta(\ell)}) - b(q_{t_\ell,\theta(\ell)}, k_{\theta(\ell)}).$$

It is easy to check that

$$\lim_{\ell\to\infty} (u_\ell - \tau^*(\alpha_{t_\ell})) = 0. \tag{2.21}$$

Then define a sequence $(N_\ell)_{\ell=1}^\infty$ by

$$N_\ell = \max\left\{1,\ [2^{n_\ell u_\ell}]\right\},$$

here $[x]$ denotes the integer part of x.

Pick $x_0 \in \mathbb{R}$ such that $\mu(B_{1/2}(x_0)) > 0$. Let $D = \bigcup_{\ell\geq 0} D_\ell$ with $D_0 = \{\emptyset\}$ and $D_\ell = \{\omega = (i_1 i_2 \cdots i_\ell) : 1 \leq i_j \leq N_j, 1 \leq j \leq \ell\}$. Similar to Step 1, we can construct a collection $\mathcal{G} = \{B_\omega : \omega \in D\}$ of closed balls of radius r_ω in $\mathbb{R}^d$ recursively, which has Moran structure and satisfies the following properties:

(q1) $B_\emptyset = B_1(x_0)$;
(q2) $r_\omega = 2^{-(n_1+\cdots+n_\ell)}$ for each $\omega \in D_\ell$;
(q3) For each $\ell \geq 1$, $\omega \in D_{\ell-1}$ and $1 \leq i \leq N_\ell$,

$$2^{-n_\ell(\tau'(q_{t_\ell,\theta(\ell)})+1/k_{\theta(\ell)})} \leq \frac{\mu(B_{\omega i})}{\mu(B_\omega)} \leq 2^{-n_\ell(\tau'(q_{t_\ell,\theta(\ell)})-1/k_{\theta(\ell)})}.$$

and

$$\mu(2B_{\omega i})/\mu(\frac{1}{2}B_{\omega i}) \leq f_{n_1+\cdots+n_\ell}(q_{t_\ell,\theta(\ell)}, k_{\theta(\ell)}) \leq n_1 + \cdots + n_\ell.$$

Let $F = \bigcap_{\ell=1}^\infty \bigcup_{\omega\in D_\ell} B_\omega$ be the Moran set associated with $\mathcal{G}$. Similar to Step 1, we can show that $F \subset E(\alpha)$ and

$$\dim_H E(\alpha) \geq \dim_H F = \liminf_{\ell\to\infty} \frac{\log(N_1 \cdots N_\ell)}{\log(2^{n_1+\cdots+n_\ell})} \geq p\tau^*(\alpha_1) + (1-p)\tau^*(\alpha_2).$$

This finishes the proof of Step 2.

Step 3. *$E(\alpha) \neq \emptyset$ if and only if $\alpha \in [\alpha_{\min}, \alpha_{\max}] \cap \mathbb{R}$. Furthermore, for any $\alpha \in [\alpha_{\min}, \alpha_{\max}] \cap \mathbb{R}$, $\dim_H E(\alpha) = \tau^*(\alpha) = \inf\{\alpha q - \tau(q) :\ q \in \mathbb{R}\}$.*

First we show that $E(\alpha) \neq \emptyset$ implies that $\alpha \in [\alpha_{\min}, \alpha_{\max}]$. Indeed, assume that $\alpha = \lim_{r\to 0} \frac{\log \mu(B_r(x))}{\log r}$ for some $x \in \mathbb{R}$. Then $\Theta(q,r) \geq \mu(B_r(x))^q$ (cf. (1.2)), which implies $\tau(q) \leq \alpha q$. Hence $\alpha \in [\alpha_{\min}, \alpha_{\max}]$.

Next we show that if $\alpha \in [\alpha_{\min}, \alpha_{\max}] \cap \mathbb{R}$, then $E(\alpha) \neq \emptyset$ and $\dim_H E(\alpha) \geq \tau^*(\alpha)$. To see this, let $\alpha \in [\alpha_{\min}, \alpha_{\max}] \cap \mathbb{R}$. Since τ is concave, there are only two possible cases: (1) $\alpha \in \overline{\{\tau'(q) : q \in \Omega\}}$; (2) $\alpha \in (\tau'(q+), \tau'(q-))$ for some $q \in \mathbb{R}$, here $\tau'(q+), \tau'(q-)$ denote the right and left derivatives of τ at q, respectively. By Step 1, we only need to consider the second case. Clearly, there exists $0 < p < 1$ such that

$$\alpha = p\tau'(q+) + (1-p)\tau'(q-).$$

Since τ is concave, there exist two sequences $(q_j)_{j=1}^{\infty}$, $(q'_j)_{j=1}^{\infty} \subset \Omega$ such that

$$q_j \searrow q, \quad q'_j \nearrow q, \quad \tau'(q_j) \nearrow \tau'(q+), \quad \tau'(q'_j) \searrow \tau'(q-)$$

as j tends to infinity. Therefore, there exists a sequence $(p_j)_{j=1}^{\infty} \subset (0,1)$ such that $\lim_{j\to\infty} p_j = p$ and

$$\alpha = p_j\tau'(q_j) + (1-p_j)\tau'(q'_j).$$

By Step 2, we have $E(\alpha) \neq \emptyset$ and

$$\dim_H E(\alpha) \geq p_j(\tau'(q_j)q_j - \tau(q_j)) + (1-p_j)(\tau'(q'_j)q'_j - \tau(q'_j)), \quad j \in \mathbb{N}.$$

Letting $j \to \infty$, we obtain

$$\dim_H E(\alpha) \geq (p\tau'(q+) + (1-p)\tau'(q-))q - \tau(q) = \alpha q - \tau(q) = \tau^*(\alpha).$$

In the end, we point out that if $\alpha \in [\alpha_{\min}, \alpha_{\max}] \cap \mathbb{R}$, then $\dim_H E(\alpha) = \tau^*(\alpha)$. This follows from the basic fact that $\dim_H E(\alpha) \leq \tau^*(\alpha)$ whenever $E(\alpha) \neq \emptyset$ (indeed, this fact holds for any compactly supported probability measure; see, e.g., Theorem 4.1 in [21]). This finishes the proof of Theorem 2.2. □

3. Self-conformal measures with the AWSC

In this section we prove Theorem 1.3. In Sect. 3.1, we introduce some notation and definitions about self-conformal measures and the asymptotically weak separation condition. In Sect. 3.2, we show that any self-conformal measure with the asymptotically weak separation condition has an asymptotically multifractal structure on $\mathbb{R}^+$; then Theorem 1.3 follows from Theorem 2.2(ii).

