Exercise 1. Show that for all $x > 0$

$$1 + \frac{x}{2} - \frac{x^2}{8} \le \sqrt{1+x} \le 1 + \frac{x}{2}.$$

Exercise 2. Let $f(x) = x^{1/3}$.

(a) Find the Taylor polynomial $p_2(x)$ of $f(x)$, centered at $x = 8$ with degree 2.

(b) Estimate $|f(x) - p_2(x)|$ for (i) $x = 8.1$ (ii) $x = 7.9$.

Exercise 3. Let $f(x) = \cos(x)$. Let $p_n(x)$ be the Taylor polynomial of $f(x)$ at $x = 1$ with degree n. Find the smallest possible n such that

$$|f(x) - p_n(x)| < 0.00001$$

for $|x - 1| < 0.1$.

Exercise 4. Find the Taylor polynomial of degree n centered at $x = c$.

1. $\frac{e^x}{1+x}$, $c = 1$, $n = 3$.
2. $\cos(1 + x^2)$, $c = 0$, $n = 10$.
3. $\arctan(x)$, $c = 0$, for general n.
4. $\sin(9x)$, $c = 0$, for general n.
5. $\ln \frac{1+x}{1-x}$, $c = 0$, for general n.
6. $\frac{2-x}{3+x}$, $c = 0$, for general n.

Exercise 5. Find the following integrations

(a) $\int (x-1)\sin(x)dx$

(b) $\int_1^4 \frac{1}{\sqrt{x}(1+\sqrt{x})}dx$.

(c) $\int \frac{x}{\sqrt{a^2-x^2}}dx$

(d) $\int \frac{dx}{(4-x^2)^{3/2}}$

(e) $\int \frac{x^5}{x^3-1}$

(f) $\int x^3 \arctan x dx$

(g) $\int \frac{x^4+x^3+6x^2+x+1}{(x+1)(x^2+1)}dx$

(h) $\int \sin(5x)\cos(3x)dx$

(i) $\int_0^{2\pi} x|\cos(x)|dx$

(j) $\int e^{|x-1|}dx$.

(k) $\int \sin^3(x)\cos^2(x)dx$

(l) $\int \sec^4(x)\tan^3(x)dx$.

(m) $\int \frac{1+\cos^2(x)}{1+\cos(x)\sin(x)}dx$ (Hint: use $t = \tan x$).

(n) $\int \frac{1+\cos(x)}{2+\sin(x)}dx$ (Hint: use $t = \tan \frac{x}{2}$.)

Exercise 6. Find $F'(x)$ for the following functions.

(a) $F(x) = \displaystyle\int_\pi^x \frac{\cos y}{y}\, dy$

(b) $F(x) = \displaystyle\int_{-\pi}^x e^{\sin 2t}\, dt$

(c) $F(x) = \displaystyle\int_x^1 \sqrt{1+t^2}\, dt$

(d) $F(x) = \displaystyle\int_0^{x^3} e^{u^2}\, du$

(e) $F(x) = \int_{-\sin x}^{\sqrt{\pi}} \cos(y^2)\, dy$

(f) $F(x) = \int_{x}^{2x} (\ln t)^2\, dt$

(g) $F(x) = \int_{x^2}^{x^3} e^{\cos u}\, du$

(h) $F(x) = \int_{-\sqrt{\ln x}}^{\sqrt{\ln x}} \frac{\sin t}{t}\, dt$

(i) $F(x) = \int_{x}^{x^2} \frac{x}{\sqrt{\ln(t)}} dt$ (Note: there is x in the integrand, not a typo)

(j) $F(x) = \int_0^x (e^{t^2} - 1) \ln(1 + x) dt.$

Exercise 7.

$$f(x) = \int_0^{\sin(x)} \frac{2\cos^2(t)}{2+t},\ g(x) = \int_0^{\sin(x)} \frac{2\cos(x)\cos(t)}{2+t} dt,\ h(t) = \int_0^{\sin(x)} \frac{\cos(t+x)}{2+t} dt.$$

Find $f'(\pi)$, $g'(\pi)$ and $h'(\pi)$.

Exercise 8.

(a) $\lim_{x \to 0^+} \frac{\int_0^x \sin(t^2) dt}{\int_0^x t\sin(t)\cos(t^2) dt}$

(b) $\lim_{x \to 0^+} \frac{\int_x^0 \sqrt{4t^2 + t^6} dt}{x^2}$

(c) $\lim_{x \to 0^+} \frac{\int_1^{3x+1} \sqrt{t^5 + t^3 + 1} dx}{\ln(x+1)}$.

