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On the Weyl–Heisenberg frames generated by simple functions ☆

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Abstract

Let $\phi(x)=\sum_{n=0}^{\infty}c_n\chi_E(x-n)$ with $\{c_n\}_{n=0}^{\infty}\in l^1$, and let $(\phi,a,1)$, $0<a\leqslant 1$ be a Weyl–Heisenberg system $\{e^{2\pi imx}\phi(x-na)\colon m,n\in\mathbb{Z}\}$. We show that if $E=[0,1]$ (and some modulo extension of E), then $(\phi,a,1)$ is a frame for each $0<a\leqslant 1$ (for certain a, respectively) if and only if the analytic function $H(z)=\sum_{n=0}^{\infty}c_nz^n$ has no zero on the unit circle $\{z\colon |z|=1\}$. These results extend the case of Casazza and Kalton (2002) [6] that $\phi(x)=\sum_{i=1}^{k}\chi_{[0,1]}(x-n_i)$ and $a=1$, which brought together the frame theory and the function theory on the closed unit disk. Our techniques of proofs are based on the Zak transform and the distribution of fractional parts of $\{na\}_{n\in\mathbb{Z}}$.

Keywords: Analytic function; Frame; Fractional part; Modulation; Translation; Zak transform; Zero

1. Introduction

Let $g\in L^2(\mathbb{R})$ and $a,b\in\mathbb{R}^+$, we use (g,a,b) to denote the *Weyl–Heisenberg system* (also called *the Gabor system*) $\{E_{mb}T_{na}g\colon m,n\in\mathbb{Z}\}$ generated by a *window function* g. Here $E_bg(t)=e^{2\pi ibt}g(t)$ is the modulation operator and $T_ag(t)=g(t-a)$ is the translation operator.

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We say that (g, a, b) is a *Weyl–Heisenberg frame* (WH-frame for short) for $L^2(\mathbb{R})$ if there exist two positive constants A, B such that

$$A\|f\|^2 \leqslant \sum_{m,n\in\mathbb{Z}} \left|\langle f, E_{mb}T_{na}g\rangle\right|^2 \leqslant B\|f\|^2 \tag{1.1}$$

holds for every $f \in L^2(\mathbb{R})$. We refer to [4,10,13] for some background materials and the recent development of this frame theory and related questions.

For any $g \in L^2(\mathbb{R})$, a fundamental problem in this area is to find all $a, b \in \mathbb{R}^+$ such that (g, a, b) generates a WH-frame for $L^2(\mathbb{R})$. One general restriction is $ab \leqslant 1$ due to the well-known density condition. However, this restriction is far from providing an answer to the fundamental problem. There is considerable amount of the literature on this problem (cf. [4–6,8,11–14, 16,20,22–24]), and at this time the problem is completely solved only for three basic functions: the Gaussian function e^{-t^2} [20,24], the hyperbolic secant $(\cosh t)^{-1}$ [17] and the one-sided exponential function $e^{-|t|}\chi_{[0,\infty)}(t)$ [15]. Here we are concerned with the case when the window function g is a linear combination of certain characteristic functions. If $ab \leqslant 1$, after rescaling we assume that $b = 1$ and $0 < a \leqslant 1$ without loss of generality. It is surprising that even for the simplest case $g = \chi_{[0,c)}$, the classification of all $a, c \in \mathbb{R}^+$ is rather difficult, and it is associated with a complicated set called Janssen's tie ([16], see also [12]). In [6], Casazza and Kalton consider $E = [0, 1) + \{n_1, \ldots, n_k\}$, a finite union of the unit length intervals with integer end points. They showed that $(\chi_E, 1, 1)$ is a frame (indeed a Riesz basis) if and only if $\sum_{i=1}^k z^{n_i}$ has no zeros on the unit circle. In this paper, we continue this investigation in two directions:

(i) we replace $\phi = \sum_{i=1}^k \chi_{[n_i,n_i+1]}$ by more general simple functions; and
(ii) we extend $a = 1$ to more general $0 < a \leqslant 1$.

Throughout we assume that $\{c_n\}_n$ are complex numbers and $\sum_{n=0}^\infty |c_n| < \infty$, and $H(z) = \sum_{n=0}^\infty c_n z^n$ for $|z| \leqslant 1$. It is clear that $H(z)$ is continuous for $\{|z| = 1\}$.