3.1. **Self-conformal measures and asymptotically weak separation condition.** Let $U \subset \mathbb{R}^d$ be an open set. A C^1-map $S : U \to \mathbb{R}^d$ is *conformal* if the differential $S'(x) : \mathbb{R}^d \to \mathbb{R}^d$ satisfies $|S'(x)y| = |S'(x)| \cdot |y| \neq 0$ for all $x \in U$ and $y \in \mathbb{R}^d$, $y \neq 0$. Furthermore, $S : U \to \mathbb{R}^d$ is *contracting* if there exists $0 < c < 1$ such that $|S(x) - S(y)| \leq c \cdot |x - y|$ for all $x, y \in U$. We say that $\{S_i : X \to X\}_{i=1}^{\ell}$ is a C^1*-conformal iterated function system* (C^1-conformal IFS) on a compact set $X \subset \mathbb{R}^d$ if each S_i extends to an injective contracting C^1-conformal map $S_i : U \to U$ on an open set $U \supset X$.

Let $\{S_i\}_{i=1}^{\ell}$ be a C^1-conformal IFS on a compact set $X \subset \mathbb{R}^d$. It is well-known, see [18], that there is a unique non-empty compact set $K \subset X$ such that $K = \bigcup_{i=1}^{\ell} S_i(K)$. Given a probability vector $(p_1, \ldots, p_\ell)$, there is a unique Borel probability measure ν satisfying

$$\nu = \sum_{i=1}^{\ell} p_i \nu \circ S_i^{-1}. \tag{3.1}$$

This measure is supported on K and it is called *self-conformal.* In particular, if the maps S_i are all similitudes, then ν is called *self-similar.*

Let $\mathcal{A} = \{1, \ldots, \ell\}$. Denote $\mathcal{A}^* = \bigcup_{n\geq 1} \mathcal{A}^n$. For $u = u_1 \ldots \circ u_k$, we write $S_u = S_{u_1} \circ \cdots S_{u_k}$, $p_u = p_{u_1} \cdots p_{u_k}$ and $K_u = S_u(K)$; in particular we let $\tilde{u}$ denote the word obtained by dropping the last letter of u. For $n \in \mathbb{N}$, denote

$$W_n := \left\{ u \in \mathcal{A}^* : \ \operatorname{diam}(K_u) \leq 2^{-n}, \operatorname{diam}(K_{\tilde{u}}) > 2^{-n} \right\}. \tag{3.2}$$

For $n \geq 0$, let

$$\mathbf{D}_n = \{[0, 2^{-n})^d + \mathbf{v} : \ \mathbf{v} \in 2^{-n}\mathbb{Z}^d\}, \tag{3.3}$$

and define

$$\tau_n(q) = \sum_{Q \in \mathbf{D}_n} \nu(Q)^q.$$

Proposition 3.1. *There is a sequence $(c_n)_{n=1}^{\infty}$ of positive numbers with*

$$\lim_{n\to\infty} \frac{1}{n} \log c_n = 0,$$

such that for any $q > 0$, $n, m \in \mathbb{N}$, and all $u \in W_n$,

$$(c_n)^{-(q+1)} \tau_m(q) \leq \sum_{Q \in \mathbf{D}_{m+n}} (\nu(S_u^{-1}Q))^q \leq (c_n)^{q+1} \tau_m(q). \tag{3.4}$$

Furthermore, the limit $\lim_{m\to\infty} \frac{\log \tau_m(q)}{-m \log 2}$ exists for each $q > 0$ and it coincides with $\tau(q) := \tau_\nu(q)$ defined as in Sec 1.

Proof. It was proved in [10, Proposition 3.3] that there exists $\beta > 0$ such that for any $\epsilon > 0$, there exists $C(\epsilon) > 0$ such that for all $q > 0$, $m, n \in \mathbb{N}$, and all $u \in W_n$,

$$\left(C(\epsilon)(1+\epsilon)^{\beta n}\right)^{-(q+1)} \tau_m(q) \leq \sum_{Q \in \mathcal{D}_{m+n}} (\nu(S_u^{-1}Q))^q \leq \left(C(\epsilon)(1+\epsilon)^{\beta n}\right)^{q+1} \tau_m(q). \tag{3.5}$$

Choose a sequence of positive numbers (ϵ_n) tending to 0 slowly enough such that $\lim_{n\to\infty} (1/n) \log C(\epsilon_n) = 0$. Let $c_n = C(\epsilon_n)(1+\epsilon_n)^{\beta n}$. Then $\lim_{n\to\infty} (\log c_n)/n = 0$, and (3.4) follows from (3.5). The existence of $\lim_{m\to\infty} \frac{\log \tau_m(q)}{-m \log 2}$ for each $q > 0$ was proved in [10, Proposition 4.3]. It is easy to check that the limit coincides with $\tau_\nu(q)$. □

We remark that Proposition 3.1 was first proved by Peres and Solomyak [30] under the bounded distortion assumption on $\{S_i\}_{i=1}^{\ell}$. In that case, the involved (c_n) in (3.4) can be replaced by a constant c.

The following definition was introduced in [10].

Definition 3.2. The IFS $\{S_i\}_{i=1}^{\ell}$ is said to satisfy the *asymptotically weak separation condition* (AWSC) if there exists a sequence (t_n) of natural numbers such that

$$\lim_{n\to\infty} \frac{1}{n} \log t_n = 0$$

and for each $n \in \mathbb{N}$ and $\widetilde{Q} \in \mathbf{D}_n$ (see (3.3) for the definition of $\mathbf{D}_n$),

$$\#\{S_u : \ u \in W_n, K_u \cap \widetilde{Q} \neq \emptyset\} \leq t_n. \tag{3.6}$$

For instance, when $\beta > 1$ is a Salem number, then an IFS $\{S_i\}_{i=1}^{\ell}$ on $\mathbb{R}$ satisfies the AWSC if each S_i has the form

$$S_i(x) = \pm\beta^{-m_i} x + d_i,$$

where $m_i \in \mathbb{N}$ and $d_i \in \mathbb{Z}[\beta]$, here $\mathbb{Z}[\beta]$ denotes the integral ring generated by β. For a proof, see [10, Proposition 5.3, Remark 5.5].

