A SEPARATION PROPERTY FOR ITERATED FUNCTION SYSTEMS OF SIMILITUDES

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ABSTRACT. Let E be the attractor of an iterated function system $\{\phi_i(x) = \rho R_i x + a_i\}_{i=1}^N$ on $\mathbb{R}^d$, where $0 < \rho < 1$, $a_i \in \mathbb{R}^d$ and R_i are orthogonal transformations on $\mathbb{R}^d$. Suppose that $\{\phi_i\}_{i=1}^N$ satisfies the open set condition, but not the strong separation condition. We show that E can not be generated by any iterated function system of similitudes satisfying the strong separation condition. This gives a partial answer to a folklore question about the separation conditions on the generating iterated function systems of self-similar sets.

1. INTRODUCTION

In this paper, we investigate the separation conditions on iterated function systems of similitudes.

By an *iterated function system* (IFS) on $\mathbb{R}^d$ we mean a finite family $\Phi = \{\phi_i\}_{i=1}^N$ of uniformly contracting mappings on $\mathbb{R}^d$ with $N > 1$. It is well known [10] that for each IFS $\Phi = \{\phi_i\}_{i=1}^N$ on $\mathbb{R}^d$, there is a unique non-empty compact set $E \subset \mathbb{R}^d$ such that

$$E = \bigcup_{i=1}^N \phi_i(E).$$

We call E the *attractor* of Φ. If each map ϕ_i in Φ is a similitude, i.e., ϕ_i is of the form

$$\phi_i(x) = \rho_i R_i x + a_i,$$

where $0 < \rho_i < 1$, $a_i \in \mathbb{R}^d$ and R_i is an orthogonal transformation on $\mathbb{R}^d$, we say that E is a *self-similar set* generated by Φ. The study of IFSs and their attractors is an important subject in fractal geometry, dynamical systems and probability (see e.g. [3, 4, 8, 10]).

One of the most fundamental conditions on IFSs of similitudes is the *open set condition* (OSC), under which the dimensions of the self-similar sets and the multifractal structure of the self-similar measures are well understood ([8, 10, 11, 12]). Recall that an IFS $\Phi = \{\phi_i\}_{i=1}^N$ of similitudes on $\mathbb{R}^d$ is said to satisfy the open set condition if there is a non-empty open set $U \subset \mathbb{R}^d$ such that $\phi_i(U)$, $i = 1, \ldots, N$, are disjoint subsets of U. Another commonly used separation condition on IFSs is the so-called *strong separation condition* (SSC). Recall that $\Phi = \{\phi_i\}_{i=1}^N$ is said to

satisfy the strong separation condition if $\phi_i(E)$, $i = 1, \ldots, N$, are disjoint, where E is the attractor of Φ.

It is easy to see that the SSC implies the OSC. Indeed if E is the attractor of an IFS $\Phi = \{\phi_i\}_{i=1}^N$ on $\mathbb{R}^d$ satisfying the SSC, letting

$$V_\epsilon(E) = \{x \in \mathbb{R}^d : |x - y| < \epsilon \text{ for some } y \in E\}$$

denote the ϵ-neighborhood of E, then one can check that $\phi_i(V_\epsilon(E))$ $(i = 1, \ldots, N)$ are disjoint subset of $V_\epsilon(E)$ for all

$$0 < \epsilon < \min_{i \neq j} \operatorname{dist}(\phi_i(E), \phi_j(E)),$$

so Φ satisfies the OSC. Meanwhile there are many examples of IFSs which satisfy the OSC but not the SSC, such as the IFS $\{x/5, (x+3)/5, (x+4)/5\}$ on $\mathbb{R}$. There are some equivalent conditions for the OSC [2, 13], but usually it is difficult to check whether these conditions hold for a given IFS. We emphasize that a self-similar set can be generated by many different IFSs of similitudes, and under mild assumptions these IFSs have a rigid algebraic structure (see, e.g., [9, 7, 5, 6, 1]).

In this paper, we consider the following folklore question about the separation conditions on the generating IFSs of self-similar sets.

Question 1.1. *Are there two IFSs Φ and Ψ of similitudes on $\mathbb{R}^d$ which generate the same self-similar set, such that Φ satisfies the OSC but not the SSC, and Ψ satisfies the SSC?*

This question was first brought to us by Ka-Sing Lau and Jun Jason Luo around 10 years ago. As they informed us, this question was also asked by Mariusz Urbański in a private communication. We remark that Question 1.1 is closely related to an open question raised by Elekes, Keleti and Máthé [7]; see Section 4.