Our main theorem is

Theorem 1.1. *Let $\phi(x) = \sum_{n=0}^\infty c_n\chi_{[0,1)}(x - n)$. Then for each $0 < a \leqslant 1$ (and hence for all $0 < a \leqslant 1$), $(\phi, a, 1)$ is a frame if and only if $H(z)$ has no zeros on $\{|z| = 1\}$.*

Let E be a Borel set in $\mathbb{R}$ and let $\phi(x) = \sum_{n=0}^\infty c_n\chi_E(x - n)$. Then the support of ϕ is contained in $\bigcup_{n=0}^\infty (E + n)$. A basic necessary condition for $(\phi, a, 1)$ to be a frame is that there exist constants C, D such that

$$0 < C \leqslant \sum_{k\in\mathbb{Z}} \left|\phi(x - ka)\right|^2 \leqslant D < \infty \quad \text{a.e.}$$

(see e.g., [7]). This implies that $E \equiv [0, 1) \pmod 1$ when $a = 1$, $c_0 = 1$ and $c_n = 0$ for $n > 0$. We say a Borel subset $E \subset \mathbb{R}$ is a *modulo-partition* of $[0, 1]$ if

$$E \equiv [0, 1] \pmod 1 \quad \text{and} \quad E \cap (E + n) = \emptyset, \quad \forall n \neq 0.$$

As a generalization of Theorem 1.1 we have

Theorem 1.2. *Let E be a modulo-partition of $[0,1]$, and let $\phi(x)=\sum_{n=0}^{\infty}c_n\chi_E(x-n)$. Then for each $q\in\mathbb{N}$, $(\phi,1/q,1)$ is a frame if and only if $H(z)=\sum_{n=0}^{\infty}c_nz^n$ has no zeros on $\{|z|=1\}$.*

We can make restrictions on the above modulo-partition set E to allow more general translations a in the system $(\phi,a,1)$.

Theorem 1.3. *Let $E=[0,c)\cup[N+c,N+1)$ with $0<c<1$ and $N\in\mathbb{N}$. Let $\phi(x)=\sum_{n=0}^{\infty}c_n\chi_E(x-n)$. Then for each $0<a\leqslant\max\{c,1-c\}$, $(\phi,a,1)$ is a frame if and only if $H(z)=\sum_{n=0}^{\infty}c_nz^n$ has no zeros on $\{|z|=1\}$.*

We also give an example to show that the condition $0<a\leqslant\max\{c,1-c\}$ in Theorem 1.3 cannot be omitted. It would be interesting to find a sharp condition for such a.

Example 1.4. Let $E=[0,\frac{1}{2})\cup[N+\frac{1}{2},N+1)$ where $N\equiv 1 \pmod 3$. Assume that $H(z)=\sum_{n=0}^{\infty}c_nz^n$ has no zeros on the unit circle, then $(\phi,\frac{3}{4},1)$ is not a frame.

In the study of WH-frames, the Zak transform has been used extensively. For example, Janssen, Casazza and the others used the transform to study the frame $(g,a,1)$ for the case $a=1$ or $a=1/q$, $q\in\mathbb{N}$, Ron and Shen [22] and Zibulski and Zeevi [25] used it to consider the case a is rational. In our investigation here, we will also use the Zak transform, together with another new technique for the irrational a on the distribution of $\{na-[na]\}_{n\in\mathbb{Z}}$ ($[x]$ is the largest integer which is less than or equal to x).

The theorems bring together frame theory and analytic function theory on the unit disk. The zeros of $H(z)=\sum_{n=0}^{\infty}c_nz^n$ on the unit circle have been studied for a long time. There is a simple sufficient condition (Kakeya theorem) to guarantee that the function $H(z)$ has no zeros on the unit disk: the coefficients $\{c_n\}_{n=0}^{\infty}$ are a decreasing sequence of positive reals such that $c_{j-1}>c_j>c_{j+1}$ for at least one $j\geqslant 1$ ([18], see also [16]). Recently the problem has also been studied extensively for polynomials with restricted coefficients (e.g., $p(z)=\sum_{n=0}^{N}a_nz^n$ where $a_n\in\{0,1\}$ (Newman polynomial) or $a_n\in\{-1,1\}$ (Littlewood polynomial) (see [1,3,21] respectively and the references therein)). There are also numerical results in connection with number theory [9] and analysis [2,19].