Remark 3.3. The AWSC is strictly weaker than the WSC introduced in [21]. To see it, for $\beta \in (1, 2)$ and $m \in \mathbb{N}$, set

$$Y^{\beta,m} := \left\{ \sum_{i=0}^{n} \epsilon_i \beta^i : \ n \in \mathbb{N}, \epsilon_i \in \{0, \pm 1, \ldots, \pm m\} \text{ for } 0 \leq i \leq n \right\}.$$

Erdös and Komornik [5] proved that if β is not a Pisot number and $m \geq \beta - \beta^{-1}$, then $Y^{\beta,m}$ contains accumulation points. This implies that the IFS $\{\lambda x, \lambda x + 1\}$ does not satisfies the WSC when $\lambda^{-1} \in (1, (\sqrt{5} + 1)/2)$ is not a Pisot number. However this IFS satisfies the AWSC when λ^{-1} is a Salem number; and there do exist infinitely many Salem numbers in $(1, (\sqrt{5} + 1)/2)$ (see, e.g., [2]).

3.2. **Asymptotically good multifractal structure.** In this subsection, we assume that $\{S_i\}_{i=1}^{\ell}$ is a C^1-conformal IFS on a compact set $X \subset \mathbb{R}^d$ which satisfies the AWSC. Let ν be a self-conformal measure associated with $\{S_i\}_{i=1}^{\ell}$ and a probability vector $(p_1, \ldots, p_\ell)$. The main result of this subsection is the following.

Theorem 3.4. *The measure ν has an asymptotically good multifractal structure over $\mathbb{R}_+$.*

To prove the above theorem, we need a simple lemma.

Lemma 3.5. *Let $q > 0$ so that $\tau'(q)$ exists and let $k \in \mathbb{N}$. Then there exist positive numbers ϵ, δ, γ and M (all depend on q, k) with $\epsilon < \min\{1, q\}$, $\delta = \min\{1/(4k), 1/(4kq)\}$, and $\gamma < 1/(4k)$, such that for any $m \geq M$,*

$$\tau_m(q) \geq 2^{-m(\tau(q)+\gamma)}, \tag{3.7}$$

$$\tau_m(q+\epsilon)\ 2^{m(\tau'(q)-\delta)\epsilon} \leq \tau_m(q)\ 2^{-m\gamma} \tag{3.8}$$

and

$$\tau_m(q-\epsilon)\ 2^{-m(\tau'(q)+\delta)\epsilon} \leq \tau_m(q)\ 2^{-m\gamma}. \tag{3.9}$$

Proof. Set $\delta = \min\{1/(4k), 1/(4kq)\}$. Since $\tau'(q)$ exists, we can pick $0 < \epsilon < \min\{1, q\}$ so that

$$(\alpha - \delta/2)\epsilon \leq |\tau(q \pm \epsilon) - \tau(q)| \leq (\alpha + \delta/2)\epsilon.$$

Set $\gamma = \min\{\epsilon\delta/8, 1/(4k)\}$. Since $\tau(u) = \lim_{n\to\infty} \tau_n(u)$ for each $u > 0$, we take M large enough such that for $m \geq M$,

$$2^{-m(\tau(u)+\gamma)} \leq \tau_m(u) \leq 2^{-m(\tau(u)-\gamma)} \quad \text{for } u = q,\ q-\epsilon,\ q+\epsilon.$$

Then we have

$$\begin{aligned}
\tau_m(q+\epsilon)2^{m(\alpha-\delta)\epsilon} &\leq 2^{-m(\tau(q+\epsilon)-\gamma)}2^{m(\alpha-\delta)\epsilon} \\
&\leq 2^{-m(\tau(q)+\gamma)}2^{-m(\tau(q+\epsilon)-\tau(q))}2^{m((\alpha-\delta)\epsilon+2\gamma)} \\
&\leq \tau_m(q)2^{-m(\alpha-\delta/2)\epsilon}2^{m((\alpha-\delta)\epsilon+2\gamma)} \\
&\leq \tau_m(q)2^{-m(\delta\epsilon/2-2\gamma)} \leq \tau_m(q)2^{-m\gamma},
\end{aligned}$$

which proves (3.8). The proof of (3.9) is essentially identical. □

The following lemma is obvious.

Lemma 3.6. *Let $q > 0$. For any $n \in \mathbb{N}$ and non-negative numbers $x_1, \ldots, x_n$,*

$$\frac{1}{n}(x_1^q + \cdots + x_n^q) \leq (x_1 + \cdots + x_n)^q \leq n^q(x_1^q + \cdots + x_n^q). \tag{3.10}$$

Proof of Theorem 3.4. Set

$$t_n = \max_{\widetilde{Q}\in\mathbf{D}_n} \#\{S_u :\ u \in W_n,\ K_u \cap \widetilde{Q} \neq \emptyset\}, \qquad n \in \mathbb{N}.$$

(See Sect. 3.1 for the notation.) Since the IFS $\{S_i\}_{i=1}^{\ell}$ is assumed to satisfy the AWSC (cf. Def. 3.4), we have

$$\lim_{n\to\infty} \frac{1}{n} \log t_n = 0.$$

For each $n \in \mathbb{N}$, define an equivalence relation on W_n by setting $u \sim v$ if and only if $S_u = S_v$. For $u \in W_n$, let $[u]$ denote the equivalence class containing u. In particular, we write

$$\overline{p}_{[u]} := \sum_{v\in[u]} p_u, \quad S_{[u]} := S_u, \quad \text{and} \quad K_{[u]} := K_u.$$

Iterating (3.1), we obtain

$$\nu = \sum_{[u]\in W_n/\sim} \overline{p}_{[u]}\, \nu \circ S_{[u]}^{-1}. \tag{3.11}$$

Recall that by Proposition 3.1, there is a sequence of positive numbers $(c_n)_{n=1}^{\infty}$ with $c_n > 1$ and $\lim_{n\to\infty}(1/n)\log c_n = 0$ such that (3.4) holds.

From now on, we fix $n \geq 0$ and $x \in \mathbb{R}$ such that $\mu(B_{2^{-n-1}}(x)) > 0$. Fix $q > 0$ so that $\tau'(q)$ exists and fix $k \in \mathbb{N}$. Let $\epsilon, \gamma, \delta, M$ be the positive numbers (depending on q, k) given in Lemma 3.5 so that (3.8)-(3.9) hold. Recall that we have the restrictions that

$$\delta = \min\left\{\frac{1}{4k}, \frac{1}{4kq}\right\}, \quad \epsilon < \min\{1, q\} \quad \text{and} \quad \gamma < \frac{1}{4k}. \tag{3.12}$$

