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THE THERMODYNAMIC FORMALISM FOR SUB-ADDITIVE POTENTIALS

Yong-Luo Cao

Department of Mathematics, Suzhou University
Suzhou, 215006, Jiang Su, P. R. China

De-Jun Feng

Department of Mathematics, The Chinese University of Hong Kong
Shatin, Hong Kong and
Department of Mathematical Sciences, Tsinghua University
Beijing, 100084, P. R. China

Wen Huang

Department of Mathematics, University of Science and Technology of China
Hefei, 230026, Anhui, P. R. China

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Abstract. The topological pressure is defined for sub-additive potentials via separated sets and open covers in general compact dynamical systems. A variational principle for the topological pressure is set up without any additional assumptions. The relations between different approaches in defining the topological pressure are discussed. The result will have some potential applications in the multifractal analysis of iterated function systems with overlaps, the distribution of Lyapunov exponents and the dimension theory in dynamical systems.

1. **Introduction and main results.** The well-known notion of topological pressure for additive potentials was first introduced by Ruelle [25] in 1973 for expansive maps acting on compact metric spaces. In the same paper he formulated a variational principle for the topological pressure. Later Walters [28] generalized these results to general continuous maps on compact metric spaces. The variational principle formulated by Walters can be stated precisely as follows: Let $T : X \to X$ be a continuous transformation on a compact metric space (X, d) and $\phi : X \to \mathbb{R}$ an arbitrary continuous function. Let $P(\phi, T)$ denote the topological pressure of ϕ (see [29]). Then

$$P(\phi, T) = \sup\left\{h_\mu(T) + \int \phi \, d\mu : \ \mu \in \mathcal{M}(X, T)\right\}, \tag{1}$$

where $\mathcal{M}(X, T)$ denotes the space of all T-invariant Borel probability measures on X endowed with the weak* topology, and $h_\mu(T)$ denotes the measure-theoretical entropy of μ. A T-invariant Borel probability measure μ such that $P(\phi, T) = h_\mu(T) + \int \phi \, d\mu$, if it exists, is called an *equilibrium state* for ϕ. The theory about

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the topological pressure, variational principle and equilibrium states plays a fundamental role in statistical mechanics, ergodic theory and dynamical systems (see, e.g., [4, 18, 26, 29]). Since the works of Bowen [5] and Ruelle [27], the topological pressure has also become a basic tool in the dimension theory in dynamical systems (see, e.g., [23, 30]). In 1984, Pesin and Pitskel' [24] defined the topological pressure of additive potentials for non-compact subsets of compact metric spaces and proved the variational principle under some supplementary conditions. Their work extended Bowen's results in [3] on topological entropy for non-compact sets.

In this paper, we generalize Ruelle and Walters's results to sub-additive potentials in general compact dynamical systems. We define the topological pressure for sub-additive potentials $\mathcal{F} = \{\log f_n\}_{n=1}^{\infty}$ and set up a variational principle between the topological pressure, measure-theoretical entropies and Lyapunov exponents. It is now $\log f_n$ which plays the role of the classical potential. More precise, $\log f_n$ plays the role of $\phi + \phi \circ T + \ldots + \phi \circ T^{n-1}$.

To formulate our results, let $T : X \to X$ be a continuous transformation on a compact metric space (X, d). A sequence $\mathcal{F} = \{\log f_n\}_{n=1}^{\infty}$ of functions on X is called *sub-additive* if each f_n is a continuous non-negative valued function on X such that

$$0 \leq f_{n+m}(x) \leq f_n(x) f_m(T^n x), \qquad \forall x \in X,\ m, n \in \mathbb{N}. \tag{2}$$

We first define the topological pressure of $\mathcal{F}$ with respect to T. As usual for any $n \in \mathbb{N}$ and $\epsilon > 0$, a set $E \subseteq X$ is said to be an (n, ϵ)-*separated subset* of X with respect to T if $\max_{0 \leq i \leq n-1} d(T^i x,\ T^i y) > \epsilon$ for any two different points $x, y \in E$. Define

$$P_n(T, \mathcal{F}, \epsilon) = \sup \left\{ \sum_{x \in E} f_n(x) :\ E \text{ is an } (n, \epsilon)\text{-separated subset of } X \right\}.$$

It is clear that $P_n(T, \mathcal{F}, \epsilon)$ is a decreasing function of ϵ. Define

$$P(T, \mathcal{F}, \epsilon) = \limsup_{n \to \infty} \frac{1}{n} \log P_n(T, \mathcal{F}, \epsilon).$$

$P(T, \mathcal{F}, \epsilon)$ is also a decreasing function of ϵ. Set $P(T, \mathcal{F}) = \lim_{\epsilon \to 0} P(T, \mathcal{F}, \epsilon)$ and we call it the *topological pressure* of $\mathcal{F}$ with respect to T. By the definition, $P(T, \mathcal{F})$ takes a value in $[-\infty, +\infty)$.

For a T-invariant Borel probability measure μ, denote

$$\mathcal{F}_*(\mu) = \lim_{n \to \infty} \frac{1}{n} \int \log f_n d\mu.$$

The existence of the above limit follows from a sub-additive argument. We call $\mathcal{F}_*(\mu)$ the *Lyapunov exponent* of $\mathcal{F}$ with respect to μ. It also takes a value in $[-\infty, +\infty)$. As a main result, we obtain the following variational principle.

Theorem 1.1. *Let $\mathcal{F} = \{\log f_n\}_{n=1}^{\infty}$ be a sub-additive potential on a compact dynamical system (X, T). Then*

$$P(T, \mathcal{F}) = \begin{cases} -\infty, & \text{if } \mathcal{F}_*(\mu) = -\infty \text{ for all } \mu \in \mathcal{M}(X, T), \\ \\ \sup \left\{ h_\mu(T) + \mathcal{F}_*(\mu) :\ \mu \in \mathcal{M}(X, T),\ \mathcal{F}_*(\mu) \neq -\infty \right\}, & \text{otherwise.} \end{cases}$$

In the above theorem we adopt the rigorous expression just in order to avoid the situation that $h_\mu(T) = +\infty$ and $\mathcal{F}_*(\mu) = -\infty$ take place simultaneously. If there is no such confusion, for instance the topological entropy of T is assumed to be finite, then we can state the variational principle simply as: $P(T,\mathcal{F}) = \sup\{h_\mu(T)+\mathcal{F}_*(\mu) : \mu \in \mathcal{M}(X,T)\}$.

Our original motivation for studying the above issue comes from the study of multifractal formalism for self-similar measures with overlaps (the reader is referred to [11, 23] for some basic information about self-similar measures and multifractals). It is known that in many interesting cases a self-similar measure with overlaps can be locally represented as infinite products of a family of matrices (see, e.g., [12, 15, 19, 20]). Thus to understand the corresponding multifractal formalism for these measures one needs to study the distribution of Lyapunov exponents for products of matrices. A similar task also arises in the study of regularity of the solutions of refinement equations in wavelets (see [6, 7, 8]). For the above purpose, a thermodynamic formalism (pressure functions, Gibbs measures and the distribution of Lyapunov exponents) for non-negative matrix-valued potentials was developed in [13, 16]. Moreover, a variational principle for products of non-negative matrices in symbolic spaces was set up by Feng in [14]. Namely, let M be a continuous non-negative matrix-valued function defined on a mixing sub-shift space and denote $\mathcal{F} = \{\log f_n\}$ where $f_n(x) = \|\prod_{i=0}^{n-1} M(T^i x)\|$, and $\|\cdot\|$ denotes the norm of matrices, then Theorem 1.1 was proved in [14] under this special setting. However due to Theorem 1.1, we have the following more general result about products of matrices:

Corollary 1.2. *Let M be a continuous function defined on (X,T) taking values in the set of all $d \times d$ (real or complex) matrices. Let $\mathcal{F} = \{\log f_n\}$ where $f_n(x) = \|\prod_{i=0}^{n-1} M(T^i x)\|$, and $\|\cdot\|$ denotes the norm of matrices, then the result of Theorem 1.1 holds.*

We point out that Falconer and Barreira had some earlier contributions in the study of thermodynamic formalism for sub-additive potentials. In [9], Falconer considered the thermodynamic formalism for sub-additive potentials on mixing repellers. He proved the variational principle for the topological pressure under some Lipschitz conditions and bounded distortion assumptions on the sub-additive potentials. More precise, he assumed that there exist constants $M, a, b > 0$ such that

$$\frac{1}{n}|\log f_n(x)| \le M, \quad \frac{1}{n}|\log f_n(x) - \log f_n(y)| \le a|x-y|, \quad \forall x, y \in X, n \in \mathbb{N}$$

and $|\log f_n(x) - \log f_n(y)| \le b$ whenever x, y belong to the same n-cylinder of the mixing repeller X. In 1996, Barreira [1] extended the work of Pesin and Pitskel' [24]. He defined the topological pressure for an arbitrary sequence of continuous functions on an arbitrary subset of compact metric space and proved the variational principle under a strong convergence assumption on the potentials. Corresponding to our setting, Barreira assumed that there exists a continuous function $\psi : X \to \mathbb{R}$ such that $\log f_{n+1} - \log(f_n \circ T)$ converges to ψ uniformly on X. We remark that the assumptions given by Falconer and Barreira are usually not satisfied by general sub-additive potentials, for instance, the one generated by the product of general matrices in Corollary 1.2.