In this paper, we are able to provide the following partially negative answer to Question 1.1, stating that there is no such pair (Φ, Ψ) with Φ being homogeneous. Recall that an IFS $\{\phi_i\}_{i=1}^N$ of similitudes is said to be *homogeneous* if all the maps ϕ_i have the same contraction ratio.

Theorem 1.2. *Let E be the attractor of an IFS $\Phi = \{\phi_i(x) = \rho R_i x + a_i\}_{i=1}^N$ on $\mathbb{R}^d$, where $0 < \rho < 1$, R_i are orthogonal transformations on $\mathbb{R}^d$ and $a_i \in \mathbb{R}^d$. Suppose that Φ satisfies the OSC, but not the SSC. Then E can not be generated by any IFS of similitudes on $\mathbb{R}^d$ satisfying the SSC.*

The proof of the above theorem is a little bit delicate. For the convenience of the readers, we illustrate some rough ideas. Suppose on the contrary that E is also generated by another IFS $\Psi = \{\psi_i\}_{j=1}^M$ of similitudes satisfying the SSC. We may assume that $r_j < \rho$ for all j, where r_j is the contraction ratio of ψ_j. For simplicity we assume that E is not contained in any hyperplane of $\mathbb{R}^d$. Choose a sufficiently small $\epsilon > 0$ and let $\mathcal{I}$ denote the collection of all homogeneous generating IFSs (of similitudes on $\mathbb{R}^d$) of E with contraction ratios lying in the interval $[\rho\epsilon, \epsilon]$. Applying a result of Elekes et al. [7] we see that $\mathcal{I}$ is finite (see Lemma 3.1). For each $\Theta = \{\theta_i\}_{i \in \mathcal{A}} \in \mathcal{I}$, we can construct M homogeneous generating IFSs $\Gamma_1, \ldots,$

Γ_M of E by $\Gamma_j = \{\psi_j^{-1} \circ \theta_i : i \in \mathcal{A}_j\}$, where $\mathcal{A}_j := \{i \in \mathcal{A} : \theta_i(E) \subset \psi_j(E)\}$. For each j we can show that there exists a positive integer k_j such that $\Phi^{k_j} \circ \Gamma_j \in \mathcal{I}$, where Φ^{k_j} denotes the k_j-th iteration of Φ. To derive a contradiction, we assign an infinite dimensional probability vector $\gamma(\Phi')$ (with finitely many non-zero entries) to each generating IFS Φ' of E satisfying the OSC; see Section 2.1. The term $\gamma(\Phi')$, which is called the characteristic vector of Φ', gives a quantitative description of the intersections between the images of E under the mappings in Φ'. Moreover we can introduce a total order relation $\preceq$ on the collection of all such vectors. A key observation is that

(1.1)
$$\gamma(\Phi') \prec \gamma(\Phi \circ \Phi')$$

for each generating IFS Φ' of E satisfying the OSC (see Lemma 2.6). Now return back to the aforementioned IFSs Θ and Γ_j $(j = 1, \ldots, M)$. From the definition of characteristic vector, we obtain an identity $\gamma(\Theta) = \sum_{j=1}^{M} r_j^s \gamma(\Gamma_j)$, where s is the Hausdorff dimension of E which satisfies $\sum_{j=1}^{M} r_j^s = 1$. As $\mathcal{I}$ is finite, we may choose $\Theta \in \mathcal{I}$ so that $\gamma(\Theta)$ is the largest in the sense that $\gamma(\Theta') \preceq \gamma(\Theta)$ for all $\Theta' \in \mathcal{I}$. Then by (1.1),

$$\gamma(\Theta) = \sum_{j=1}^{M} r_j^s \gamma(\Gamma_j) \prec \sum_{j=1}^{M} r_j^s \gamma(\Phi^{k_j} \circ \Gamma_j) \preceq \sum_{j=1}^{M} r_j^s \gamma(\Theta) = \gamma(\Theta),$$

leading to a contradiction $\gamma(\Theta) \prec \gamma(\Theta)$.

The paper is organized as follows. In Section 2 we introduce the definition of characteristic vector for each IFS of similitudes, and prove Lemma 2.6. In Section 3 we prove Theorem 1.2. In Section 4 we give a final remark.

2. Characteristic vectors of IFSs and a key property

2.1. **Characteristic vectors of IFSs of similitudes.**
In this subsection, we will assign an infinite-dimensional probability vector to each IFS of similitudes on $\mathbb{R}^d$.