Denote

$$A = \frac{\nu(B_{2^{-n}}(x))}{\nu(B_{2^{-n-1}}(x))}.$$

For convenience, denote $r = 2^{-n}$. Let n' be the unique integer satisfying

$$r/16 < 2^{-n'}\sqrt{d} \leq r/8. \tag{3.13}$$

Clearly

$$0 < n' - n < 4 + \frac{\log d}{2\log 2}. \tag{3.14}$$

A simple geometric argument shows that $B_r(x)$ intersects at most

$$\left(\frac{2r}{2^{-n'}} + 1\right)^d \leq (32\sqrt{d})^d$$

elements in $\mathbf{D}_{n'}$. Hence we have

$$\#\{[u] \in W_{n'}/\sim\colon\ K_{[u]} \cap B_r(x) \neq \emptyset\} \leq (32\sqrt{d})^d t_{n'} =: \tilde{t}_{n'}. \tag{3.15}$$

Pick $[u_0] \in W_{n'}/\sim$ such that $K_{[u_0]} \cap B_{r/2}(x) \neq \emptyset$ and

$$\overline{p}_{[u_0]} = \max\{\overline{p}_{[u]}:\ [u] \in W_{n'}/\sim, K_u \cap B_{r/2}(x) \neq \emptyset\}.$$

By (3.11),

$$\sum_{[u]\in W_{n'}/\sim,\ K_{[u]}\cap B_{r/2}(x)\neq\emptyset} \overline{p}_{[u]} \geq \nu(B_{r/2}(x)) = \frac{\nu(B_r(x))}{A}.$$

Therefore we have

$$\overline{p}_{[u_0]} \geq \frac{\nu(B_r(x))}{\tilde{t}_{n'} A}. \tag{3.16}$$

Set

$$\Gamma = \{[u] \in W_{n'}/\sim, K_{[u]} \cap B_{7r/8}(x) \neq \emptyset\}.$$

By (3.15), $\#\Gamma \leq \tilde{t}_{n'}$. Now define a measure η on $\mathbb{R}^d$ by

$$\eta = \sum_{[u]\in\Gamma} \overline{p}_{[u]}\ \nu \circ S_{[u]}^{-1}.$$

Then by (3.11), the restrictions of η and ν on $B_{7r/8}(x)$ coincide, i.e., $\eta|_{B_{7r/8}(x)} = \nu|_{B_{7r/8}(x)}$. By (3.13), $K_{[u]} \subset B_r(x)$ for all $[u] \in \Gamma$, hence by (3.11),

$$\sum_{[u]\in\Gamma} \overline{p}_{[u]} \leq \nu(B_r(x)). \tag{3.17}$$

Let $m' \in \mathbb{N}$. Denote

$$\tau_{n'+m'}(F, q) = \sum_{Q\in\mathbf{D}_{n'+m'}:\, Q\subset F} \nu(Q)^q, \qquad F \subset \mathbb{R}^d.$$

Since $K_{[u_0]} \cap B_{r/2}(x) \neq \emptyset$, by (3.13), for all those $Q \in \mathbf{D}_{n'+m'}$ with $Q \cap K_{[u_0]} \neq \emptyset$, we have $Q \subset B_{3r/4}(x)$. Hence we have

$$\begin{aligned} \tau_{n'+m'}(B_{3r/4}(x), q) &\geq \sum_{Q\in\mathbf{D}_{n'+m'}:\, Q\cap K_{[u_0]}\neq\emptyset} \nu(Q)^q \\ &\geq \sum_{Q\in\mathbf{D}_{n'+m'},\, Q\cap K_{u_0}\neq\emptyset} (\overline{p}_{[u_0]})^q\ (\nu \circ S_{[u_0]}^{-1}(Q))^q \\ &= (\overline{p}_{[u_0]})^q \sum_{Q\in\mathbf{D}_{n'+m'}} (\nu \circ S_{u_0}^{-1}(Q))^q \\ &\geq (c_{n'})^{-(q+1)} (\overline{p}_{[u_0]})^q \tau_{m'}(q) \qquad \text{(by (3.4))} \\ &\geq (c_{n'}\tilde{t}_{n'}A)^{-q-1} \nu(B_r(x))^q \tau_{m'}(q) \qquad \text{(by (3.16))}. \end{aligned} \tag{3.18}$$

On the other hand, we have

$$
\begin{aligned}
\tau_{n'+m'}(B_{7r/8}(x),q) &= \sum_{Q\in\mathbf{D}_{n'+m'}:\, Q\subset B_{7r/8}(x)} \nu(Q)^q \\
&= \sum_{Q\in\mathbf{D}_{n'+m'}:\, Q\subset B_{7r/8}(x)} \eta(Q)^q \le \sum_{Q\in\mathbf{D}_{n'+m'}} \eta(Q)^q \\
&= \sum_{Q\in\mathbf{D}_{n'+m'}} \sum_{[u]\in\Gamma} \Big(\overline{p}_{[u]}\ \nu\circ S_{[u]}^{-1}(Q)\Big)^q \\
&\le \sum_{Q\in\mathbf{D}_{n'+m'}} (\tilde{t}_{n'})^q \sum_{[u]\in\Gamma} (\overline{p}_{[u]})^q\ \nu\circ S_u^{-1}(Q)^q \qquad \text{(by (3.10))} \\
&\le (\tilde{t}_{n'})^q \sum_{[u]\in\Gamma} (\overline{p}_{[u]})^q \sum_{Q\in\mathbf{D}_{n'+m'}} \nu\circ S_u^{-1}(Q)^q \\
&\le (c_{n'}\tilde{t}_{n'})^{q+1}\nu(B_r(x))^q\tau_{m'}(q) \qquad \text{(by (3.4), (3.17))}.
\end{aligned}
\tag{3.19}
$$

Combining (3.18) with (3.19) yields

$$
\tau_{n'+m'}(B_{7r/8}(x),q) \le \tau_{n'+m'}(B_{3r/4}(x),q)\cdot(c_{n'}\tilde{t}_{n'}A)^{2q+2}, \quad \forall m'\in\mathbb{N}. \tag{3.20}
$$

We remark that in (3.18)-(3.20), q can be replaced by any positive number.

From now on, assume that

$$
\begin{aligned}
m' \ge h_n = h_n(q,k) :=& M + \frac{2q+3}{\gamma}\Big(\log(4c_{n'}\tilde{t}_{n'}) + \log A \\
&+ \log(8^{1/q}\cdot 5^{d(q+1)/q})\Big),
\end{aligned}
\tag{3.21}
$$

where γ and M are the positive numbers given in Lemma 3.5 (they depend on q and k).

