Ex. (X,Y) has joint distribution $p(x,y)$. Show

$$E(X/X+Y=z) = \int_{-\infty}^{\infty} x\, p(x, z-x)\,dx \Big/ \int_{-\infty}^{\infty} p(x,z-x)\,dx.$$

Pf. (i) Set $Z = X+Y$, it is direct to check

Z has density $p_Z(z) = \int p(x, z-x)\,dx$

(ii) Write $\varphi(Z) = E(X/Z) = E(X/\mathcal{G})$, where $\mathcal{G}$ is generated by Z, and generating set is $\Lambda_z = \{\omega : Z(\omega) \le z\}$.

Then

$$\int_{\Lambda_z} \varphi(Z)\,dP = \int_{-\infty}^{z} \varphi(z')\,p_Z(z')\,dz' \qquad (*)$$

On the other hand, by def of conditional exp.

$$\int_{\Lambda_z} \varphi(Z)\,dP = \int_{\Lambda_z} X\,dP = \int_{X+Y\le z} X\,dP$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{z-x} x\,p(x,y)\,dy\,dx$$

$$\overset{(\text{let } z'=y-x)}{=} \int_{-\infty}^{\infty}\int_{-\infty}^{z} x\,p(x,z'-x)\,dz'\,dx$$

$$= \int_{-\infty}^{z}\left(\int_{-\infty}^{\infty} x\,p(x,z'-x)\,dx\right)dz'$$

Since z is arbitrary, compare with $(*)$, we have

$$E(X/X+Y=z) = \varphi(z) = \left(\int_{-\infty}^{\infty} x\,p(x,z-x)\,dx\right)\Big/ p_Z(z)$$