The major virtue of our result is that we don't need *any* additional assumptions on the compact continuous dynamical system (X,T) nor *any* regularity assumption on $\mathcal{F}$. Our definition of the topological pressure for sub-additive potentials in this section adopts an approach via separated sets, which is similar to the classical additive case (see Walters' book [29]). It is proved to be equivalent to Falconer's definition in the mixing repeller case. Nevertheless our definition is different from that of Barreira through open covers. We don't know if in general these two definitions are equivalent. However, we do prove that they are equivalent in some situations (for instance, when the entropy map is upper semi-continuous and the topological entropy is bounded). In section 4, we provide another definition of the topological pressure via open covers, which is equivalent to the one given in this section.

Our proof of Theorem 1.1 is a generalization of Misiurewicz's elegant proof of the classical variational principle [21]. The principal applications of Theorem 1.1 are closely related to the dimension estimates for a broad class of Cantor-like sets and dynamical repellers [1, 11], the multifractal structure of iterated function systems with overlaps [12, 20], the distribution of Lyapunov exponents of a dynamical system [9, 13].

After a first version of this paper was completed, we were informed that Barreira [2] and Mummert [22] independently considered the thermodynamic formalism of almost-additive potentials, where $\mathcal{F}=\{\log f_n\}$ is assumed to satisfy

$$0<c^{-1}f_n(x)f_m(T^nx)\le f_{n+m}(x)\le cf_n(x)f_m(T^nx).$$

They proved the variational principle under an additional tempered variation condition and gave some conditions for the existence and uniqueness of equilibrium states and Gibbs measures. We were also informed that Käenmäki [17] proved the variational principle on full shift symbolic spaces for a class of special sub-additive potentials $\mathcal{F}=\{\log f_n\}$ (where $f_n(x)$ were assumed to depend upon only the first n coordinates of x) using the upper semi-continuity of the entropy map on symbolic spaces, she also showed that for typical self-affine sets there exists an ergodic invariant measure having the same Hausdorff dimension as the set itself.

The paper is arranged in the following manner: in section 2 we provide some useful lemmas, in section 3 we prove Theorem 1.1, and in section 4 we discuss the relations between different approaches in the definition of the topological pressure.

2. **Some Lemmas.** In this section, we give some lemmas which are needed in our proof of Theorem 1.1.

Let $\mathcal{F}=\{\log f_n\}_{n=1}^{\infty}$ be a sequence of functions defined on a compact metric space (X,d), and $T:\ X\to X$ a continuous map. Assume that $\mathcal{F}$ is sub-additive. Let $P(T,\mathcal{F})$ be defined as in section 1. We begin with the following lemma.

Lemma 2.1. *For any $k\in\mathbb{N}$, we have*

$$P\left(T^k,\mathcal{F}^{(k)}\right)=kP(T,\mathcal{F}),$$

where $T^k:=\underbrace{T\circ\cdots\circ T}_{k\text{ times}}$ *and* $\mathcal{F}^{(k)}:=\{\log f_{kn}\}_{n=1}^{\infty}$.

Proof. Fix $k\in\mathbb{N}$. Observe that if E is an (n,ϵ)-separated subset of X with respect to T^k, then E must be an (nk,ϵ)-separated subset of X with respect to T. It follows that

$$\begin{aligned}P_n\left(T^k,\mathcal{F}^{(k)},\epsilon\right) &= \sup\left\{\sum_{x\in E}f_{kn}(x):\ E\text{ is }(n,\epsilon)\text{-separated w.r.t }T^k\right\}\\ &\le \sup\left\{\sum_{x\in E}f_{kn}(x):\ E\text{ is }(nk,\epsilon)\text{-separated w.r.t }T\right\}\\ &= P_{kn}(T,\mathcal{F},\epsilon),\end{aligned}$$

from which we deduce that $P(T^k, \mathcal{F}^{(k)}) \le kP(T, \mathcal{F})$.

To show the reverse inequality, for any $\epsilon > 0$ we choose $\delta > 0$ small enough such that

$$d(x, y) \le \delta \Longrightarrow \max_{1 \le i \le k-1} d(T^i x, T^i y) < \epsilon. \tag{3}$$

Define $C = \max(1, \sup_{x \in X} f_1(x))$. Then $1 \le C < \infty$. Moreover

$$f_n(x) \le f_1(x) f_1(Tx) \ldots f_1(T^{n-1}x) \le C^n$$

for all $x \in X$ and $n \in \mathbb{N}$. Now for any given natural number n, let ℓ be an arbitrary integer in $[kn, k(n+1))$. Observe that by (3), any (ℓ, ϵ)-separated subset of X with respect to T must be an (n, δ)-separated subset of X with respect to T^k. This observation together with $f_\ell(x) \le f_{nk}(x) f_{\ell - nk}(T^{nk}x) \le C^k f_{nk}(x)$ yields

$$\begin{aligned} P_\ell(T, \mathcal{F}, \epsilon) &= \sup \left\{ \sum_{x \in E} f_\ell(x) : \ E \text{ is } (\ell, \epsilon)\text{-separated w.r.t } T \right\} \\ &\le \sup \left\{ \sum_{x \in E} C^k f_{kn}(x) : \ E \text{ is } (n, \delta)\text{-separated w.r.t } T^k \right\} \\ &= C^k P_n(T^k, \mathcal{F}^{(k)}, \delta). \end{aligned}$$

This implies $kP(T, \mathcal{F}) \le P(T^k, \mathcal{F}^{(k)})$. □

Lemma 2.2. *Let n, k be two positive integers with $n \ge 2k$. For any $x \in X$, we have*

$$(f_n(x))^k \le C^{2k^2} \prod_{j=0}^{n-k} f_k(T^j x),$$

where $C = \max(1, \sup_{x \in X} f_1(x))$.

Proof. Let $x \in X$. For $j = 0, 1, \ldots, k-1$, we have

$$f_n(x) \le f_j(x) f_{n-j}(T^j x) \le C^j f_{n-j}(T^j x).$$

Hence

$$(f_n(x))^k \le \prod_{j=0}^{k-1} C^j f_{n-j}(T^j x) \le C^{k^2} \prod_{j=0}^{k-1} f_{n-j}(T^j x). \tag{4}$$

Observe that for any given j between 0 and $k-1$,

$$f_{n-j}(T^j x) \le \left(\prod_{\ell=0}^{t_j - 1} f_k(T^{k\ell + j}x) \right) f_{n-j-kt_j}(T^{kt_j + j}x) \le C^k \left(\prod_{\ell=0}^{t_j - 1} f_k(T^{k\ell + j}x) \right),$$

where t_j is the largest integer t so that $kt + j \le n$. Combining the above inequality with (4) yields

$$(f_n(x))^k \le C^{2k^2} \prod_{j=0}^{k-1} \prod_{l=0}^{t_j - 1} f_k(T^{k\ell + j}x) = C^{2k^2} \prod_{j=0}^{n-k} f_k(T^j x).$$

This finishes the proof of the lemma. □

After a first version of this paper was completed, we were informed that an analogous inequality, $(f_n(x))^k \le C^{3k^2} \prod_{j=0}^{n-1} f_k(T^j x)$, was obtained by Käenmäki (see [17, Lemma 2.2]).

Lemma 2.3. *Suppose $\{\nu_n\}_{n=1}^{\infty}$ is a sequence in $\mathcal{M}(X)$, where $\mathcal{M}(X)$ denotes the space of all Borel probability measures on X with the weak* topology. We form the new sequence $\{\mu_n\}_{n=1}^{\infty}$ by $\mu_n = \frac{1}{n}\sum_{i=0}^{n-1}\nu_n \circ T^{-i}$. Assume that μ_{n_i} converges to μ in $\mathcal{M}(X)$ for some subsequence $\{n_i\}$ of natural numbers. Then $\mu \in \mathcal{M}(X,T)$, and moreover*

$$\limsup_{i\to\infty} \frac{1}{n_i}\int \log f_{n_i}(x)\, d\nu_{n_i}(x) \le \mathcal{F}_*(\mu).$$

Proof. The statement $\mu \in \mathcal{M}(X,T)$ is well-known (see [29, Theorem 6.9] for a proof). To show the desired inequality, we fix an $k \in \mathbb{N}$. For $n \ge 2k$, by Lemma 2.2 we have

$$\begin{aligned}
\frac{1}{n}\int \log f_n(x)\, d\nu_n(x) &= \frac{1}{kn}\int \log\left(f_n(x)\right)^k\, d\nu_n(x)\\
&\le \frac{1}{kn}\left(\int \sum_{j=0}^{n-k} \log f_k(T^j x)\, d\nu_n(x) + 2k^2\log C\right)\\
&= \frac{n-k+1}{kn}\int \log f_k\, d\mu_{n,k} + \frac{2k\log C}{n},
\end{aligned}$$

where $\mu_{n,k} = \frac{1}{n-k+1}\sum_{j=0}^{n-k}\nu_n \circ T^{-i}$. In particular,

$$\frac{1}{n_i}\int \log f_{n_i}\, d\nu_{n_i} \le \frac{n_i-k+1}{n_i k}\int \log f_k\, d\mu_{n_i,k} + \frac{2k\log C}{n_i}. \tag{5}$$