It is easy to see that

$$
2^{2^{m'-1}} \ge (c_{n'}\tilde{t}_{n'}A)^{2q+2}.
$$

By (3.20), there exists $1\le j\le 2^{m'-1}$ such that

$$
\tau_{n'+m'}(B_{3r/4+j\cdot 2^{-(n'+m'-1)}\sqrt{d}}(x),q) \le 2\tau_{n'+m'}(B_{3r/4+(j-1)\cdot 2^{-(n'+m'-1)}\sqrt{d}}(x),q).
$$

(Otherwise,

$$
\begin{aligned}
\tau_{n'+m'}(B_{7r/8}(x),q) &\ge \tau_{n'+m'}(B_{3r/4+2^{m'-1}\cdot 2^{-(n'+m'-1)}\sqrt{d}}(x),q) \\
&\ge 2\tau_{n'+m'}(B_{3r/4+(2^{m'-1}-1)\cdot 2^{-(n'+m'-1)}\sqrt{d}}(x),q) \\
&\ge \cdots \\
&\ge 2^{2^{m'-1}}\tau_{n'+m'}(B_{3r/4}(x),q),
\end{aligned}
$$

which contradicts (3.20).) Fix such j and take

$$
r' = 3r/4 + (j-1)\cdot 2^{-(n'+m'-1)}\sqrt{d}.
$$

Then

$$\tau_{n'+m'}(B_{r'+2^{-(n'+m'-1)}\sqrt{d}}(x), q) \leq 2\tau_{n'+m'}(B_{r'}(x), q). \tag{3.22}$$

Now define

$$\mathcal{F} = \{Q \in \mathbf{D}_{n'+m'} : \ Q \subset B_{7r/8}(x), \ \nu(Q) < \nu(B_r(x)) \cdot 2^{-m'(\alpha+\delta)}\},$$
$$\mathcal{F}' = \{Q \in \mathbf{D}_{n'+m'} : \ Q \subset B_{7r/8}(x), \ \nu(Q) > \nu(B_r(x)) \cdot 2^{-m'(\alpha-\delta)}\}.$$

Then we have the estimation

$$\begin{aligned}
\sum_{Q\in\mathcal{F}} \nu(Q)^q &\leq \nu(B_r(x))^\epsilon 2^{-m'(\alpha+\delta)\epsilon} \sum_{Q\in\mathcal{F}} \nu(Q)^{q-\epsilon} \\
&\leq \nu(B_r(x))^\epsilon 2^{-m'(\alpha+\delta)\epsilon} \tau_{n'+m'}(B_{7r/8}(x), q-\epsilon) \\
&\leq (c_{n'}\tilde{t}_{n'})^{q-\epsilon+1} 2^{-m'(\alpha+\delta)\epsilon} \nu(B_r(x))^q \tau_{m'}(q-\epsilon) \\
&\qquad \text{(by applying (3.19), in which } q \text{ is replaced by } q-\epsilon) \\
&\leq (c_{n'}\tilde{t}_{n'})^{q+2} \nu(B_r(x))^q \tau_{m'}(q) 2^{-m'\gamma} \qquad \text{(by (3.9))} \\
&\leq (c_{n'}\tilde{t}_{n'}A)^{2q+3} 2^{-m'\gamma} \tau_{n'+m'}(B_{3r/4}(x), q) \qquad \text{(by (3.18))} \\
&\leq \frac{1}{4}\tau_{n'+m'}(B_{3r/4}(x), q) \qquad \text{(by (3.21))}.
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
\sum_{Q\in\mathcal{F}'} \nu(Q)^q &\leq \nu(B_r(x))^{-\epsilon}\, 2^{m'(\alpha-\delta)\epsilon} \sum_{Q\in\mathcal{F}'} \nu(Q)^{q+\epsilon} \\
&\leq \nu(B_r(x))^{-\epsilon}\, 2^{m'(\alpha-\delta)\epsilon}\, \tau_{n'+m'}(B_{7r/8}(x), q+\epsilon) \\
&\leq (c_{n'}\tilde{t}_{n'})^{q+\epsilon+1}\, 2^{m'(\alpha-\delta)\epsilon} \nu(B_r(x))^q \tau_{m'}(q+\epsilon) \\
&\qquad \text{(by applying (3.19), in which } q \text{ is replaced by } q+\epsilon) \\
&\leq (c_{n'}\tilde{t}_{n'})^{q+2} \nu(B_r(x))^q \tau_{m'}(q) 2^{-m'\gamma} \qquad \text{(by (3.8))} \\
&\leq (c_{n'}A\tilde{t}_{n'})^{2q+3} 2^{-m'\gamma} \tau_{n'+m'}(B_{3r/4}(x), q) \qquad \text{(by (3.18))} \\
&\leq \frac{1}{4}\tau_{n'+m'}(B_{3r/4}(x), q).
\end{aligned}$$

For any $Q \in \mathbf{D}_{n'+m'}$, we denote by

$$Q^* = \prod_{s=1}^{d} \left[\frac{i_s - 2}{2^{n'+m'}}, \frac{i_s+3}{2^{n'+m'}}\right) \quad \text{if} \quad Q = \prod_{s=1}^{d} \left[\frac{i_s}{2^{n'+m'}}, \frac{i_s+1}{2^{n'+m'}}\right).$$

Clearly, Q^* contains exactly 5^d many elements in $\mathbf{D}_{n'+m'}$. Set

$$T := 8^{1/q} \cdot 5^{d(q+1)/q} \qquad \text{and}$$
$$\mathcal{F}'' = \{Q \in \mathbf{D}_{n'+m'} : \ Q \subset B_{r'}(x), \ \nu(Q^*) > T\nu(Q)\}.$$

Then

$$\begin{aligned}
\sum_{Q\in\mathcal{F}''}\nu(Q)^q &\le \sum_{Q\in\mathbf{D}_{n'+m'}:\, Q\subset B_{r'}(x)} T^{-q}\nu(Q^*)^q \\
&\le T^{-q}5^{d(q+1)}\tau_{n'+m'}(B_{r'+2^{-(n'+m'-1)}\sqrt{d}}(x),q) \qquad \text{(by (3.10))} \\
&\le 2\cdot T^{-q}5^{d(q+1)}\tau_{n'+m'}(B_{r'}(x),q) \qquad \text{(by (3.22))} \\
&= \frac{1}{4}\tau_{n'+m'}(B_{r'}(x),q).
\end{aligned}$$

Let

$$\mathcal{P}=\{Q\in\mathbf{D}_{n'+m'}:\ Q\subset B_{r'}(x), \nu(Q^*)\le T\nu(Q) \text{ and } 2^{-m'(\alpha+\delta)}\le \nu(Q)/\nu(B_r(x))\le 2^{-m'(\alpha-\delta)}\}.$$