To begin with, let $\Phi = \{\phi_i(x) = \rho_i R_i x + a_i\}_{i=1}^{N}$ be an IFS of similitudes on $\mathbb{R}^d$ and let $E = E_\Phi$ be the attractor of Φ. We introduce a binary relation $\sim_\Phi$ on the set $V := V_\Phi = \{1, \ldots, N\}$ by

$$i \sim_\Phi j \quad \text{if } \phi_i(E) \cap \phi_j(E) \neq \emptyset.$$

We say that i is *adjacent* to j if $i \sim_\Phi j$. Moreover, we say that i is *connected* to j, if there exist $x_1, \ldots, x_n \in \{1, \ldots, N\}$ with $x_1 = i$ and $x_n = j$ such that x_k is adjacent to x_{k+1} for $k = 1, \ldots, n - 1$. A subset V' of V is said to be *connected* if each two elements of V' are connected to each other.

Clearly, V can be written as the union of a number of disjoint maximal connected subsets, each of them is called a *connected component* of V. Moreover a connected component of V with cardinality n is called an *n-component* of V.

It is direct to check that $\Lambda \subset V$ is an n-component of V if and only if the following two properties hold:

(i) $C_\Lambda \cap C_{V \setminus \Lambda} = \emptyset$, where $C_\Lambda := \bigcup_{i \in \Lambda} \phi_i(E)$.

(ii) $C_{\Lambda'} \cap C_{V \setminus \Lambda'} \neq \emptyset$ for every nonempty proper subset Λ' of Λ.

Let s denote the similarity dimension of Φ, i.e., s is the unique number so that $\sum_{i=1}^{N} \rho_i^s = 1$. We define an infinite dimensional vector $\gamma(\Phi) = (\gamma_n^\Phi)_{n=1}^\infty$ by

$$\gamma_n^\Phi = \sum_{\Lambda : \, \Lambda \text{ is a } n\text{-component of V}} \sum_{i \in \Lambda} \rho_i^s,$$

where we adopt the convention that $\gamma_n^\Phi = 0$ if V has no any n-component. Clearly $\gamma_n^\Phi = 0$ for all $n > N$ and $\sum_{n=1}^\infty \gamma_n^\Phi = 1$.

Definition 2.1. *We call $\gamma(\Phi) = (\gamma_n^\Phi)_{n=1}^\infty$ the characteristic vector of Φ.*

Below we give some simple examples to illustrate the characteristic vector $\gamma(\Phi)$ of Φ.

Example 2.2. *If Φ satisfies the SSC, then $\gamma(\Phi) = (1, 0, \ldots)$.*

Example 2.3. *Suppose that the attractor $E = E_\Phi$ of $\Phi = \{\phi_i\}_{i=1}^N$ is connected, then $\gamma_N^\Phi = 1$ and $\gamma_n^\Phi = 0$ for all $n \neq N$.*

Example 2.4. *Let $\Phi = \{\phi_i\}_{i=1}^3$ be an IFS on $\mathbb{R}$ defined by*

$$\phi_1(x) = \frac{x}{5}, \quad \phi_2(x) = \frac{x+3}{5}, \quad \phi_3 = \frac{x+4}{5}.$$

Then $\gamma(\Phi) = (1/3, 2/3, 0, \ldots)$.

2.2. A key property of characteristic vectors.
In order to compare two characteristic vectors of IFSs, we write

$$\Omega = \left\{ (x_n)_{n=1}^\infty \in \mathbb{R}^\mathbb{N} : \ x_n \neq 0 \text{ for at most finitely many } n \right\}.$$

Clearly, Ω is a vector space over $\mathbb{R}$ and $\gamma(\Phi) \in \Omega$ for each IFS Φ of similitudes on $\mathbb{R}^d$. Now we introduce a relation $\prec$ on Ω by

$x \prec y$ if there exists $m \geq 1$ such that $x_m < y_m$ and $x_n = y_n$ for all $n > m$,

where $x = (x_n)_{n=1}^\infty$ and $y = (y_n)_{n=1}^\infty$. We write $x \preceq y$ if $x \prec y$ or $x = y$. Clearly Ω is totally ordered in the sense that for any $x, y \in \Omega$, we have either $x = y$, or $x \prec y$, or $y \prec x$. Moreover, if $x \preceq y$ then $ax \preceq ay$ for all $a > 0$; and if $x \preceq y$, $u \preceq v$ then $x + u \preceq y + v$.