Note that for any $g \in C(X)$,

$$\begin{aligned}
\left| n\int g\, d\mu_n - (n-k+1)\int g\, d\mu_{n,k}\right| &= \left|\sum_{i=n-k+1}^{n-1}\int g(T^i x)\, d\nu_n(x)\right|\\
&\le (k-1)\sup_{x\in X}|g(x)|.
\end{aligned}$$

It follows that $\lim_{n\to\infty}(\int g\, d\mu_n - \int g\, d\mu_{n,k}) = 0$. Since $\lim_{i\to\infty}\mu_{n_i} = \mu$, we have $\lim_{i\to\infty}\mu_{n_i,k} = \mu$. Notice that f_k is a non-negative continuous function on X. We have

$$\begin{aligned}
\limsup_{i\to\infty}\int \log f_k\, d\mu_{n_i,k} &\le \lim_{\epsilon\downarrow 0}\limsup_{i\to\infty}\int \log(f_k+\epsilon)\, d\mu_{n_i,k}\\
&= \lim_{\epsilon\downarrow 0}\int \log(f_k+\epsilon)\, d\mu\\
&= \int \log f_k\, d\mu.
\end{aligned}$$

Combining it with (5) yields

$$\limsup_{i\to\infty}\frac{1}{n_i}\int \log f_{n_i}\, d\nu_{n_i} \le \frac{1}{k}\int \log f_k\, d\mu.$$

Now the desired inequality follows by letting $k \to \infty$. □

Lemma 2.4. *Let $\nu \in \mathcal{M}(X)$. Suppose $\xi = \{A_1,\ldots,A_k\}$ is a partition of $(X,\mathcal{B}(X))$. Then for any positive integers n,ℓ with $n \ge 2\ell$, we have*

$$\frac{1}{n}H_\nu\left(\bigvee_{i=0}^{n-1}T^{-i}\xi\right) \le \frac{1}{\ell}H_{\nu_n}\left(\bigvee_{i=0}^{\ell-1}T^{-i}\xi\right) + \frac{2\ell}{n}\log k,$$

where $\nu_n = \frac{1}{n}\sum_{i=0}^{n-1}\nu \circ T^{-i}$.

Proof. The above result is contained implicitly in the proof of the classical variational principle given by Misiurewicz (see [29]). In the following we present a complete proof for the reader's convenience.

For $0 \le j \le \ell-1$, let t_j denote the largest integer t so that $t\ell + j \le n$. Then we have

$$\bigvee_{i=0}^{n-1} T^{-i}\xi = \bigvee_{r=0}^{t_j-1} T^{-r\ell-j} \bigvee_{i=0}^{\ell-1} T^{-i}\xi \vee \bigvee_{m\in S_j} T^{-m}\xi$$

for any $0 \le j \le \ell-1$, where S_j is a subset of $\{0,\dots,n-1\}$ with cardinality at most 2ℓ. Hence

$$H_\nu\left(\bigvee_{i=0}^{n-1} T^{-i}\xi\right) \le \sum_{r=0}^{t_j-1} H_\nu\left(T^{-r\ell-j}\bigvee_{i=0}^{\ell-1} T^{-i}\xi\right) + 2\ell\log k.$$

Summing this over j from 0 to $\ell-1$ gives

$$\begin{aligned}
\ell H_\nu\left(\bigvee_{i=0}^{n-1} T^{-i}\xi\right) &\le \sum_{j=0}^{\ell-1}\sum_{r=0}^{t_j-1} H_\nu\left(T^{-r\ell-j}\bigvee_{i=0}^{\ell-1} T^{-i}\xi\right) + 2\ell^2\log k \\
&= \sum_{p=0}^{n-\ell} H_\nu\left(T^{-p}\bigvee_{i=0}^{\ell-1} T^{-i}\xi\right) + 2\ell^2\log k \\
&\le \sum_{p=0}^{n-1} H_\nu\left(T^{-p}\bigvee_{i=0}^{\ell-1} T^{-i}\xi\right) + 2\ell^2\log k \\
&= \sum_{p=0}^{n-1} H_{\nu\circ T^{-p}}\left(\bigvee_{i=0}^{\ell-1} T^{-i}\xi\right) + 2\ell^2\log k \\
&\le nH_{\nu_n}\left(\bigvee_{i=0}^{\ell-1} T^{-i}\xi\right) + 2\ell^2\log k,
\end{aligned}$$

where the last inequality has used the concavity of the function $\phi(x) = -x\log x$. This implies the desired inequality. □

3. **The proof of Theorem 1.1.** In this section we give the proof of Theorem 1.1, which is influenced by Misiurewicz's elegant proof of the classical variational principle [21].

Proof of Theorem 1.1. We divide the proof into three small steps:

Step 1: $P(T,\mathcal{F}) \ge h_\mu(T) + \mathcal{F}_*(\mu)$, $\forall \mu \in \mathcal{M}(X,T)$ *with* $\mathcal{F}_*(\mu) \ne -\infty$.

Suppose μ is an element in $\mathcal{M}(X,T)$ satisfying $\mathcal{F}_*(\mu) \ne -\infty$. Let $\xi = \{A_1,\dots,A_k\}$ be a partition of $(X,\mathcal{B}(X))$. Let $\alpha > 0$ be given. Choose $\epsilon > 0$ so that $\epsilon k\log k < \alpha$. Since μ is regular there are compact sets $B_j \subseteq A_j$ with $\mu(A_j\backslash B_j) < \epsilon$, $1 \le j \le k$. Let η be the partition $\{B_0, B_1,\dots,B_k\}$, where $B_0 = X\backslash\bigcup_{j=1}^k B_j$. Then a direct check shows $H_\mu(\xi/\eta) < \epsilon k\log k < \alpha$ (see [29, Page 189] for details). Now set $b = \min_{1\le i\ne j\le k} d(B_i,B_j) > 0$. Pick $\delta > 0$ so that $\delta < b/2$.

Let $n \in \mathbb{N}$. For each $C \in \bigvee_{j=0}^{n-1} T^{-j}\eta$, choose some $x(C) \in \mathrm{Closure}(C)$ such that $f_n(x(C)) = \sup_{y\in C} f_n(y)$. We claim that for each $C \in \bigvee_{j=0}^{n-1} T^{-j}\eta$, there are at most 2^n many different $\tilde{C}$'s in $\bigvee_{j=0}^{n-1} T^{-j}\eta$ such that

$$d_n(x(C), x(\tilde{C})) := \max_{0\le j\le n-1} d\left(T^j(x(C)),\ T^j(x(\tilde{C}))\right) < \delta.$$

To see this claim, for each $C \in \bigvee_{j=0}^{n-1} T^{-j}\eta$ we pick the unique index $(i_0(C), i_1(C), \ldots, i_{n-1}(C)) \in \{0, 1, \ldots, k\}^n$ such that

$$C = B_{i_0(C)} \cap T^{-1}B_{i_1(C)} \cap \cdots \cap T^{-(n-1)}B_{i_{n-1}(C)}.$$

Now fix a $C \in \bigvee_{j=0}^{n-1} T^{-j}\eta$ and let $\mathcal{Y}$ denote the collection of all $\tilde{C} \in \bigvee_{j=0}^{n-1} T^{-j}\eta$ with $d_n(x(C), x(\tilde{C})) < \delta$. Then we have

$$\#\left\{i_\ell(\tilde{C}) : \tilde{C} \in \mathcal{Y}\right\} \le 2, \qquad \ell = 0, 1, \ldots, n-1. \tag{6}$$

To see this inequality, we assume on the contrary that there exists $0 \le \ell \le n-1$ and $\tilde{C}_1, \tilde{C}_2, \tilde{C}_3 \in \mathcal{Y}$ such that $i_\ell(\tilde{C}_1), i_\ell(\tilde{C}_2), i_\ell(\tilde{C}_3)$ are distinct. Without loss of generality, we assume $i_\ell(\tilde{C}_1) \ne 0$ and $i_\ell(\tilde{C}_2) \ne 0$. This implies

$$\begin{aligned} d_n\left(x(\tilde{C}_1),\ x(\tilde{C}_2)\right) &\ge d\left(T^\ell(x(\tilde{C}_1)),\ T^\ell(x(\tilde{C}_2))\right) \ge d\left(B_{i_\ell(\tilde{C}_1)},\ B_{i_\ell(\tilde{C}_2))}\right) \\ &\ge b > 2\delta > d_n\left(x(C),\ x(\tilde{C}_1)\right) + d_n\left(x(C),\ x(\tilde{C}_2)\right), \end{aligned}$$

which leads to a contradiction. Thus (6) is true, from which the claim follows.