For the organization of the paper, we prove the necessity of the theorems, and also the upper frame bound of the sufficiency in Section 2. The main task is to establish the positive lower frame bound of the sufficiency. This is proved in Section 3 for the rational translations, and in Section 4 for the irrational translations.

2. Preliminaries

For $f\in L^2(\mathbb{R})$, we define the *Zak transform* of f by [4,13]:

$$Z_f(x,t)=\sum_{k\in\mathbb{Z}}f(x+k)e^{-2\pi ikt},\quad x,t\in[0,1].$$

It is clear that the series converges in $L^2(Q)$ where $Q=[0,1)\times[0,1)$. Note that Z_f is *quasi-periodic* in the following sense:

$$Z_f(x+1,t)=e^{2\pi it}Z_f(x,t),\qquad Z_f(x,t+1)=Z_f(x,t).$$

Hence it is sufficient to consider $(x,t) \in Q$. By definition, we have $Z_{E_m T_n \chi_{[0,1)}}(x,t) = e^{2\pi i(nx+mt)}$ for $m, n \in \mathbb{Z}$ and $(x,t) \in Q$, it follows that the Zak transform is a unitary map from $L^2(\mathbb{R})$ onto $L^2(Q)$ [4,13].

The following lemma will be used throughout the paper.

Lemma 2.1. *Let $f, g \in L^2(\mathbb{R})$, then*

$$\sum_{m,n\in\mathbb{Z}} \left|\langle f, E_m T_{na} g\rangle\right|^2 = \sum_{n\in\mathbb{Z}} \int_0^1 dx \left| \int_0^1 Z_f(x,t)\overline{Z_g}(x+na,t)\,dt\right|^2.$$

Proof. It follows from the definition of the Zak transform that $Z_{E_m T_{na} g}(x,t) = Z_g(x - na, t)e^{2\pi imx}$. Hence

$$\begin{aligned}
\sum_{m,n} \left|\langle f, E_m T_{na} g\rangle\right|^2 &= \sum_{m,n} \left|\langle Z_f, Z_{E_m T_{na} g}\rangle\right|^2 \\
&= \sum_n \sum_m \left| \int_0^1 \left(\int_0^1 Z_f(x,t)\overline{Z_g}(x-na,t)\,dt\right) e^{-2\pi imx}\,dx\right|^2 \\
&= \sum_n \int_0^1 \left| \int_0^1 Z_f(x,t)\overline{Z_g}(x-na,t)\,dt\right|^2 dx. \qquad \square
\end{aligned}$$

Let E be a bounded Borel subset in $\mathbb{R}$. Define

$$\phi(x) = \sum_{n=0}^{\infty} c_n \chi_E(x-n) \tag{2.1}$$

where all c_n are complex numbers and $\sum_{n=0}^{\infty} |c_n| < \infty$. Let $H(z) = \sum_{n=0}^{\infty} c_n z^n$, then H is a continuous function on $\{|z| = 1\}$, and

$$Z_\phi(x,t) = Z_{\chi_E}(x,t)\sum_{n=0}^{\infty} c_n e^{-2\pi int} = Z_{\chi_E}(x,t)H\big(e^{-2\pi it}\big), \quad (x,t) \in Q. \tag{2.2}$$

To abbreviate the notations in Lemma 2.1, we will fix $g = \phi$, and use $S(f)$ to denote the two sums there. Let

$$F(x; s, k) = \int_0^1 Z_f(x,t)\overline{Z_\phi}(x+s,t)e^{-2\pi ikt}\,dt. \tag{2.3}$$

Hence the quasi-periodic property implies that $F(x; \ell, k) = F(x; 0, k+\ell)$ for any integer ℓ, and the above lemma can be rewritten as

$$S(f)=\sum_n \int_0^1 \left|F(x;na,0)\right|^2 dx. \tag{2.4}$$

For the above $H(z)$, we let

$$A=\min_{|z|=1}\left|H(z)\right| \quad \text{and} \quad B=\max_{|z|=1}\left|H(z)\right|. \tag{2.5}$$

It is easy to check that for any bounded Borel set E, $\phi(x)=\sum_{n=0}^{\infty} c_n\chi_E(x-n)$ lies in the Wiener amalgam space $W(L^\infty,\ell^1)$, it follows that $(\phi,a,1)$ is a Bessel sequence [10, Proposition 6.2.2]. The following proposition states this fact. We give a direct proof for the convenience of the reader.