We have

$$\begin{aligned}
\sum_{Q\in\mathcal{P}}\nu(Q)^q &\ge \sum_{Q\in\mathbf{D}_{n'+m'}:\, Q\subset B_{r'}(x)}\nu(Q)^q-\sum_{Q\in\mathcal{F}\cup\mathcal{F}'\cup\mathcal{F}''}\nu(Q)^q \\
&= \tau_{n'+m'}(B_{r'}(x),q)-\sum_{Q\in\mathcal{F}\cup\mathcal{F}'\cup\mathcal{F}''}\nu(Q)^q \\
&\ge \frac{1}{4}\tau_{n'+m'}(B_{r'}(x),q)\ge\frac{1}{4}\tau_{n'+m'}(B_{3r/4}(x),q) \\
&\ge \frac{1}{4}(c_{n'}\tilde{t}_{n'}A)^{-q-1}\nu(B_r(x))^q\tau_{m'}(q) \qquad \text{(by (3.18))} \\
&\ge 2^{-m'/(4k)}\nu(B_r(x))^q2^{-m'(\tau(q)+\gamma)}>0 \qquad \text{(by (3.21), (3.7))}.
\end{aligned}$$

Clearly $\#\mathcal{P}\ge 1$. Since $\nu(Q)\le\nu(B_r(x))2^{-m'(\alpha-\delta)}$ for each $Q\in\mathcal{P}$, we have

$$\begin{aligned}
\#\mathcal{P} &\ge \nu(B_r(x))^{-q}2^{qm'(\alpha-\delta)}\sum_{Q\in\mathcal{P}}\nu(Q)^q \\
&\ge 2^{m'(\alpha q-\tau(q)-\delta q-\gamma-\frac{1}{4k})}\ge 2^{m'(\alpha q-\tau(q)-\frac{3}{4k})} \qquad \text{(by (3.12))} \\
&\ge 5^d2^{m(\alpha q-\tau(q)-\frac{1}{k})},
\end{aligned}$$

with $m:=m'+n'-n$. Clearly $n+m=n'+m'$.

A simple geometric argument shows that there exists a family $\mathcal{P}'\subset\mathcal{P}$ with

$$\#\mathcal{P}'\ge 5^{-d}(\#\mathcal{P})\ge 2^{m(\alpha q-\tau(q)-\frac{1}{k})},$$

such that the set in $\{Q^*:\ Q\in\mathcal{P}'\}$ are disjoint. Pick a large number C (independent of $n+m$) such that each $Q\in\mathbf{D}_{n+m}$ can be covered by C many balls of radius of 2^{-n-m-1}. Then for any $Q\in\mathcal{P}'$, we can pick a ball $B_{2^{-n-m-1}}(y_Q))$ with $y_Q\in Q$ such that $\nu(B_{2^{-n-m-1}}(y_Q))\ge\nu(Q)/C$. Note that $Q\subset B_{2^{-n-m}}(y_Q)$ and $B_{2^{-n-m+1}}(y_Q)\subset Q^*$. We have

$$\frac{\nu(B_{2^{-n-m+1}}(y_Q))}{\nu(B_{2^{-n-m-1}}(y_Q))}\le CT \tag{3.23}$$

and

$$(3.24) \qquad 2^{-m(\alpha+\frac{1}{k})} \le \frac{\nu(Q)}{\nu(B_{2^{-n}}(x))} \le \frac{\nu(B_{2^{-n-m}}(y_Q))}{\nu(B_{2^{-n}}(x))} \le \frac{T\nu(Q)}{\nu(B_{2^{-n}}(x))} \le 2^{-m(\alpha-\frac{1}{k})}.$$

Hence we have shown that when $n \ge 0$ and $x \in \mathbb{R}^d$ are given so that $\nu(B_{2^{-n-1}}(x)) > 0$, for any $q \in \Omega_+$ and $k > 0$, if $m \ge h_n + n' - n$, where h_n is defined as in (3.21), then there exist a disjoint family of balls $\{B_{2^{-n-m'}}(y_Q) : \ Q \in \mathcal{P}'\}$ contained in $B_{2^{-n}}(x)$, with $\#\mathcal{P}' \ge 2^{m(\alpha q - \tau(q) - \frac{1}{k})}$ and (3.23)-(3.24) hold. This implies that ν has an asymptotically good multifractal structure on $\mathbb{R}_+$. □

4. The proof of Theorem 1.2

We first give a simple lemma.

Lemma 4.1. *Assume that μ is a self-similar measure associated with an IFS $\{S_i(x) = \rho x + a_i\}_{i=1}^{\ell}$ on $\mathbb{R}$ and a probability vector $(p_1, \dots, p_\ell)$. Let K be the attractor of $\{S_i\}_{i=1}^{\ell}$. Then we have the following properties.*

(i) *If $\dim_H K = 1$, then $\tau_\mu(0+) \ge 1$.*
(ii) *If $p_i > \rho$ for some $1 \le i \le \ell$, then $\tau'_\mu(+\infty) \le \log p_i / \log \rho < 1$.*

Proof. To prove (i), assume that $\dim_H K = 1$. Then it can be checked directly that $\tau_\mu(0) = -1$. Now let $0 < q < 1$. By the concavity of x^q on $(0, +\infty)$, we have

$$\sum_{Q \in \mathbf{D}_n} \mu(Q)^q = \sum_{Q \in \mathbf{D}_n :\, Q \cap K \ne \emptyset} \mu(Q)^q \le v_n^{1-q},$$

where $v_n = \#\{Q \in \mathbf{D}_n : \ Q \cap K \ne \emptyset\}$. Since $v_n \le c 2^n$ for some constant $c > 0$, we derive that $\tau_\mu(q) \ge q - 1$ and hence

$$\tau_\mu(0+) = \lim_{q \to 0+} \frac{\tau_\mu(q) - \tau_\mu(0)}{q} \ge 1.$$

To show (ii), assume that $p_1 > \rho$ without loss of generality. Then $\mu(S_1^n(K)) \ge p_1^n$ for each $n \ge 1$, where S_1^n denotes the n-th composition of S_1. It follows that for $q > 0$, $\Theta_\mu(q; \rho^n \mathrm{diam}(K)) \ge \mu(S_1^n(K))^q \ge p_1^{nq}$. Hence $\tau_\mu(q) \le q \log p_1 / \log \rho$, which implies that $\tau'_\mu(+\infty) \le \log p_1 / \log \rho < 1$. □

Lemma 4.2. *For $n \ge 4$, let β_n be the largest real root of the polynomial $Q_n(x) = x^n - x^{n-1} - \cdots - x + 1$. Then $\beta_n^{n+1} > 2^n$ for $n \ge 5$.*