In the following we construct an (n, δ)-separated subset E of X with respect to T such that

$$2^n \sum_{y\in E} f_n(y) \ge \sum_{C\in\bigvee_{j=0}^{n-1} T^{-j}\eta} f_n(x(C)). \tag{7}$$

To achieve this purpose we first choose an element $C_1 \in \bigvee_{j=0}^{n-1} T^{-j}\eta$ such that

$$f_n(x(C_1)) = \max_{C\in\bigvee_{j=0}^{n-1} T^{-j}\eta} f_n(x(C)).$$

Let $\mathcal{Y}_1$ denote the collection of all $\tilde{C} \in \bigvee_{j=0}^{n-1} T^{-j}\eta$ with $d_n(x(C_1), x(\tilde{C})) < \delta$. Then the cardinality of $\mathcal{Y}_1$ does not exceed 2^n. If the collection $\bigvee_{j=0}^{n-1} T^{-j}\eta\backslash\mathcal{Y}_1$ is not empty, we choose an element $C_2 \in \bigvee_{j=0}^{n-1} T^{-j}\eta\backslash\mathcal{Y}_1$ such that $f_n(x(C_2)) = \max_{C\in\bigvee_{j=0}^{n-1} T^{-j}\eta\backslash\mathcal{Y}_1} f_n(x(C))$. Let $\mathcal{Y}_2$ denote the collection of $\tilde{C} \in \bigvee_{j=0}^{n-1} T^{-j}\eta\backslash\mathcal{Y}_1$ with $d_n(x(C_2), x(\tilde{C})) < \delta$. We continue this process. More precise, in step m we choose an element $C_m \in \bigvee_{j=0}^{n-1} T^{-j}\eta\backslash\bigcup_{j=1}^{m-1}\mathcal{Y}_j$ such that

$$f_n(x(C_m)) = \max_{C\in\bigvee_{j=0}^{n-1} T^{-j}\eta\backslash\bigcup_{j=1}^{m-1}\mathcal{Y}_j} f_n(x(C)).$$

Let $\mathcal{Y}_m$ denote the collection of all $\tilde{C} \in \bigvee_{j=0}^{n-1} T^{-j}\eta\backslash\bigcup_{j=1}^{m-1}\mathcal{Y}_j$ with $d_n(x(C_m), x(\tilde{C})) < \delta$. Since the partition $\bigvee_{j=0}^{n-1} T^{-j}\eta$ is finite, the above process will stop at some step ℓ. Denote $E = \{x(C_j) :\ j = 1, \ldots, \ell\}$. Then E is (n, δ)-separated and

$$\sum_{y\in E} f_n(y) = \sum_{j=1}^{\ell} f_n(x(C_j)) \ge \sum_{j=1}^{\ell} 2^{-n} \sum_{C\in\mathcal{Y}_j} f_n(x(C)) = 2^{-n} \sum_{C\in\bigvee_{j=0}^{n-1} T^{-j}\eta} f_n(x(C)),$$

from which (7) follows.

Recall $\mu \in \mathcal{M}(X,T)$ and $\mathcal{F}_*(\mu) \neq -\infty$. We have

$$\begin{aligned}
& \frac{1}{n} H_\mu\left(\bigvee_{j=0}^{n-1} T^{-j}\eta\right) + \frac{1}{n}\int \log f_n(x)\, d\mu(x) \\
\leq\ & \frac{1}{n} \sum_{C \in \bigvee_{j=0}^{n-1} T^{-j}\eta} \mu(C)\left(\log f_n(x(C)) - \log\mu(C)\right) \\
\leq\ & \frac{1}{n} \log \sum_{C \in \bigvee_{j=0}^{n-1} T^{-j}\eta} f_n(x(C)) \\
\leq\ & \frac{1}{n} \log\left(2^n \sum_{y\in E} f_n(y)\right) \\
\leq\ & \log 2 + \frac{1}{n}\log P_n(T,\mathcal{F},\delta).
\end{aligned}$$

In the second inequality above, we have used the basic inequality $\sum_{i=1}^m p_i(c_i - \log p_i) \leq \log\sum_{i=1}^m e^{c_i}$, where $c_i \in \mathbb{R}$, $p_i \geq 0$ and $\sum_{i=1}^m p_i = 1$ (see [4, p. 4] for a proof). Now taking $n \to \infty$ and $\delta \to 0$, we obtain $h_\mu(T,\eta) + \mathcal{F}_*(\mu) \leq \log 2 + P(T,\mathcal{F})$. Hence

$$h_\mu(T,\xi) + \mathcal{F}_*(\mu) \leq h_\mu(T,\eta) + H_\mu(\xi/\eta) + \mathcal{F}_*(\mu) \leq \log 2 + \alpha + P(T,\mathcal{F}).$$

Since ξ and α are arbitrary, we have

$$h_\mu(T) + \mathcal{F}_*(\mu) \leq \log 2 + P(T,\mathcal{F}).$$

This holds for all continuous maps T and sub-additive potentials $\mathcal{F}$. Thus we can apply it to T^k and $\mathcal{F}^{(k)}$ to obtain

$$k(h_\mu(T) + \mathcal{F}_*(\mu)) = h_\mu(T^k) + \mathcal{F}_*^{(k)}(\mu) \leq \log 2 + P(T^k, \mathcal{F}^{(k)}) = \log 2 + kP(T,\mathcal{F}),$$

where the last equality follows from Lemma 2.1. Here we have used the fact that $\mu \in \mathcal{M}(X,T^k)$ and $\mathcal{F}_*^{(k)}(\mu) = k\mathcal{F}_*(\mu) \neq -\infty$. Since k is arbitrary, we have $h_\mu(T) + \mathcal{F}_*(\mu) \leq P(T,\mathcal{F})$.

Step 2: If $P(T,\mathcal{F}) \neq -\infty$, then for any small enough $\epsilon > 0$, there exists a $\mu \in \mathcal{M}(X,T)$ such that $\mathcal{F}_*(\mu) \neq -\infty$ and $h_\mu(T) + \mathcal{F}_*(\mu) \geq P(T,\mathcal{F},\epsilon)$.

Let $\epsilon > 0$ be small enough such that $P(T,\mathcal{F},\epsilon) \neq -\infty$. For any $n \in \mathbb{N}$, let E_n be an (n,ϵ)-separated subset of X with respect to T such that $f_n(y) > 0$ for all $y \in E_n$ and

$$\log \sum_{y\in E_n} f_n(y) \geq \log P_n(T,\mathcal{F},\epsilon) - 1.$$

Let $\sigma_n \in \mathcal{M}(X)$ be the atomic measure concentrated on E_n by the formula

$$\sigma_n = \frac{\sum_{y\in E_n} f_n(y)\delta_y}{\sum_{y\in E_n} f_n(y)},$$

where δ_y denote the Dirac measure at y. Let $\mu_n = \frac{1}{n}\sum_{i=0}^{n-1} \sigma_n \circ T^{-i}$. Since $\mathcal{M}(X)$ is compact we can choose a subsequence $\{n_i\}$ of natural numbers such that

$$\lim_{i\to\infty} \frac{1}{n_i} \log P_{n_i}(T,\mathcal{F},\epsilon) = P(T,\mathcal{F},\epsilon)$$

and $\{\mu_{n_i}\}$ converges in $\mathcal{M}(X)$ to some $\mu \in \mathcal{M}(X)$. By Lemma 2.3, $\mu \in \mathcal{M}(X,T)$.

Choose a partition $\xi = \{A_1, \ldots, A_k\}$ of $(X, \mathcal{B}(X))$ so that $\mathrm{diam}(A_i) < \epsilon$ and $\mu(\partial A_i) = 0$ for $1 \le i \le k$. Such a partition does exist (see [29, Lemma 8.5] for a proof). Since each element of $\bigvee_{j=0}^{n-1} T^{-j}\xi$ contains at most one point of E_n, we have

$$\begin{aligned} H_{\sigma_n}\left(\bigvee_{j=0}^{n-1} T^{-j}\xi\right) + \int \log f_n \, d\sigma_n &= \sum_{y \in E_n} \sigma_n(\{y\}) \left(\log f_n(y) - \log \sigma_n(\{y\})\right) \\ &= \sum_{y \in E_n} \sigma_n(\{y\}) \log \left(\sum_{z \in E_n} f_n(z)\right) \\ &= \log \sum_{z \in E_n} f_n(z) \ge \log P_n(T, \mathcal{F}, \epsilon) - 1. \end{aligned}$$

Thus

$$\frac{1}{n} H_{\sigma_n}\left(\bigvee_{j=0}^{n-1} T^{-j}\xi\right) + \frac{1}{n}\int \log f_n \, d\sigma_n \ge \frac{1}{n}\log P_n(T, \mathcal{F}, \epsilon) - \frac{1}{n}. \tag{8}$$

Fix $\ell \in \mathbb{N}$. By Lemma 2.4, we have for $n_i \ge 2\ell$,

$$\frac{1}{n_i} H_{\sigma_{n_i}}\left(\bigvee_{j=0}^{n_i-1} T^{-j}\xi\right) \le \frac{1}{\ell} H_{\mu_{n_i}}\left(\bigvee_{j=0}^{\ell-1} T^{-j}\xi\right) + \frac{2\ell}{n_i}\log k. \tag{9}$$

Since $\mu(\partial A_i) = 0$ and $\{\mu_{n_i}\}$ converges to μ , we have

$$\lim_{i\to\infty} \frac{1}{\ell} H_{\mu_{n_i}}\left(\bigvee_{j=0}^{\ell-1} T^{-j}\xi\right) = \frac{1}{\ell} H_{\mu}\left(\bigvee_{j=0}^{\ell-1} T^{-j}\xi\right).$$

This together with (9) gives

$$\limsup_{i\to\infty} \frac{1}{n_i} H_{\sigma_{n_i}}\left(\bigvee_{j=0}^{n_i-1} T^{-j}\xi\right) \le \frac{1}{\ell} H_{\mu}\left(\bigvee_{j=0}^{\ell-1} T^{-j}\xi\right).$$

Meanwhile by Lemma 2.3,

$$\limsup_{i\to\infty} \frac{1}{n_i}\int \log f_{n_i} \, d\sigma_{n_i} \le \mathcal{F}_*(\mu).$$

Combining these two inequalities with (8) yields

$$\frac{1}{\ell} H_{\mu}\left(\bigvee_{i=0}^{\ell-1} T^{-i}\xi\right) + \mathcal{F}_*(\mu) \ge P(T, \mathcal{F}, \epsilon).$$

Since $H_\mu\left(\bigvee_{i=0}^{\ell-1} T^{-i}\xi\right) \in [0, +\infty)$ and $P(T, \mathcal{F}, \epsilon) \ne -\infty$, we have $\mathcal{F}_*(\mu) \ne -\infty$. Taking $\ell \to +\infty$ yields

$$h_\mu(T) + \mathcal{F}_*(\mu) \ge h_\mu(T, \xi) + \mathcal{F}_*(\mu) = \lim_{\ell\to\infty} \frac{1}{\ell} H_\mu\left(\bigvee_{i=0}^{\ell-1} T^{-i}\xi\right) + \mathcal{F}_*(\mu) \ge P(T, \mathcal{F}, \epsilon).$$

Step 3: $P(T, \mathcal{F}) = -\infty$ if and only if $\mathcal{F}_*(\mu) = -\infty$ for all $\mu \in \mathcal{M}(X, T)$.