For two IFSs $\Phi = \{\phi_i\}_{i=1}^N$ and $\Psi = \{\psi_j\}_{j=1}^M$ on $\mathbb{R}^d$, the composition of Φ and Ψ is a new IFS on $\mathbb{R}^d$ given by

$$\Phi \circ \Psi = \{\phi_i \circ \psi_j : 1 \leq i \leq N, \ 1 \leq j \in M\}.$$

We begin with a simple lemma.

Lemma 2.5. *Let $\Phi = \{\phi_i\}_{i=1}^N$ and $\Psi = \{\psi_j\}_{j=1}^M$ be two IFSs of similitudes on $\mathbb{R}^d$ satisfying the OSC. Suppose that Φ and Ψ generate the same self-similar set E, i.e., $E_\Phi = E_\Psi = E$. Then $\Phi \circ \Psi$ is also a generating IFS of E satisfying the OSC.*

Proof. Since $E_\Phi = E_\Psi = E$, we have $E = \bigcup_{i=1}^N \phi_i(E) = \bigcup_{j=1}^M \psi_i(E)$. It follows that

$$\bigcup_{i=1}^N \bigcup_{j=1}^M \phi_i(\psi_j(E)) = \bigcup_{i=1}^N \phi_i \left(\bigcup_{j=1}^M \psi_j(E) \right) = \bigcup_{i=1}^N \phi_i(E) = E,$$

which implies $E_{\Phi \circ \Psi} = E$.

To see that $\Phi \circ \Psi$ satisfies the OSC, let s be the Hausdorff dimension of E. Clearly, Φ and Ψ have the same similarity dimension s and $\mathcal{H}^s(E) > 0$, where $\mathcal{H}^s$ stands for the s-dimensional Hausdorff measure; see [8, 10]. It follows that $\Phi \circ \Psi$ also has the similarity dimension s. Since $\mathcal{H}^s(E) > 0$, by [13, Theorem 2.1] $\Phi \circ \Psi$ satisfies the OSC. $\qquad\square$

The following lemma plays a key role in the proof of Theorem 1.2.

Lemma 2.6. *Let $\Phi = \{\phi_i\}_{i=1}^N$ and $\Psi = \{\psi_j\}_{j=1}^M$ be two IFSs of similitudes on $\mathbb{R}^d$ satisfying the OSC. Suppose that Φ and Ψ generate the same self-similar set E. Moreover, suppose that Φ does not satisfy the SSC. Then $\gamma(\Psi) \prec \gamma(\Phi \circ \Psi)$.*

Proof. By Lemma 2.5, $\Phi \circ \Psi$ is a generating IFS of E satisfying the OSC. Write

$$V_\Psi = \{1, \ldots, M\} \text{ and } V_{\Phi \circ \Psi} = \{(i,j): \ i = 1, \ldots, N, \ j = 1, \ldots, M\}.$$

Define the binary relations $\sim_\Psi$ and $\sim_{\Phi \circ \Psi}$ on V_Ψ and $V_{\Phi \circ \Psi}$ respectively as in Section 2.1. For $n \geq 1$, let $\mathcal{C}_n$ denote the collection of n-components of V_Ψ with respect to $\sim_\Psi$, and $\mathcal{D}_n$ the collection of n-components of $V_{\Phi \circ \Psi}$ with respect to $\sim_{\Phi \circ \Psi}$; see Section 2.1 for the relevant definitions.

Let us first give a simple observation. Suppose that two elements $j, k \in V_\Psi$ are connected to each other, i.e., there exist $j_1, \ldots, j_s \in V_\Psi$, with $j_1 = j$ and $j_s = k$, such that $\psi_{j_\ell}(E) \cap \psi_{j_{\ell+1}}(E) \neq \emptyset$ for all $\ell = 1, \ldots, s-1$. Clearly for each $i \in \{1, \ldots, N\}$, $\phi_i \circ \psi_{j_\ell}(E) \cap \phi_i \circ \psi_{j_{\ell+1}}(E) \neq \emptyset$ for all $\ell = 1, \ldots, s-1$; therefore the elements (i,j) and (i,k) in $V_{\Phi \circ \Psi}$ are connected to each other.

According to the above observation, for every n-component Λ of V_Ψ and $i \in \{1, \ldots, N\}$, the set $\{i\} \times \Lambda$ is connected with respect to $\sim_{\Phi \circ \Psi}$, hence it is either an n-component of $V_{\Phi \circ \Psi}$ or a proper subset of an n_1-component of $V_{\Phi \circ \Psi}$ for some $n_1 > n$.