Proof. Multiplying $x - 1$ by $Q_n(x)$ yields

$$(x-1)Q_n(x) = x^{n+1} - 2x^n + 2x - 1.$$

Table 1. Elements in $[I]$

122122122211112	122122211112221
122122211121112	122211112221221
122211121112221	122211121121112
211112221221221	211121112221221
211121121112221	211121121121112

Hence $(2-\beta_n)\beta_n^n = 2\beta_n - 1$. Now assume that $n \geq 5$. It is easy to check that $\beta_n > 1.8$. Hence $2-\beta_n = \frac{2\beta_n - 1}{\beta_n^n} < 3 \times 1.8^{-n}$. Let $\epsilon_n = 2 - \beta_n$. Then $(n+1)\epsilon_n \leq (n+1) \times 3 \times 1.8^{-n} < 1$. By the Mean Value Theorem,

$$(2-\epsilon_n)^{n+1} = 2^{n+1} - (n+1)\epsilon_n \xi_n^n \geq 2^{n+1} - (n+1)\epsilon_n 2^n > 2^n.$$

That is, $\beta_n^{n+1} > 2^n$. □

Proof of Theorem 1.2. Assume $\lambda = \beta_n^{-1}$, $n \geq 4$. Iterate (1.1) k-times to get

$$\nu_\lambda = \sum_{I \in \mathcal{A}^k} \frac{1}{2^k} \nu_\lambda \circ S_I^{-1}, \tag{4.1}$$

where $\mathcal{A} = \{1, 2\}$. Define an equivalence relation $\sim$ on $\mathcal{A}^k$ $I \sim J$ if and only if $S_I = S_J$. For $I \in \mathcal{A}^k$, let I denote the equivalence class that contains I. Then (4.1) can be rewritten as

$$\nu_\lambda = \sum_{[I] \in \mathcal{A}^k/\sim} \frac{\#[I]}{2^k} \nu_\lambda \circ S_{[I]}^{-1}, \tag{4.2}$$

where $\#[I]$ denotes the cardinality of the equivalence class $[I]$. To prove Theorem 1.2, according to Lemma 4.1, it suffices to show that there exists $k \in \mathbb{N}$ and $I \in \mathcal{A}^k$ such that $\frac{\#[I]}{2^k} > \lambda^k$. We prove this fact by considering two different cases separately: $n \geq 5$ and $n = 4$. In the first case, we take $k = n+1$ and $I = 1\underbrace{2\cdots 2}_{n-1}1$. It is easy to see that $I \sim 2\underbrace{1\cdots 1}_{n-1}2$, and hence $\#[I] \geq 2$. Then the inequality $\frac{\#[I]}{2^k} > \lambda^k$ follows from Lemma 4.2. Next we consider the case $n = 4$. Take $k = 15$ and let

$$I = 122211121112221.$$

A direct computation shows that $\#[I] = 10$ (see Table 1) and $\frac{\#[I]}{2^k} > \lambda^k$. □

5. Absolutely continuous self-similar measures with non-trivial range of local dimensions

In this section, we show the existence of an absolutely continuous self-similar measure on $\mathbb{R}$ with non-trivial range of local dimensions. Indeed, we have the following result.

Proposition 5.1. *For $\lambda, u \in (0,1)$, let $\Phi_{\lambda,u} := \{S_i\}_{i=1}^3$ be the IFS on $\mathbb{R}$ given by*

$$S_1(x) = \lambda x, \quad S_2(x) = \lambda x + u, \qquad S_3(x) = \lambda x + 1.$$

Let $\mu_{\lambda,u}$ be the self-similar measure associated with $\Phi_{\lambda,u}$ and the probability vector $\{1/4, 5/12, 1/3\}$, i.e., $\mu = \mu_{\lambda,u}$ satisfies

$$\mu = \frac{1}{4}\mu \circ S_1^{-1} + \frac{5}{12}\mu \circ S_2^{-1} + \frac{1}{3}\mu \circ S_3^{-1}.$$

Then for $\mathcal{L}^2$-a.e. $(\lambda, u) \in (0.3405, 0.3439) \times (1/3, 1/2)$, $\mu_{\lambda,u}$ is absolutely continuous, and the range of local dimensions of $\mu_{\lambda,u}$ contains a non-degenerate interval, on which the multifractal formalism for $\mu_{\lambda,u}$ is valid.

Proof. For $q > 0$, let $\tau(q, \lambda, u)$ denote the L^q spectrum of $\mu_{\lambda,u}$. Applying Theorem 6.2 by Falconer in [7], for each $0 < \lambda < 1/2$, we have for $\mathcal{L}$-a.e. $u \in (0,1)$,

$$\tau(q, \lambda, u) = \min\left\{\frac{\log\left((1/4)^q + (5/12)^q + (1/3)^q\right)}{\log \lambda},\ q - 1\right\}, \quad 1 < q < 2.$$

Write $f(q) = (1/4)^q + (5/12)^q + (1/3)^q$. Clearly $f(1) = 1$. It is easily checked that $\log f(q)$ is strictly convex over $q > 0$ and hence $\frac{\log f(q)}{q-1}$ is strictly increasing over $q > 1$. Note that $f(1.5)^{1/(1.5-1)} = f(1.5)^2 \approx 0.34387$. Hence for $0 < \lambda < 0.3438$ and $q > 1.5$,

$$g(q, \lambda) := \frac{\log\left((1/4)^q + (5/12)^q + (1/3)^q\right)}{\log \lambda} > q - 1.$$

Therefore for every $0 < \lambda < 0.3438$, we have for $\mathcal{L}$-a.e. $u \in (0,1)$, $\tau(q, \lambda, u) = g(q, \lambda)$ for $1.5 < q < 2$; clearly, g is differentiable in q, thus by Theorem 1.1 in [10], the range of local dimensions of $\mu_{\lambda,u}$ contains the non-degenerate interval $\{\frac{dg(q,\lambda)}{dq} :\ 1.5 < q < 2\}$, on which the multifractal formalism for $\mu_{\lambda,u}$ is valid.

To complete the proof of the proposition, it suffices to show that for every $u \in (1/3, 1/2)$, $\mu_{\lambda,u}$ is absolutely continuous for $\mathcal{L}$-a.e. $\lambda \in (0.3405, 0.3438)$. This is done by simply applying a general result by Peres and Solomyak (see Theorem 1.3 in [29]). The transversality condition needed there holds since $\lambda(\sqrt{3}+1) < 1$ (see the remark after Theorem 1.3 in [29]) and $0.3405 > (1/4)^{1/4}(5/12)^{5/12}(1/3)^{1/3} \approx 0.34042$. $\square$

We end the paper by posing the following unsolved questions:

(i) Does Theorem 1.1 hold for all $\lambda \in (1/2, 1)$? Moreover, does Theorem 1.3 hold for all self-conformal measures?