By step 1 we have $P(T, \mathcal{F}) \ge \mathcal{F}_*(\mu)$ for all $\mu \in \mathcal{M}(X, T)$ with $\mathcal{F}_*(\mu) \ne -\infty$, which shows the necessity. The sufficiency is implied by step 2 (since if $P(T, \mathcal{F}) \ne -\infty$, then by step 2 there exists some μ with $\mathcal{F}_*(\mu) \ne -\infty$). This finishes the proof the theorem.

4. **Other definitions of topological pressures for sub-additive potentials.** In this section, we first discuss the definitions of topological pressures given by Falconer [9] and Barreira [1]. Then we provide an equivalent definition via open covers, following the idea of Bowen used in his definition of topological entropy for non-compact sets.

In [9], Falconer gave a definition of topological pressures on mixing repellers. Without loss of generality, we only consider one-sided sub-shifts of finite type rather than mixing repellers. Let (Σ_A, σ) be a one-sided sub-shift space over an alphabet $\{1, \ldots, m\}$, where $m \geq 2$ (see [4]). As usual Σ_A is endowed with the metric $d(x, y) = m^{-n+1}$ where $x = (x_k)_{k=1}^{\infty}$, $y = (y_k)_{k=1}^{\infty}$ and n is the smallest of the k such that $x_k \neq y_k$. For any admissible string $I = i_1 \ldots i_n$ of length n over the letters $\{1, \ldots, m\}$, denote $[I] = \{(x_k) \in \Sigma_A : \ x_j = i_j \text{ for } 1 \leq j \leq n\}$. The $[I]$ is called an n-cylinder in Σ_A.

Let $\mathcal{F}$ be a sub-additive sequence of functions on Σ_A. In [9], Falconer defined the topological pressure of $\mathcal{F}$ by

$$FP(\sigma, \mathcal{F}) = \lim_{n\to\infty} \frac{1}{n} \log FP_n(\sigma, \mathcal{F}) \quad \text{and} \quad FP_n(\sigma, \mathcal{F}) = \sum_{[I]} \sup_{x\in[I]} f_n(x),$$

where the summation is taken over the collection of all n-cylinders in Σ_A. It is not so hard to see that in this special case, $FP_n(\sigma, \mathcal{F})$ is identical to $P_n(\sigma, \mathcal{F}, 1/m)$, and $P_n(\sigma, \mathcal{F}, m^{-k}) = FP_{n+k-1}(\sigma, \mathcal{F})$ for all $k \in \mathbb{N}$. This implies $FP(\sigma, \mathcal{F}) = P(\sigma, \mathcal{F})$.

Now let us turn to Barreira's approach in defining pressures for sub-additive potentials via open covers. As in the previous sections, let T be a continuous map acting on a compact metric space (X, d). Let $\mathcal{F} = \{\log f_n\}_{n=1}^{\infty}$ be a sub-additive sequence of functions defined on X. Suppose that $\mathcal{U}$ is a finite open cover of the space X. Denote $\operatorname{diam}(\mathcal{U}) := \max\{\operatorname{diam}(U) : U \in \mathcal{U}\}$. For $n \geq 1$ we denote by $\mathcal{W}_n(\mathcal{U})$ the collection of strings $\mathbf{U} = U_1 \ldots U_n$ with $U_i \in \mathcal{U}$. For $\mathbf{U} \in \mathcal{W}_n(\mathcal{U})$ we call the integer $m(\mathbf{U}) = n$ the *length of* $\mathbf{U}$ and define

$$\begin{aligned} X(\mathbf{U}) &= U_1 \cap T^{-1}U_2 \cap \ldots \cap T^{-(n-1)}U_n \\ &= \left\{x \in X : \ T^{j-1}x \in U_j \text{ for } j = 1, \ldots, n\right\}. \end{aligned}$$

We say that $\Gamma \subset \bigcup_{n\geq 1} \mathcal{W}_n(\mathcal{U})$ covers X if $\bigcup_{\mathbf{U}\in\Gamma} X(\mathbf{U}) = X$. For each $\mathbf{U} \in \mathcal{W}_n(\mathcal{U})$, we write $f(\mathbf{U}) = \sup_{x\in X(\mathbf{U})} f_n(x)$ when $X(\mathbf{U}) \neq \emptyset$ and $f(\mathbf{U}) = -\infty$ otherwise. For $s \in \mathbb{R}$, define

$$M(T, s, \mathcal{F}, \mathcal{U}) = \lim_{n\to\infty} \inf_{\Gamma_n} \sum_{\mathbf{U}\in\Gamma_n} e^{-sm(\mathbf{U})} f(\mathbf{U}),$$

where the infimum is taken over all $\Gamma_n \subset \bigcup_{j\geq n} \mathcal{W}_j(\mathcal{U})$ that cover X. Likewise, we define

$$\underline{M}(T, s, \mathcal{F}, \mathcal{U}) = \liminf_{n\to\infty} \inf_{\Gamma_n} \sum_{\mathbf{U}\in\Gamma_n} e^{-sm(\mathbf{U})} f(\mathbf{U}),$$

$$\overline{M}(T, s, \mathcal{F}, \mathcal{U}) = \limsup_{n\to\infty} \inf_{\Gamma_n} \sum_{\mathbf{U}\in\Gamma_n} e^{-sm(\mathbf{U})} f(\mathbf{U}),$$

where the infimum is taken over all $\Gamma_n \subset \mathcal{W}_n(\mathcal{U})$ that cover X. Define

$$P^*(T, \mathcal{F}, \mathcal{U}) = \inf\{s : \ M(T, s, \mathcal{F}, \mathcal{U}) = 0\} = \sup\{s : \ M(T, s, \mathcal{F}, \mathcal{U}) = +\infty\},$$

$$\underline{CP^*}(T, \mathcal{F}, \mathcal{U}) = \inf\{s : \ \underline{M}(T, s, \mathcal{F}, \mathcal{U}) = 0\} = \sup\{s : \ \underline{M}(T, s, \mathcal{F}, \mathcal{U}) = +\infty\},$$

$$\overline{CP^*}(T, \mathcal{F}, \mathcal{U}) = \inf\{s : \ \overline{M}(T, s, \mathcal{F}, \mathcal{U}) = 0\} = \sup\{s : \ \overline{M}(T, s, \mathcal{F}, \mathcal{U}) = +\infty\}.$$

Define

$$P^*(T,\mathcal{F}) = \liminf_{\operatorname{diam}(\mathcal{U})\to 0} P^*(T,\mathcal{F},\mathcal{U}),$$

$$\underline{CP^*}(T,\mathcal{F}) = \liminf_{\operatorname{diam}(\mathcal{U})\to 0} \underline{CP^*}(T,\mathcal{F},\mathcal{U}),$$

$$\overline{CP^*}(T,\mathcal{F}) = \liminf_{\operatorname{diam}(\mathcal{U})\to 0} \overline{CP^*}(T,\mathcal{F},\mathcal{U}).$$

Barreira named $P^*(T,\mathcal{F})$ the *topological pressure*, $\underline{CP^*}(T,\mathcal{F})$ and $\overline{CP^*}(T,\mathcal{F})$ the *lower and upper topological pressures of* $\mathcal{F}$. In the following we prove

Lemma 4.1. *For any finite open cover* $\mathcal{U}$ *of* X*, we have*

$$P^*(T,\mathcal{F},\mathcal{U}) = \underline{CP^*}(T,\mathcal{F},\mathcal{U}) = \overline{CP^*}(T,\mathcal{F},\mathcal{U}).$$

Proof. The lemma was first proved by Barreira (see [1, Theorem 1.6]) under an additional assumption that $0 < f_n < Cf_{n+1}$ for a constant $C > 0$. In the following we modify Barreira's argument to obtain the general result.