We claim that for some $n \in \mathbb{N}$, there are an n-component Λ of V_Ψ and $i \in \{1, \ldots, N\}$ such that the set $\{i\} \times \Lambda$ is not an n-component of $V_{\Phi \circ \Psi}$. To prove this claim we use contradiction. Suppose on the contrary that the claim is false. Then for each $i \in \{1, \ldots, N\}$, $n \in \mathbb{N}$ and $\Lambda \in \mathcal{C}_n$, we have $\{i\} \times \Lambda \in \mathcal{D}_n$, which implies that for each $1 \leq i' \leq N$ with $i' \neq i$,

$$\phi_i \left(\bigcup_{j \in \Lambda} \psi_j(E) \right) \cap \phi_{i'}(E) = \phi_i \left(\bigcup_{j \in \Lambda} \psi_j(E) \right) \cap \phi_{i'} \left(\bigcup_{k=1}^M \psi_k(E) \right) = \emptyset.$$

Taking the union over all connected components of V_Ψ yields that for $i, i' \in \{1, \ldots, N\}$ with $i \neq i'$,

$$\phi_i(E) \cap \phi_{i'}(E) = \phi_i \left(\bigcup_{\Lambda \in \bigcup_{n \geq 1} \mathcal{C}_n} \bigcup_{j \in \Lambda} \psi_j(E) \right) \cap \phi_{i'}(E) = \emptyset,$$

which contradicts the assumption that $\Phi = \{\phi_i\}_{i=1}^N$ does not satisfy the SSC. This completes the proof of the claim.

Let n_0 be the largest integer such that there exist an n_0-component Λ_0 of V_Ψ and $i_0 \in \{1, \ldots N\}$ such that the set $\{i_0\} \times \Lambda_0$ is not an n_0-component of $V_{\Phi \circ \Psi}$. Then as we pointed out above, $\{i_0\} \times \Lambda_0$ is a proper subset of an n_1-component D_0 of $V_{\Phi \circ \Psi}$ for some $n_1 > n_0$. Clearly D_0 is not of the form $\{i\} \times \Lambda$ with $1 \leq i \leq N$ and $\Lambda \in \mathcal{C}_{n_1}$.

Now for $i \in \{1, \ldots, N\}$ and $j \in \{1, \ldots, M\}$, let ρ_i and r_j denote the contraction ratios of ϕ_i and ψ_j, respectively. Let s denote the Hausdorff dimension of E. Then for each $n \geq n_1$ and $\Lambda \in \mathcal{C}_n$, $\{i\} \times \Lambda \in \mathcal{D}_n$ for each $1 \leq i \leq N$. It follows that for $n \geq n_1$,

$$(2.1) \qquad \gamma_n^{\Phi \circ \Psi} = \sum_{D \in \mathcal{D}_n} \sum_{(i,j) \in D} (\rho_i r_j)^s \geq \sum_{i=1}^N \sum_{\Lambda \in \mathcal{C}_n} \sum_{j \in \Lambda} (\rho_i r_j)^s = \sum_{\Lambda \in \mathcal{C}_n} \sum_{j \in \Lambda} r_j^s = \gamma_n^\Psi.$$

Recall that $\mathcal{D}_{n_1}$ contains an element D_0 which is not of the form $\{i\} \times \Lambda$ with $1 \leq i \leq N$ and $\Lambda \in \mathcal{C}_{n_1}$. Hence

$$\gamma_{n_1}^{\Phi \circ \Psi} = \sum_{D \in \mathcal{D}_{n_1}} \sum_{(i,j) \in D} (\rho_i r_j)^s$$

$$(2.2) \qquad \geq \sum_{(i',j') \in D_0} (\rho_{i'} r_{j'})^s + \sum_{i=1}^N \sum_{\Lambda \in \mathcal{C}_{n_1}} \sum_{j \in \Lambda} (\rho_i r_j)^s$$

$$= \sum_{(i',j') \in D_0} (\rho_{i'} r_{j'})^s + \gamma_{n_1}^\Psi$$

$$> \gamma_{n_1}^\Psi.$$

Combining (2.1) and (2.2) yields that $\gamma(\Psi) \prec \gamma(\Phi \circ \Psi)$. $\qquad\square$

3. The proof of Theorem 1.2

In this section we prove Theorem 1.2. For a compact set $E \subset \mathbb{R}^d$, let $\mathcal{I}_E$ denote the collection of all homogeneous IFSs of similitudes on $\mathbb{R}^d$ that generate E and satisfy the OSC. Write

$$(3.1) \qquad\qquad \mathcal{I}_{E,a,b} = \{\Phi \in \mathcal{I}_E : \ a \leq \rho_\Phi < b\}$$

for $0 < a < b < 1$, where ρ_Φ denotes the common contraction ratio of the maps in Φ.