(d) $\lim_{x \to 0^+} \frac{\int_0^{\sin(x)} t\sin(\sin(t)) dt}{x^3}$

(e) $\lim_{x \to 0} \left(\frac{1}{x^4} \int_0^x (e^{t^2} - 1) \ln(1+t) dt \right)$

(f) $\lim_{x \to 0} \left(\frac{1}{x^4} \int_0^x (e^{t^2} - 1) \ln(1+x) dt \right)$

Exercise 9. (Level 3)
Prove the following reduction formulas.

(a) $I_n = \int x^n e^{ax} dx;\ I_n = \frac{x^n e^{ax}}{a} - \frac{n}{a} I_{n-1},\ n \geq 1$

(b) $I_n = \int \sin^n x dx;\ I_n = -\frac{\cos x \sin^{n-1} x}{n} + \frac{n-1}{n} I_{n-2},\ n \geq 2$

(c) $I_n = \int \cos^n x dx;\ I_n = \frac{\sin x \cos^{n-1} x}{n} + \frac{n-1}{n} I_{n-2},\ n \geq 2$

(d) $I_n = \int \frac{1}{\sin^n x} dx;\ I_n = -\frac{\cos x}{(n-1)\sin^{n-1} x} + \frac{n-2}{n-1} I_{n-2},\ n \geq 2$

(e) $I_n = \int x^n \cos x dx;\ I_n = x^n \sin x + n x^{n-1} \cos x - n(n-1) I_{n-2},\ n \geq 2$

(f) $I_n = \int \frac{dx}{(x^2 - a^2)^n};\ I_n = -\frac{x}{2a^2(n-1)(x^2-a^2)^{n-1}} + \frac{2n-3}{2a^2(n-1)} I_{n-1},\ n \geq 1$

(g) $I_n = \int \frac{x^n dx}{\sqrt{x+a}}$; $I_n = \frac{2x^n\sqrt{x+a}}{2n+1} - \frac{2an}{2n+1} I_{n-1}$, $n \geq 1$

(h) $I_n = \int (\ln x)^n dx$; $I_n = x(\ln x)^n - nI_{n-1}$, $n \geq 1$.

(i) $\int \sin^n x \cos^m x \, dx = \frac{\sin^{n+1} x \cos^{m-1} x}{n+m} + \frac{m-1}{n+m} \int \sin^n x \cos^{m-2} x \, dx$, where m and n are natural numbers and $m \geq 2$.

(j) Show that for any integer $n \geq 2$,

$$\int \sec^n x \, dx = \frac{\sec^{n-2} x \tan x}{n-1} + \frac{n-2}{n-1} \int \sec^{n-2} x \, dx$$

Exercise 10. Let $I_m = \int_0^{\pi/2} \cos^m t \, dt$ where $m = 0, 1, 2, \ldots$.

(a) (i) Evaluate I_0 and I_1.

(ii) Show that $I_m = \frac{m-1}{m} I_{m-2}$ for $m \geq 2$.

Hence, evaluate I_{2n} and I_{2n+1} for $n \geq 1$.

(b) Show that $I_{2n-1} \geq I_{2n} \geq I_{2n+1}$ for $n \geq 1$.

(c) Let $A_n = \frac{1}{2n+1} \left[\frac{2 \cdot 4 \cdot 6 \cdots (2n)}{1 \cdot 3 \cdot 5 \cdots (2n-1)} \right]^2$ where $n = 0, 1, 2, \ldots$.

(i) Using (a) and (b), show that $\frac{2n+1}{2n} A_n \geq \frac{\pi}{2} \geq A_n$.

(ii) Show that $\{A_n\}$ is monotonic increasing.

(iii) Evaluate $\lim_{n\to\infty} \frac{1}{2n+1} \left[\frac{2 \cdot 4 \cdot 6 \cdots (2n)}{1 \cdot 3 \cdot 5 \cdots (2n-1)} \right]^2$.

Exercise 11. FOr each non-negative real numbers α, β, define

$$I_{\alpha,\beta} = \int_0^1 x^\alpha (1-x)^\beta dx.$$

(a) Show that whenever $\alpha \geq 0$, $\beta \geq 1$,

$$(\alpha + A) I_{\alpha,\beta} = \beta I_{\alpha+1,\beta-1}.$$

Here A is an integer whose value you have to determine explicitly.

(b) Hence, or otherwise, show that whenever m, n are positive integers

$$I_{m,n} = \frac{m!}{n!}(m+n+B)!.$$

Here B is an integer whose value you have to determine explicitly.

Exercise 12. Evaluate the following integrals of rational functions.