Fix a finite open cover $\mathcal{U}$ of X. By the definitions we have immediately that

$$P^*(T,\mathcal{F},\mathcal{U}) \le \underline{CP^*}(T,\mathcal{F},\mathcal{U}) \le \overline{CP^*}(T,\mathcal{F},\mathcal{U}).$$

Thus to prove the lemma it suffices to show $P^*(T,\mathcal{F},\mathcal{U}) \ge \overline{CP^*}(T,\mathcal{F},\mathcal{U})$. To see this inequality we may assume $P^*(T,\mathcal{F},\mathcal{U}) < +\infty$. Take any $s \in \mathbb{R}$ such that $s > P^*(T,\mathcal{F},\mathcal{U})$. Then there exists a $\Gamma \subset \bigcup_{n\ge 1} \mathcal{W}_n(\mathcal{U})$ which covers X and

$$N(\Gamma) := \sum_{\mathbf{U}\in\Gamma} e^{-sm(\mathbf{U})} f(\mathbf{U}) < 1.$$

Since X is compact, the Γ can be assumed to be finite. Define $\Gamma^n = \{\mathbf{U}_1 \dots \mathbf{U}_n : \mathbf{U}_i \in \Gamma\}$ and $\Gamma^\infty = \bigcup_{n\ge 1} \Gamma^n$. Since $\mathcal{F}$ is sub-additive, we have

$$N(\Gamma^n) := \sum_{\mathbf{U}\in\Gamma^n} e^{-sm(\mathbf{U})} f(\mathbf{U}) \le N(\Gamma)^n,$$

and

$$N(\Gamma^\infty) := \sum_{\mathbf{U}\in\Gamma^\infty} e^{-sm(\mathbf{U})} f(\mathbf{U}) \le \sum_{n=1}^{\infty} N(\Gamma)^n < \frac{1}{1-N(\Gamma)}.$$

Define $K = \max\{m(\mathbf{U}) : \mathbf{U} \in \Gamma\}$. For any integer $n \ge 2K$, define

$$\Lambda_n = \left\{ \mathbf{U}_1\mathbf{U}_2 \dots \mathbf{U}_k : \mathbf{U}_i \in \Gamma \text{ for } 1 \le i \le k, \ \sum_{i=1}^{k-1} m(\mathbf{U}_i) < n-K \le \sum_{i=1}^{k} m(\mathbf{U}_i) \right\}.$$

It is clear $n - K \le m(\mathbf{U}) < n$ for any $\mathbf{U} \in \Lambda_n$. Furthermore $\Lambda_n \subset \Gamma^\infty$ and Λ_n covers X. Now define

$$\tilde{\Lambda}_n = \left\{ \mathbf{U}\mathbf{V} : \mathbf{U} \in \Lambda_n, \mathbf{V} \in \mathcal{W}_{n-m(\mathbf{U})}(\mathcal{U}) \right\}.$$

Then $\tilde{\Lambda}_n \subset \mathcal{W}_n(\mathcal{U})$ covers X. Observe that

$$\begin{aligned} & \sum_{\mathbf{U}\in\Lambda_n, \mathbf{V}\in\mathcal{W}_{n-m(\mathbf{U})}(\mathcal{U})} e^{-sm(\mathbf{UV})} f(\mathbf{UV}) \\ \leq & \left(\sum_{\mathbf{U}\in\Lambda_n} e^{-sm(\mathbf{U})} f(\mathbf{U})\right) \left(\sum_{\mathbf{V}\in\bigcup_{j=1}^K \mathcal{W}_j(\mathcal{U})} e^{-sm(\mathbf{V})} f(\mathbf{V})\right) \\ \leq & N(\Gamma^\infty) \left(\sum_{\mathbf{V}\in\bigcup_{j=1}^K \mathcal{W}_j(\mathcal{U})} e^{-sm(\mathbf{V})} f(\mathbf{V})\right) < \infty. \end{aligned}$$

We have $\overline{M}(T, s, \mathcal{F}, \mathcal{U}) < \infty$. Hence $\overline{CP^*}(T, \mathcal{F}, \mathcal{U}) \leq s$. It follows $\overline{CP^*}(T, \mathcal{F}, \mathcal{U}) \leq P^*(T, \mathcal{F}, \mathcal{U})$. This finishes the proof of the lemma. $\square$

As a direct corollary, we have

Corollary 4.2. $P^*(T, \mathcal{F}) = \underline{CP^*}(T, \mathcal{F}) = \overline{CP^*}(T, \mathcal{F})$.

Now we consider the connection between $P^*(T, \mathcal{F})$ and $P(T, \mathcal{F})$.

Lemma 4.3. $P^*(T, \mathcal{F}) \geq P(T, \mathcal{F})$.

Proof. The following proof is a slightly modified version of the proof of Theorem 9.2 of [29]. For any open cover $\mathcal{U}$ of X, denote

$$Q_n(T, \mathcal{F}, \mathcal{U}) = \inf_{\Gamma} \sum_{\mathbf{U}\in\Gamma} f(\mathbf{U}).$$

where the infimum is taken over all $\Gamma \subset \mathcal{W}_n(\mathcal{U})$ that cover X. It is clear that

$$\overline{CP^*}(T, \mathcal{F}, \mathcal{U}) = \limsup_{n\to\infty} \frac{1}{n} \log Q_n(T, \mathcal{F}, \mathcal{U}).$$

For any $\epsilon > 0$, suppose $\mathcal{U}$ is an open cover of X with $\mathrm{diam}(\mathcal{U}) < \epsilon$. Let E be an (n, ϵ)-separated subset of X with respect to T. Since no members of $\bigvee_{i=0}^{n-1} T^{-i}\mathcal{U}$ contains two elements of E we have $\sum_{x\in E} f_n(x) \leq Q_n(T, \mathcal{F}, \mathcal{U})$, therefore $P_n(T, \mathcal{F}, \epsilon) \leq Q_n(T, \mathcal{F}, \mathcal{U})$. Letting $\epsilon \to 0$ we obtain $P(T, \mathcal{F}) \leq \overline{CP^*}(T, \mathcal{F}) = P^*(T, \mathcal{F})$. $\square$

We don't know whether the equality $P^*(T, \mathcal{F}) = P(T, \mathcal{F})$ always holds under the above general setting. However we can show the equality in several cases, see Propositions 4.4-4.7.

Proposition 4.4. *Assume the topological entropy $h(T) < \infty$ and the entropy map $\mu \mapsto h_\mu(T)$ is upper semi-continuous. Then $P^*(T, \mathcal{F}) = P(T, \mathcal{F})$.*

Proof. We divide the proof into several small steps.
Step 1. $P^*(T, \mathcal{F}) \leq \sup\{h_\mu(T) + \int \frac{1}{k} \log(f_k + \epsilon) d\mu\}$ *for any* $\epsilon > 0$ *and* $k \in \mathbb{N}$.

Fix $\epsilon > 0$ and $k \in \mathbb{N}$. Define $g(x) = (f_k(x) + \epsilon)^{1/k}$. Set $\mathcal{G} = \{g_n\}_{n=1}^\infty$, where $g_n(x) = \prod_{j=0}^{n-1} g(T^j x)$. Then by Walters' variational principle (see (1) and [29, Theorem 9.4]),

$$P^*(T, \mathcal{G}) = \sup \left\{ h_\mu(T) + \int \frac{1}{k} \log(f_k + \epsilon) d\mu : \ \mu \in \mathcal{M}(X, T) \right\}.$$

Set $C = \max(1, \sup_{x\in X} f_1(x))$. Then by Lemma 2.2,

$$f_n(x) \leq C^{2k} g_{n-k}(x) \leq C^{2k} \epsilon^{-k} g_n(x), \qquad \forall x \in X, \ n \in \mathbb{N}.$$

It follows $P^*(T,\mathcal{F}) \le P^*(T,\mathcal{G})$, and thus

$$P^*(T,\mathcal{F}) \le \sup\left\{h_\mu(T) + \int \frac{1}{k}\log(f_k+\epsilon)d\mu,\ \mu\in\mathcal{M}(X,T)\right\}.$$

Step 2. For any $k\in\mathbb{N}$,

$$\begin{aligned}&\lim_{\epsilon\downarrow 0}\sup\left\{h_\mu(T) + \int \frac{1}{k}\log(f_k+\epsilon)d\mu,\ \mu\in\mathcal{M}(X,T)\right\}\\ =\ &\sup\left\{h_\mu(T) + \int \frac{1}{k}\log f_k d\mu,\ \mu\in\mathcal{M}(X,T)\right\}.\end{aligned}$$

In this step, we will use the assumption $h_\mu(T) < \infty$ and the entropy map is upper semi-continuous. Denote by A the value of the first limit. Then there exist a convergent sequence $\{\mu_n\}$ in $\mathcal{M}(X,T)$, and $\{\epsilon_n\}\downarrow 0$ such that $\lim_{n\to\infty} h_{\mu_n}(T)$ equals a finite value (denoted by B) and furthermore

$$\lim_{n\to\infty}\int \frac{1}{k}\log(f_k+\epsilon_n)d\mu_n = A-B.$$

Let ν denote the limit of $\{\mu_n\}$. For any fixed $\epsilon>0$,

$$\lim_{n\to\infty}\int \frac{1}{k}\log(f_k+\epsilon_n)d\mu_n \le \lim_{n\to\infty}\int \frac{1}{k}\log(f_k+\epsilon)d\mu_n.$$

Therefore

$$\lim_{n\to\infty}\int \frac{1}{k}\log(f_k+\epsilon_n)d\mu_n \le \int \frac{1}{k}\log(f_k+\epsilon)d\nu.$$

Letting $\epsilon\downarrow 0$, we obtain

$$\lim_{n\to\infty}\int \frac{1}{k}\log(f_k+\epsilon_n)d\mu_n \le \int \frac{1}{k}\log f_k d\nu.$$