Lemma 3.1. *Let E be the attractor of an IFS of similitudes on $\mathbb{R}^d$ satisfying the SSC. Suppose that E is not contained in any hyperplane of $\mathbb{R}^d$. Then for any $0 < a < b < 1$, $\mathcal{I}_{E,a,b}$ is a finite set.*

Proof. Let $\mathrm{Sim}(d)$ denote the collection of all similarity maps of $\mathbb{R}^d$. Notice that each element ϕ in $\mathrm{Sim}(d)$ is an affine map on $\mathbb{R}^d$ which is of the form $\phi(x) = Ax + a$, where A is a $d \times d$ matrix and $a \in \mathbb{R}^d$, hence ϕ can be viewed as a point in $\mathbb{R}^{d^2+d}$. Therefore $\mathrm{Sim}(d)$ can be viewed as a metric subspace of $\mathbb{R}^{d^2+d}$, where we endow $\mathbb{R}^{d^2+d}$ with the usual Euclidean metric.

For $\delta > 0$, write

$$\mathcal{S}_{E,\delta} = \{\phi \in \mathrm{Sim}(d) : \phi(E) \supset E \text{ and } \rho_\phi \leq \delta\},$$

where ρ_ϕ denotes the similarity ratio of ϕ. To prove the conclusion of the lemma, it suffices to show that $\mathcal{S}_{E,\delta}$ is finite for every $\delta > 0$.

Since E is a compact subset of $\mathbb{R}^d$, it follows that $\mathcal{S}_{E,\delta}$ is a compact subset of $\mathrm{Sim}(d)$ for each $\delta > 0$. Meanwhile by [7, Proposition 4.3(i)], under the assumptions of the lemma on E, $\{\phi \in \mathrm{Sim}(d) : \phi(E) \supset E\}$ is a discrete subset of $\mathrm{Sim}(d)$. Hence $\mathcal{S}_{E,\delta}$ is both discrete and compact, so it is finite. $\square$

Now we are ready to prove Theorem 1.2.

Proof of Theorem 1.2. We first assume that E is not contained in a hyperplane of $\mathbb{R}^d$. To prove the theorem we use contradiction. Suppose on the contrary that E can be generated by an IFS $\Psi = \{\psi_j\}_{j=1}^M$ of similitudes on $\mathbb{R}^d$ which satisfies the SSC. Below we derive a contradiction.

Replacing Ψ by its n-th iteration $\Psi^n := \underbrace{\Psi \circ \cdots \circ \Psi}_{n}$ for a large n if necessary, we may assume that $r_j < \rho$ for all $1 \leq j \leq M$, where r_j denotes the contraction ratio of ψ_j. Set

$$(3.2) \qquad \delta := \min_{j \neq j'} \mathrm{dist}\left(\psi_j(E), \psi_{j'}(E)\right).$$

Since Ψ satisfies the SSC, we have $\delta > 0$. Clearly $\delta < \mathrm{diam}(E)$.

Pick a large positive integer ℓ so that $\rho^\ell < \min_{1 \leq j \leq M} r_j$. Then fix a small $\epsilon > 0$ such that

$$(3.3) \qquad \epsilon \leq \frac{\rho^\ell \delta}{\mathrm{diam}(E)}.$$

Clearly $\epsilon < \rho^\ell$ since $\delta < \mathrm{diam}(E)$. Let $\mathcal{I} := \mathcal{I}_{E,\rho\epsilon,\epsilon}$ denote the collection of homogeneous generating IFSs Φ' on $\mathbb{R}^d$ of E satisfying the OSC and $\rho\epsilon \leq \rho_{\Phi'} < \epsilon$. By Lemma 3.1, $\mathcal{I}$ is a finite set. Let k be the unique integer so that $\rho\epsilon \leq \rho^k < \epsilon$. Then $k > \ell \geq 1$ and $\Phi^k \in \mathcal{I}$, so $\mathcal{I}$ is non-empty. Since $\mathcal{I}$ is a nonempty finite set, there exists $\Theta \in \mathcal{I}$ whose characteristic vector $\gamma(\Theta)$ is the largest in the sense that $\gamma(\Phi') \preceq \gamma(\Theta)$ for all $\Phi' \in \mathcal{I}$.