(a) $\int \frac{x^2dx}{1-x^2}$

(b) $\int \frac{x^3}{3+x}\,dx$

(c) $\int \frac{(1+x)^2}{1+x^2}\,dx$

(d) $\int \frac{dx}{x^2+2x-3}$

(e) $\int \frac{dx}{(x^2-2)(x^2+3)}$

(f) $\int \frac{x^2+1}{(x+1)^2(x-1)}, dx$

(g) $\int \frac{x^2}{(x^2-3x+2)^2}, dx$

(h) $\int \frac{x^2+5x+4}{x^4+5x^2+4}, dx$

(i) $\int \frac{dx}{(x+1)(x^2+1)}$

(j) $\int \frac{2x^3-4x^2-x-3}{x^2-2x-3}\,dx$

(k) $\int \frac{4-2x}{(x^2+1)(x-1)^2}\,dx$

(l) $\int \frac{dx}{x(x^2+1)^2}$

(m) $\int \frac{x^2dx}{(x-1)(x-2)(x-3)}$

(n) $\int \frac{xdx}{x^2(x^2-2x+2)}$

Exercise 13. (Level 3)
Compute the indefinite integrals below:

(a) i. $\int \frac{2x+4}{x-2}dx$ ii. $\int \frac{x^2+1}{x+1}dx$ iii. $\int \frac{x^3}{x-1}dx$

(b) i. $\int \frac{(x+1)dx}{(x-1)^2}$

ii. $\int \frac{(x+6)dx}{(x+2)(x-3)}$

iii. $\int \frac{4dx}{x^2-4}$

iv. $\int \frac{x^2dx}{x^2-4}$

v. $\int \frac{(x^2+1)dx}{x^2+3x+2}$

vi. $\int \frac{(2x^2-2)dx}{2x^2-5x+2}$

vii. $\int \frac{dx}{x^2+4}$

viii. $\int \frac{xdx}{x^2+4}$

ix. $\int \frac{x^2dx}{x^2+4}dx$

x. $\int \frac{(2x+5)dx}{x^2-2x+10}$

xi. $\int \frac{(x^2+15)dx}{x^2-2x+10}$

xii. $\int \frac{(-x+1)dx}{2x^2+4x+5}$

Exercise 14. (Level 3)
Compute the indefinite integrals below:

(a) $\int \frac{(x-1)dx}{(x+3)^3}$

(b) $\int \frac{(2x^2-3x+3)dx}{(x-1)^3}$

(c) $\int \frac{(3x^2-4x+4)dx}{x(x-1)(x-2)}$

(d) $\int \frac{(x^3-4x^2-x+2)dx}{x(x^2-1)}$

(e) $\int \frac{(10x^2-10x-20)dx}{2x^3+3x^2-2x}$

(f) $\int \frac{(x^2-2)dx}{x(x-1)^2}$

(g) $\int \frac{(3x^2-4x+2)dx}{(x-2)(x+1)^2}$

(h) $\int \frac{(x^2-x+2)dx}{x^3-4x^2+4x}$

(i) $\displaystyle\int \frac{(4x^2+x+12)dx}{x(x^2+4)}$

(j) $\displaystyle\int \frac{(-x+3)dx}{x^3+x^2+x+1}$

(k) $\displaystyle\int \frac{3dx}{x^3+1}$

(l) $\displaystyle\int \frac{(2x^4+x^3+3x^2-3x)dx}{x^3-1}$

Exercise 15. (Level 3)

Compute the indefinite integrals below:

(a) $\displaystyle\int \frac{(x^3+4x^2-2x-1)dx}{x^2(x-1)(x+1)}$

(b) $\displaystyle\int \frac{4x^2dx}{x^4-1}$

(c) $\displaystyle\int \frac{(4x^2+8x+2)dx}{(x+1)^2(x^2+4x+5)}$

(d) $\displaystyle\int \frac{x^6+2x^4+2x^2+2x+2}{(x^2+1)^2}dx$

(e) $\displaystyle\int \frac{8x^2dx}{x^4+4}$

(f) $\displaystyle\int \frac{(-2x^3+2x+4)dx}{x^6-x^2}$

Exercise 16. (Level 3)

Evaluate the following integrals.

(a) $\displaystyle\int \frac{dx}{\sin^3 x}$

(b) $\displaystyle\int \frac{dx}{1+\sin x}$

(c) $\displaystyle\int \frac{dx}{\sin x\cos^4 x}$

(d) $\displaystyle\int \frac{dx}{2+\sin x}$

(e) $\displaystyle\int \frac{1-\cos x}{3+\cos x}dx$

(f) $\displaystyle\int \frac{\cos x+1}{\sin x+\cos x}dx$