On the other hand, $h_\nu(T) \ge \lim_{n\to\infty} h_{\mu_n}(T) = B$. The above two inequalities imply

$$h_\nu(T) + \int \frac{1}{k}\log f_k d\nu \ge B + (A-B) = A.$$

Thus the desired equality follows.
Step 3. $\lim_{k\to\infty}\sup\{h_\mu(T)+\int\frac{1}{k}\log f_k d\mu,\ \mu\in\mathcal{M}(X,T)\} = \sup\{h_\mu(T)+\mathcal{F}_*(\mu),\ \mu\in\mathcal{M}(X,T)\}$.
The direction "$\ge$" is clear. To show the other direction, we will also use the assumption that $h(T)<\infty$ and the entropy map is upper semi-continuous. We denote by A the value of the first limit. Then there exist $\{\mu_n\}\to\nu$ in $\mathcal{M}(X,T)$, and $\{k_n\}\uparrow +\infty$ such that $\lim_{n\to\infty} h_{\mu_n}(T)$ equals a finite value (denoted by B) and furthermore

$$\lim_{n\to\infty}\int \frac{1}{k_n}\log f_{k_n} d\mu_n = A-B.$$

Fix $\ell\in\mathbb{N}$. Then by Lemma 2.2, for large $n\in\mathbb{N}$ we have

$$(f_{k_n}(x))^\ell \le C^{2\ell^2}\prod_{j=0}^{k_n-\ell} f_\ell(T^j x)$$

and thus

$$\frac{1}{k_n}\log f_{k_n}(x) \le \frac{2\ell\log C}{k_n} + \frac{\sum_{i=0}^{k_n-\ell}\log f_\ell(T^i x)}{k_n\ell}, \qquad \forall x\in X.$$

Therefore

$$\int \frac{1}{k_n} \log f_{k_n} d\mu_n \le \frac{2\ell \log C}{k_n} + \int \frac{k_n - \ell + 1}{k_n \ell} \log f_\ell \, d\mu_n.$$

Taking $n \to +\infty$ yields $A - B \le \int \frac{1}{\ell} \log f_\ell d\nu$. Then letting $\ell \to +\infty$ we obtain $A - B \le \mathcal{F}_*(\nu)$. Since the entropy map is upper semi-continuous, we have $h_\nu(T) \ge \lim_{n\to\infty} h_{\mu_n}(T) = B$. Thus $A \le h_\nu(T) + \mathcal{F}_*(\nu)$. This finishes the proof of step 3.

Step 4. $P^*(T, \mathcal{F}) = P(T, \mathcal{F})$.

By Lemma 4.3 it suffices to prove $P^*(T, \mathcal{F}) \le P(T, \mathcal{F})$. To see this, by step 1-3, we have

$$P^*(T, \mathcal{F}) \le \sup\{h_\mu(T) + \mathcal{F}_*(\mu) : \ \mu \in \mathcal{M}(X, T)\}.$$

Since $h(T) < \infty$, by Theorem 1.1 we have $P(T, \mathcal{F}) = \sup\{h_\mu(T) + \mathcal{F}_*(\mu) : \ \mu \in \mathcal{M}(X, T)\}$, from which $P^*(T, \mathcal{F}) \le P(T, \mathcal{F})$ follows. This finishes the proof of the proposition. □

Proposition 4.5. *Assume the dynamical system (X, T) satisfies the following doubling property: there exists a sequence $\{r_n\} \subset \mathbb{N}$ such that $\lim_{n\to\infty} \log r_n / n = 0$, and for any $n \in \mathbb{N}$, $x \in X$ and $\epsilon > 0$, the Bowen ball*

$$B_n(x, \epsilon) := \{y \in X : \ d(T^i x, T^i y) < \epsilon \text{ for } 1 \le i \le n - 1\}$$

can be covered by at most r_n many Bowen balls $B_n(y_i, \epsilon/2)$, $i = 1, \ldots, r_n$. Then $P^*(T, \mathcal{F}) = P(T, \mathcal{F})$.

Proof. For any open cover $\mathcal{U}$ of X, denote

$$Q_n(T, \mathcal{F}, \mathcal{U}) = \inf_{\Gamma} \sum_{\mathbf{U} \in \Gamma} f(\mathbf{U}).$$

where the infimum is taken over all $\Gamma \subset \mathcal{W}_n(\mathcal{U})$ that cover X. It is easily checked that $Q_{n+m}(T, \mathcal{F}, \mathcal{U}) \le Q_n(T, \mathcal{F}, \mathcal{U}) Q_m(T, \mathcal{F}, \mathcal{U})$. Thus we have

$$P^*(T, \mathcal{F}, \mathcal{U}) = \overline{CP^*}(T, \mathcal{F}, \mathcal{U}) = \inf_n \frac{1}{n} \log Q_n(T, \mathcal{F}, \mathcal{U}).$$

In the following we show that for any $n \in \mathbb{N}$ and $\epsilon > 0$, there exists an open cover $\mathcal{U}$ of X with $\mathrm{diam}(\mathcal{U}) \le \epsilon$ such that

$$Q_n(T, \mathcal{F}, \mathcal{U}) \le r_n^4 P_n(T, \mathcal{F}, \epsilon). \tag{10}$$

Since $\lim_{n\to\infty} \log r_n / n = 0$, the above inequality implies $P^*(T, \mathcal{F}) \le P(T, \mathcal{F})$. By Lemma 4.3, we have $P^*(T, \mathcal{F}) = P(T, \mathcal{F})$.

To show (10), we choose an (n, ϵ)-separated subset E of X with maximal cardinality. Then the family of Bowen balls $\{B_n(x, \epsilon) : \ x \in E\}$ covers X. For any $x \in E$, pick $y(x) \in \mathrm{Closure}(B_n(x, \epsilon))$ such that

$$f_n(y(x)) = \sup_{y \in B_n(x, \epsilon)} f_n(y).$$

Define $\mathcal{U} = \{B(T^i x, \epsilon) : \ x \in E, 1 \le i \le n-1\}$, where $B(x, \delta) := \{y \in X : \ d(x, y) < \delta\}$. Then $\mathcal{U}$ covers X and $\mathrm{diam}(\mathcal{U}) \le \epsilon$. Since

$$X = \bigcup_{x \in E} B_n(x, \epsilon) = \bigcup_{x \in E} \bigcap_{i=0}^{n-1} T^{-i} B(T^i x, \epsilon),$$

we have

$$Q_n(T, \mathcal{F}, \mathcal{U}) \le \sum_{x \in E} f_n(y(x)). \tag{11}$$

To prove (10) we first claim that for any $\underline{x} \in E$ there are at most r_n^4 many $x \in E$ such that $d_n(y(x), y(\underline{x})) \le \epsilon$. To show this, let $x_1, \ldots, x_\ell$ be ℓ many different points in E such that $d_n(y(x_i), y(\underline{x})) \le \epsilon$ for $1 \le i \le \ell$. Then $B_n(\overline{x}, 4\epsilon) \supset \bigcup_{i=1}^{\ell} B_n(x_i, \epsilon)$. By the assumed doubling property, $B_n(\overline{x}, 4\epsilon)$ can be covered by at most r_n^4 many Bowen balls $B_n(z, \epsilon/4)$'s. Since any $B_n(z, \epsilon/4)$ intersects at most one of the Bowen balls $B_n(x_i, \epsilon/4)$, we obtain $\ell \le r_n^4$. Now we pick $z_1 \in E$ such that $f_n(y(z_1)) = \max\{f_n(y(x)) :\ x \in E\}$. Then we get a set E_1 by removing all those points x in E such that $B_n(y(z_1), y(x)) \le \epsilon$. If $E_1 \neq \emptyset$, we pick $z_2 \in E_1$ such that $f_n(y(z_2)) = \max\{f_n(y(x)) :\ x \in E_1\}$. Then we get a set E_2 by removing all those points x in E_1 such that $B_n(y(z_2), y(x)) \le \epsilon$. Continuing this process (which will stop at some step p), we obtain an (n, ϵ)-separated subset $\{y(z_i) :\ 1 \le i \le p\}$ and

$$\sum_{i=1}^{p} f_n(y(z_i)) \ge r_n^{-4} \sum_{x \in E} f_n(y(x)).$$

Combining it with (11) yields (10). This finishes the proof. □

Proposition 4.6. *Let (X, T) be a dynamical system such that for any $\epsilon > 0$, there exists a partition $\mathcal{U} = \{U_1, \ldots, U_k\}$ of X such that $\operatorname{diam}(\mathcal{U}) < \epsilon$ and all U_i are open. Then $P^*(T, \mathcal{F}) = P(T, \mathcal{F})$.*

Proof. Assume $\mathcal{U} = \{U_1, \ldots, U_k\}$ is an open partition of X. Then all U_i are both open and closed. Set

$$\gamma = \min_{1 \le i < j \le k} d(U_i, U_j).$$

Then $\gamma > 0$. For any $0 < \delta < \gamma$ and $\ell \in \mathbb{N}$, one has

$$\inf_{\Gamma_\ell} \sum_{\mathbf{U} \in \Gamma_\ell} e^{-sm(\mathbf{U})} f(\mathbf{U}) \le e^{-s\ell} P_\ell(T, \mathcal{F}, \delta),$$

where the infimum is taken over all $\Gamma_\ell \subset \mathcal{W}_\ell(\mathcal{U})$ that cover X. This implies $\underline{CP^*}(T, \mathcal{F}, \mathcal{U}) \le P(T, \mathcal{F}, \delta)$. Hence one has $P^*(T, \mathcal{F}, \mathcal{U}) = \underline{CP^*}(T, \mathcal{F}, \mathcal{U}) \le P(T, \mathcal{F}, \delta)$, moreover $P^*(T, \mathcal{F}, \mathcal{U}) \le P(T, \mathcal{F})$. Since the diameter of $\mathcal{U}$ can be arbitrarily small, we have $P^*(T, \mathcal{F}) \le P(T, \mathcal{F})$. This finishes the proof. □

If we put some bounded distortion assumptions on $\mathcal{F}$, then we can also show the equivalence of these two definitions.