(ii) Is it always true that $\tau'_{\nu_\lambda}(+\infty) < 1$ when λ^{-1} is a Salem number?

We remark that the inequality in (ii) always holds in the case that λ^{-1} is a Pisot number in $(1,2)$; because in the Pisot case, $\tau'_{\nu_\lambda}(1) = \dim_H \nu_\lambda < 1$ (cf. [8]), hence $\tau'_{\nu_\lambda}(+\infty) \le \tau'_{\nu_\lambda}(1) < 1$.

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References

[1] J. Barral and I. Bhouri, Multifractal analysis for projections of Gibbs and related measures. *Ergodic Theory Dynam. Systems.* At press. doi:10.1017/S0143385710000143.

[2] P. Borwein, *Computational excursions in analysis and number theory.* CMS Books in Mathematics. Springer-Verlag, 2002.

[3] R. Cawley and R. D. Mauldin, Multifractal decompositions of Moran fractals. *Adv. Math.* 92 (1992), 196–236.

[4] P. Erdös, On a family of symmetric Bernoulli convolutions. *Amer. J. Math.* **61** (1939), 974–976.

[5] P. Erdos and V. Komornik, Developments in non-integer bases. *Acta Math. Hungar.* **79** (1998), 57–83.

[6] K. J. Falconer, *Fractal geometry, mathematical foundations and applications.* Wiley, 1990.

[7] K. J. Falconer, Generalized dimensions of measures on self-affine sets. *Nonlinearity* **12** (1999), 877–891.

[8] D. J. Feng, The smoothness of L^q-spectrum of self-similar measures with overlaps. *J. Lond. Math. Soc. (2)* **68** (2003), 102–118.

[9] D. J. Feng, The limited Rademacher functions and Bernoulli convolutions associated with Pisot numbers. *Adv. Math.* **195** (2005), 24-101.

[10] D. J. Feng, Gibbs properties of self-conformal measures and the multifractal formalism. *Ergodic Theory Dynam. Systems* **27** (2007), 787–812.

[11] D. J. Feng, Lyapunov exponent for products of matrices and multifractal analysis. Part II: General matrices. *Israel J. of Math.* **170** (2009), 355-394.

[12] D. J. Feng and K. S. Lau, Multifractal formalism for self-similar measures with weak separation condition. *J. Math. Pures Appl.* **92** (2009), 407–428.

[13] D. J. Feng, K. S. Lau and J. Wu, Ergodic limits on the conformal repellers. *Adv. Math.* **169** (2002), 58–91.

[14] D. J. Feng and E. Olivier, Multifractal analysis of the weak Gibbs measures and phase transition–applications to some Bernoulli convolutions. *Ergodic Theory Dynam. Systems.* **23** (2003), 1751 - 1784.

[15] D. J. Feng and Y. Wang, Bernoulli convolutions associated with certain non-Pisot numbers. *Adv. Math.* **187** (2004), 173–194.

[16] T. C. Halsey, M. H. Jensen, L. P. Kadanoff, I. Procaccia and B. I. Shraiman, Fractal measures and their singularities: the characterization of strange sets. *Phys. Rev. A* (3) **33** (1986), 1141–1151.

[17] T. Y. Hu, The local dimensions of the Bernoulli convolution associated with the golden number. *Trans. Amer. Math. Soc.* **349** (1997), 2917–2940.

[18] J. E. Hutchinson, Fractals and self-similarity. *Indiana Univ. Math. J.* **30** (1981), 713–747.

[19] S. P. Lalley, Random series in powers of algebraic integers: Hausdorff dimension of the limit distribution. *J. London Math. Soc.* (2) **57** (1998), 629–654.

[20] K. S. Lau and S. M. Ngai, The L^q-dimension of the Bernoulli convolution associated with the golden number. *Studia Math.* **131** (1998), 225–251.

[21] K. S. Lau and S. M. Ngai, Multifractal measures and a weak separation condition, *Adv. Math.* **141** (1999), 45-96.

[22] K. S. Lau and S. M. Ngai, The L^q-dimension of the Bernoulli convolution associated with P. V. numbers. *Osaka J. Math.* **36** (1999), 993–1010.

[23] K. S. Lau and X. Y. Wang, Some exceptional phenomena in multifrcatal formalism. Part I. *Asian J. Math.* **9** (2005), 275–294.
[24] F. Ledrappier and A. Porzio, On the multifractal analysis of Bernoulli convolutions. I. Large-Deviation Results, II. Dimensions, *J. Statist. Phys.* **82** (1996), 367–420.
[25] E. Olivier, N. Sidorov and A. Thomas, On the Gibbs properties of Bernoulli convolutions related to β-numeration in multinacci bases. *Monatsh. Math.* **145** (2005), 145–174.
[26] L. Olsen, A multifractal formalism. *Adv. Math.* **116** (1995), 82–196.
[27] N. Patzschke, Self-conformal multifractal measures. *Adv. in Appl. Math.* **19** (1997), 486–513.
[28] Y. Peres, W. Schlag and B. Solomyak, Sixty years of Bernoulli convolutions. Fractal geometry and stochastics, II (Greifswald/Koserow, 1998), 39–65, Progr. Probab., 46, Birkhäuser, Basel, 2000.
[29] Y. Peres and B. Solomyak, Self-similar measures and intersections of Cantor sets. *Trans. Amer. Math. Soc.* **350** (1998), 4065–4087.
[30] Y. Peres and B. Solomyak, Existence of L^q dimensions and entropy dimension for self-conformal measures. *Indiana Univ. Math. J.* **49** (2000), 1603–1621.
[31] Ya. Pesin, *Dimension theory in dynamical systems. Contemporary views and applications.* University of Chicago Press, 1997.
[32] A. Porzio, On the regularity of the multifractal spectrum of Bernoulli convolutions. *J. Stat. Phys.* **91** (1998), 17–29.
[33] R. T. Rockafellar, *Convex analysis*, Princeton University Press, 1970.
[34] P. Shmerkin, A modified multifractal formalism for a class of self-similar measures with overlap. *Asian J. Math.* **9** (2005), 323–348.
[35] B. Solomyak, On the random series $\sum \pm\lambda^n$ (an Erdös problem). *Ann. of Math.* **142** (1995), 611–625.
[36] B. Solomyak, Private communication.
[37] B. Testud, Phase transitions for the multifractal analysis of self-similar measures. *Nonlinearity* **19** (2006), 1201–1217.

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