Keep in mind that $\rho_\Theta \in [\rho\epsilon, \epsilon)$ since $\Theta \in \mathcal{I}$. Write $\Theta = \{\theta_i\}_{i\in\mathcal{A}}$. By (3.3),

$$\text{(3.4)} \qquad \text{diam}(\theta_i(E)) = \rho_\Theta \text{diam}(E) < \epsilon\,\text{diam}(E) \le \rho^\ell \delta < \delta,$$

and

$$\text{(3.5)} \qquad \rho_\Theta < \epsilon < \rho^\ell < \min_{1\le j\le M} r_j.$$

Since $\bigcup_{i\in\mathcal{A}} \theta_i(E) = E = \bigcup_{j=1}^M \psi_j(E)$, by (3.2) and (3.4) we see that if $\theta_i(E) \cap \psi_j(E) \ne \emptyset$ then $\theta_i(E) \subset \psi_j(E)$. Due to this, we can partition $\mathcal{A}$ into M disjoint subsets $\mathcal{A}_1, \ldots, \mathcal{A}_M$ by setting

$$\mathcal{A}_j := \{i \in \mathcal{A}\colon \theta_i(E) \subset \psi_j(E)\}, \quad j = 1, \ldots, M,$$

and moreover,

$$\text{(3.6)} \qquad \bigcup_{i\in\mathcal{A}_j} \theta_i(E) = \psi_j(E), \quad j = 1, \ldots, M.$$

For $j = 1, \ldots, M$, let

$$\Gamma_j = \left\{ \psi_j^{-1} \circ \theta_i : i \in \mathcal{A}_j \right\}.$$

By (3.6) and (3.5), for each $1 \le j \le M$, Γ_j is a homogeneous generating IFS of E satisfying the OSC, and its contraction ratio ρ_{Γ_j} satisfies

$$\text{(3.7)} \qquad \rho_{\Gamma_j} = r_j^{-1}\rho_\Theta \ge r_j^{-1}\rho\epsilon > \epsilon,$$

where we have used $\rho_\Theta \ge \rho\epsilon$ and $r_j < \rho$ in the last two inequalities. For each $1 \le j \le M$, let k_j be the unique integer such that

$$\rho^{k_j}\rho_{\Gamma_j} \in [\rho\epsilon, \epsilon),$$

then $k_j \ge 1$ by (3.7), and $\Phi^{k_j} \circ \Gamma_j \in \mathcal{I}$, where Φ^{k_j} denotes the k_j-th iteration of Φ.

Next we compare the characteristic vectors of the IFSs Θ and Γ_j ($j = 1, \ldots, M$). Since Ψ satisfies the SSC, by (3.6) we see that $\theta_i(E) \cap \theta_{i'}(E) = \emptyset$ if $i \in \mathcal{A}_j$ and $i' \in \mathcal{A}_{j'}$ for some $j \ne j'$. It follows that for $n \in \mathbb{N}$, every n-component of $\mathcal{A}$ (with respect to $\sim_\Theta$) is totally contained in $\mathcal{A}_j$ for some j. Hence for each $n \in \mathbb{N}$,

$$
\begin{aligned}
\text{(3.8)} \qquad \gamma_n^\Theta &= \sum_{\Lambda:\,\Lambda \text{ is an } n\text{-component of } \mathcal{A}} \sum_{i\in\Lambda} \rho_\Theta^s \\
&= \sum_{j=1}^M \sum_{\Lambda:\,\Lambda \text{ is an } n\text{-component of } \mathcal{A}_j} \sum_{i\in\Lambda} \rho_\Theta^s.
\end{aligned}
$$

Meanwhile by (3.6), it is easy to see that each n-component of $\mathcal{A}_j$ with respect to $\sim_\Theta$ is an n-component of $\mathcal{A}_j$ with respect to $\sim_{\Gamma_j}$, and vice versa. It follows that for $j = 1, \ldots, M$,

$$\text{(3.9)} \qquad \gamma_n^{\Gamma_j} = \sum_{\Lambda:\,\Lambda \text{ is an } n\text{-component of } \mathcal{A}_j} \sum_{i\in\Lambda} (r_j^{-1}\rho_\Theta)^s.$$