Proposition 4.7. *Assume that $\mathcal{F} = \{\log f_n\}_{n=1}^{\infty}$ satisfies the following additional assumptions:*

(i) *$f_n(x) > 0$ for all $x \in X$ and $n \in \mathbb{N}$.*

(ii)

$$\limsup_{\operatorname{diam}(\mathcal{U}) \to 0} \limsup_{n \to \infty} \frac{\log \gamma_n(\mathcal{F}, \mathcal{U})}{n} = 0,$$

where for any finite open cover $\mathcal{U}$, $\gamma_n(\mathcal{F}, \mathcal{U})$ is defined by

$$\gamma_n(\mathcal{F}, \mathcal{U}) = \sup\{f_n(x)/f_n(y) :\ x, y \in X(\mathbf{U}) \text{ for some } \mathbf{U} \in \mathcal{W}_n(\mathcal{U})\}.$$

Then we have $P^*(T, \mathcal{F}) = P(T, \mathcal{F})$.

Proof. We take some arguments similar to that in the proof of Proposition 4.5. Under the above assumptions, instead of (10) we may show that for any finite open cover $\mathcal{U}$ of X and $n \in \mathbb{N}$,

$$Q_n(T, \mathcal{F}, \mathcal{U}) \le \gamma_n(\mathcal{F}, \mathcal{U}) P_n(T, \mathcal{F}, \epsilon).$$

To show this, as in the proof of Proposition 4.5, we have

$$Q_n(T,\mathcal{F},\mathcal{U}) \le \sum_{x\in E} f_n(y(x)) \le \gamma_n(\mathcal{F},\mathcal{U}) \sum_{x\in E} f_n(x) \le \gamma_n(\mathcal{F},\mathcal{U}) P_n(T,\mathcal{F},\epsilon). \quad (12)$$

This finishes the proof. □

In the remaining part of this section, we will follow Bowen's idea in [3] to give an equivalent definition of topological pressure via open covers.

Let (X,T) be a compact dynamical system. Let $\mathcal{C}_X$ denote the set of all finite Borel covers of X. Let $\mathcal{F}=\{\log f_n\}_{n=1}^{\infty}$ be a sub-additive potential. For any open cover $\mathcal{U}$ of X, we define

$$P_n^{**}(T,\mathcal{F},\mathcal{U}) = \min_{\alpha\in\mathcal{C}_X,\ \alpha\succeq\bigvee_{i=0}^{n-1}T^{-i}\mathcal{U}} \sum_{A\in\alpha} \sup_{x\in A} f_n(x),$$

where $\alpha \succeq \bigvee_{i=0}^{n-1} T^{-i}\mathcal{U}$ means that for each $A\in\alpha$, there exists $\mathbf{U}\in\mathcal{W}_n(\mathcal{U})$ such that $A\subseteq X(\mathbf{U})$. Furthermore define

$$P^{**}(T,\mathcal{F},\mathcal{U}) = \inf_{n\ge 1} \frac{1}{n}\log P_n^{**}(T,\mathcal{F},\mathcal{U}),$$

and

$$P^{**}(T,\mathcal{F}) = \sup_{\mathcal{U}} P^{**}(T,\mathcal{F},\mathcal{U}). \quad (13)$$

where $\mathcal{U}$ ranges over all open covers of X.

Proposition 4.8. $P^{**}(T,\mathcal{F}) = P(T,\mathcal{F})$.

Proof. We divide the proof into several small parts:

Step 1. $P^{**}(T,\mathcal{F}) = \lim_{\mathrm{diam}(\mathcal{U})\to 0} P^{**}(T,\mathcal{F},\mathcal{U})$. It comes from the fact that $P^{**}(T,\mathcal{F},\mathcal{U}) \le P^{**}(T,\mathcal{F},\mathcal{V})$, whenever $\mathcal{U}$ and $\mathcal{V}$ are two open covers of X and $\mathrm{diam}(\mathcal{V})$ is less than the Lebesgue number of $\mathcal{U}$.

Step 2. $P^{**}(T,\mathcal{F},\mathcal{U}) = \lim_{n\to\infty}\frac{1}{n}\log P_n^{**}(T,\mathcal{F},\mathcal{U})$. It follows from the fact that $P_n^{**}(T,\mathcal{F},\mathcal{U})$ is sub-multiplicative, i.e, $P_{n+m}^{**}(T,\mathcal{F},\mathcal{U}) \le P_n^{**}(T,\mathcal{F},\mathcal{U})P_m^{**}(T,\mathcal{F},\mathcal{U})$ for all $n,m\in\mathbb{N}$. To show this fact, observe that for any $\alpha,\beta\in\mathcal{C}_X$ with $\alpha\succeq\bigvee_{i=0}^{n-1}T^{-i}\mathcal{U}$ and $\beta\succeq\bigvee_{i=0}^{m-1}T^{-i}\mathcal{U}$, we have $\alpha\vee T^{-n}\beta\in\mathcal{C}_X$, $\alpha\vee T^{-n}\beta\succeq\bigvee_{i=0}^{n+m-1}T^{-i}\mathcal{U}$, and furthermore

$$\sum_{C\in\alpha\vee T^{-n}\beta} \sup_{x\in C} f_{n+m}(x) \le \sum_{A\in\alpha}\sup_{x\in A} f_n(x) \sum_{B\in\beta}\sup_{y\in B} f_m(y).$$

Step 3. For any $\epsilon>0$, if $\mathcal{U}$ is an open cover of X with $\mathrm{diam}(\mathcal{U})<\epsilon$*, then* $P^{**}(T,\mathcal{F},\mathcal{U}) \ge P(T,\mathcal{F},\epsilon)$. To show this fact, it suffices to note that for any $\alpha\in\mathcal{C}_X$ with $\alpha\succeq\bigvee_{i=0}^{n-1}T^{-i}\mathcal{U}$ and any (n,ϵ)-separated set E of X, each $e\in E$ is contained exactly in one element $A\in\alpha$.

Step 4. For any $n\in\mathbb{N}$ and $\epsilon>0$, there exists an open cover $\mathcal{U}$ of X such that $\mathrm{diam}(\mathcal{U})\le\epsilon$ *and* $P_n(T,\mathcal{F},\epsilon)\ge P_n^{**}(T,\mathcal{F},\mathcal{U})$. To prove this statement, let $n\in\mathbb{N}$ and $\epsilon>0$. Pick $x_1\in X$ with $f_n(x_1)=\sup_{x\in X} f_n(x)$, pick $x_2\in X\backslash B_n(x_1,\epsilon)$ such that $f_n(x_2)=\sup_{x\in X\backslash B_n(x_1,\epsilon)} f_n(x)$, pick $x_3\in X\backslash\bigcup_{i=1}^2 B_n(x_i,\epsilon)$ such that $f_n(x_3)=\sup_{x\in X\backslash\bigcup_{i=1}^2 B_n(x_i,\epsilon)} f_n(x)$, and so on. We obtain an (n,ϵ)-separated set $\{x_1,x_2,\cdots,x_\ell\}$ of maximal cardinality. Let α be the partition

$$\left\{ B_n(x_1,\epsilon), B_n(x_2,\epsilon)\backslash B_n(x_1,\epsilon), B_n(x_3,\epsilon)\backslash\bigcup_{i=1}^2 B_n(x_i,\epsilon),\ldots \right\}.$$

Then $\sum_{A\in\alpha}\sup_{x\in A} f_n(x)=\sum_{i=1}^{\ell} f_n(x_i)\le P_n(T,\mathcal{F},\epsilon)$. Let $\mathcal{U}=\{B(T^j x_i,\epsilon):\ 0\le j\le n-1, 1\le i\le \ell\}$. We have $\operatorname{diam}(\mathcal{U})\le\epsilon$, $\alpha\succeq\bigvee_{i=0}^{n-1}T^{-i}\mathcal{U}$ and $P_n(T,\mathcal{F},\epsilon)\ge P_n^{**}(T,\mathcal{F},\mathcal{U})$.

By the statements in step 1 and step 3, we have $P^{**}(T,\mathcal{F})\ge P(T,\mathcal{F})$. Whilst by the statements in step 2 and step 4, for any $\epsilon>0$, there exists an open cover $\mathcal{U}$ with diameter not exceeding ϵ such that $P^{**}(T,\mathcal{F},\mathcal{U})\le P(T,\mathcal{F},\epsilon)$. This fact, together with the result in step 1, gives $P^{**}(T,\mathcal{F})\le P(T,\mathcal{F})$ when we let ϵ go to 0. This finishes the proof of the proposition. $\square$

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E-mail address: `ylcao@suda.edu.cn; djfeng@math.cuhk.edu.hk; wenh@mail.ustc.edu.cn`