Proposition 2.2. *Let E be a bounded Borel set and let $\phi(x)=\sum_{n=0}^{\infty} c_n\chi_E(x-n)$. Then for each $0<a\leqslant 1$, there exists $C>0$ dependent on a such that*

$$\sum_{m,n\in\mathbb{Z}} \left|\langle f, E_m T_{na}\phi\rangle\right|^2 \leqslant C\|f\|^2 \quad \forall f\in L^2(\mathbb{R}).$$

Proof. Since E can be decomposed as $E=\bigcup_{n=-N}^{N}(E_n+n)$ for some $N\in\mathbb{N}$ where each $E_n\subseteq[0,1)$, we have $Z_{\chi_E}(x,t)=\sum_{n=-N}^{N}\chi_{E_n}(x)e^{-2\pi int}$ for $(x,t)\in Q$. Note that, for any $0\leqslant\alpha<1$,

$$Z_{\chi_E}(x+\alpha,t)=\begin{cases} Z_{\chi_E}(x+\alpha,t), & 0\leqslant x<1-\alpha;\\ Z_{\chi_E}(x+\alpha-1,t)e^{2\pi it}, & 1-\alpha\leqslant x<1.\end{cases}$$

For $0\leqslant x<1-\alpha$, we have

$$\begin{aligned}\left|F(x;\alpha,k)\right|^2 &= \left|\int_0^1 Z_f(x,t)\sum_{n=-N}^{N}\chi_{E_n}(x+\alpha)e^{2\pi int}\overline{H}\left(e^{-2\pi it}\right)e^{-2\pi ikt}\,dt\right|^2\\ &\leqslant (2N+1)\sum_{n=-N}^{N}\left|\int_0^1 Z_f(x,t)e^{2\pi i(n-k)t}\overline{H}\left(e^{-2\pi it}\right)dt\right|^2.\end{aligned}$$

Similarly for $1-\alpha<x\leqslant 1$, we have

$$\left|F(x;\alpha,k)\right|^2 \leqslant (2N+1)\sum_{n=-N}^{N}\left|\int_0^1 Z_f(x,t)e^{2\pi i(n-k-1)t}\overline{H}\left(e^{-2\pi it}\right)dt\right|^2.$$

Hence

$$\int_0^1 \left|F(x;\alpha,k)\right|^2 dx \leqslant (2N+1)\sum_{n=-N}^{N}\int_0^1 dx\left(\left|\int_0^1 Z_f(x,t)e^{2\pi i(n-k)t}\overline{H}\left(e^{-2\pi it}\right)dt\right|^2\right.$$

$$+\left|\int_0^1 Z_f(x,t)e^{2\pi i(n-k-1)t}\overline{H}\big(e^{-2\pi it}\big)\,dt\right|^2\Bigg).$$

Let $\mathcal{F}_k = \{na-[na]\colon [na]=k,\ n\in\mathbb{Z}\}$ where $[na]$ is the largest integer $\leqslant na$, then $1\leqslant \#\mathcal{F}_k(a)\leqslant 1+a^{-1}$. We have, from the above,

$$\begin{aligned}S(f) &= \sum_{k\in\mathbb{Z}}\sum_{\alpha\in\mathcal{F}_k}\int_0^1 \big|F(x;\alpha,k)\big|^2\,dx \\ &\leqslant C_1\int_0^1 dx\int_0^1 \big|Z_f(x,t)\overline{H}\big(e^{-2\pi it}\big)\big|^2\,dt \leqslant C_1B^2\|f\|^2,\end{aligned}$$

where $C_1 = 2(2N+1)^2(1+a^{-1})$. $\square$

Proposition 2.3. *Let E be a bounded Borel set and let $\phi(x)=\sum_{n=0}^{\infty}c_n\chi_E(x-n)$. For each $0<a\leqslant 1$, suppose there exists $C>0$ such that*

$$\sum_{m,n\in\mathbb{Z}}\big|\langle f, E_mT_{na}\phi\rangle\big|^2 \geqslant C\|f\|^2\quad \forall f\in L^2(\mathbb{R}).$$

Then $\min_{|z|=1}|H(z)| = A > 0$ *where $H(z)$ is given in* (2.2).