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Combining (3.8) with (3.9) yields that $\gamma_n^\Theta = \sum_{j=1}^M r_j^s \gamma_n^{\Gamma_j}$ for each $n \in \mathbb{N}$. Hence

$$(3.10) \qquad \gamma(\Theta) = \sum_{j=1}^M r_j^s \gamma(\Gamma_j).$$

Recall that for each $1 \le j \le M$, there exists $k_j \in \mathbb{N}$ such that $\Phi^{k_j} \circ \Gamma_j \in \mathcal{I}$. This implies that $\gamma(\Phi^{k_j} \circ \Gamma_j) \preceq \gamma(\Theta)$. Since Φ satisfies the OSC but not the SSC, by Lemma 2.6 we have

$$\gamma(\Gamma_j) \prec \gamma(\Phi \circ \Gamma_j) \prec \cdots \prec \gamma(\Phi^{k_j} \circ \Gamma_j) \preceq \gamma(\Theta)$$

for $j = 1, \ldots, M$. Combining this with (3.10) yields that

$$\gamma(\Theta) \prec \sum_{j=1}^M r_j^s \gamma(\Theta) = \gamma(\Theta),$$

which leads to a contradiction. This proves the theorem under the assumption that E is not contained in a hyperplane of $\mathbb{R}^d$.

Finally we consider the general case when E may be contained in a hyperplane of $\mathbb{R}^d$. Suppose on the contrary that E can be generated by an IFS $\Psi = \{\psi_j\}_{j=1}^M$ of similitudes on $\mathbb{R}^d$ which satisfies the SSC. Let H be the affine hull of E, i.e., H is the affine subspace of $\mathbb{R}^d$ with the smallest dimension that covers E. Then it is direct to check that the following properties hold:

  (i) E is not contained in a proper affine subspace of H;
  (ii) $\phi_i(H) = H$, $\psi_j(H) = H$ for all $1 \le i \le N$ and $1 \le j \le M$;
  (iii) Let $\widehat{\phi}_i, \widehat{\psi}_j$ denote the restrictions of ϕ_i and ψ_j on H. Then $\widehat{\Phi} = \{\widehat{\phi}_i\}_{i=1}^N$ and $\widehat{\Psi} = \{\widehat{\psi}_j\}_{j=1}^M$ are IFSs of similitudes on H that generate E, and moreover, $\widehat{\Phi}$ is a homogeneous IFS satisfying the OSC, and $\widehat{\Psi}$ satisfies the SSC.

Then we can derive a contradiction by following the previous argument (in which Φ, Ψ and $\mathbb{R}^d$ are replaced by $\widehat{\Phi}$, $\widehat{\Psi}$ and H, respectively). This completes the proof of the theorem. $\qquad\square$

4. A final remark

We remark that Question 1.1 is closely related to the following question raised by Elekes, Keleti and Máthé.

Question 4.1 ([7, Question 9.3]). *Let $E \subset \mathbb{R}^d$ be a self-similar set generated by an IFS $\Psi = \{\psi_j\}_{j=1}^M$ satisfying the SSC and let f be a similitude such that $f(E) \subset E$. Does this imply that $f(E)$ is a relative open set in E (or in other words $f(E)$ is a finite union of elementary pieces of E)?*

Here an elementary piece of E means a set of the form $\psi_{j_1} \circ \cdots \circ \psi_{j_m}(E)$, where $j_1, \ldots, j_m \in \{1, \ldots, M\}$. To our best knowledge, so far Question 4.1 still remains open. We remark that an affirmative answer to Question 4.1 would yield a negative answer to Question 1.1. To see this, suppose that E is the attractor of an IFS

$\Psi = \{\psi_j\}_{j=1}^M$ of similitudes satisfying the SSC, and that the answer to Question 4.1 is affirmative. Notice that the collection of elementary pieces of E (with respect to the IFS Ψ) has the following net structure: for any two given elementary pieces E_1 and E_2, one has either $E_1 \cap E_2 = \emptyset$, or $E_1 \subset E_2$, or $E_1 \supset E_2$. Hence if $f(E) \subset E$ for a similitude f, then $f(E)$ is a finite union of disjoint elementary pieces of E. As a consequence, if f_1 and f_2 are two similitudes mapping E into itself, then either $f_1(E) \cap f_2(E) = \emptyset$, or $f_1(E) \cap f_2(E)$ contains an elementary piece of E which implies that $\mathcal{H}^s(f_1(E) \cap f_2(E)) > 0$, where s denotes the Hausdorff dimension of E. Now if $\Phi = \{\phi_i\}_{i=1}^N$ is another generating IFS (of similitudes) of E satisfying the OSC, then $\mathcal{H}^s(\phi_i(E) \cap \phi_j(E)) = 0$ for all $i \neq j$ (see [10]) which forces that $\phi_i(E) \cap \phi_j(E) = \emptyset$ for all $i \neq j$, that is, Φ satisfies the SSC. Therefore there is no generating IFS (of similitudes) of E which satisfies the OSC but not the SSC.

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