Proof. If $A=0$, there exists $t_0\in[0,1)$ such that $H(e^{-2\pi it_0})=0$. For any $\epsilon>0$, we have $|H(e^{-2\pi it})|\leqslant\epsilon$ when $t\in I$ a sub-interval of $[0,1)$. Let $f\in L^2(\mathbb{R})$ be such that $Z_f(x,t)=\chi_I(t)$. By the same argument as in Proposition 2.2, we have

$$S(f)\leqslant C_1\int_0^1 dx\int_I \big|Z_f(x,t)\overline{H}\big(e^{-2\pi it}\big)\big|^2\,dt\leqslant \epsilon^2C_1\|f\|^2,$$

where C_1 is the same as in the above proposition, this contradicts the hypothesis. $\square$

3. Proofs for rational translations

We first consider Theorem 1.2 with $\phi(x)=\sum_{n=0}^{\infty}c_n\chi_E(x-n)$ where E is a modulo-partition of $[0,1]$. In that case $a=1/q$ in the WH-system $(\phi,a,1)$.

Proof of Theorem 1.2. We see that the modulo-partition property implies that $\bigcup_{n\in Z}(E+n)=\mathbb{R}$ is a disjoint union. Hence for $x\in[0,1)$, there exists a unique integer $\eta(x)$ such that $x+\eta(x)\in E$, then

$$Z_{\chi_E}(x,t)=e^{-2\pi i\eta(x)t},\quad x,t\in[0,1). \tag{3.1}$$

Note that $na=j/q+k$ for some j and k. By Lemma 2.1 and (2.3),

$$
\begin{aligned}
S(f) &= \sum_{n\in\mathbb{Z}} \int_0^1 \bigl|F(x;na,0)\bigr|^2\,dx = \sum_{k\in\mathbb{Z}} \sum_{j=0}^{q-1} \int_0^1 \bigl|F(x;\,j/q,k)\bigr|^2\,dx \\
&= \sum_{j=0}^{q-1} \sum_{k\in\mathbb{Z}} \int_0^{1-j/q} \left| \int_0^1 Z_f(x,t)\overline{H}\bigl(e^{-2\pi i t}\bigr) e^{-2\pi i (k-\eta(x+j/q))t}\,dt \right|^2 dx \\
&\quad + \sum_{j=0}^{q-1} \sum_{k\in\mathbb{Z}} \int_{1-j/q}^{1} \left| \int_0^1 Z_f(x,t)\overline{H}\bigl(e^{-2\pi i t}\bigr) e^{-2\pi i (k-\eta(x+j/q-1)+1)t}\,dt \right|^2 dx \quad \bigl(\text{by (3.1)}\bigr) \\
&= q \sum_{k\in\mathbb{Z}} \int_0^1 \left| \int_0^1 Z_f(x,t)\overline{H}\bigl(e^{-2\pi i t}\bigr) e^{-2\pi i k t}\,dt \right|^2 dx \\
&= q \int_Q \bigl|Z_f \overline{H}\bigl(e^{-2\pi i t}\bigr)\bigr|^2.
\end{aligned}
$$

Let $A = \min_{|z|=1} |H(z)|$ and $B = \max_{|z|=1} |H(z)|$ as in (2.5). If $A > 0$, then the above yields $qA^2\|f\|^2 \leqslant S(f) \leqslant qB^2\|f\|^2$, and the sufficiency follows. The necessity follows from the above identity and the proof of Proposition 2.3. (Note that Proposition 2.3 can be applied directly if E is bounded, but cannot be applied if E is unbounded.) □

To prove Theorems 1.1 and 1.3 for the rational a, we need one more technique. If $a = p/q$ where p, q are co-prime, then for $0 \leqslant i, j \leqslant q-1$, there exist $0 \leqslant d_{ij} \leqslant p$ and $0 \leqslant r_{ij} \leqslant q-1$ such that

$$i + jp = d_{ij}q + r_{ij}.$$

It is clear that for each i, $\{r_{ij}\}_{j=0}^{q-1} = \{0, 1, \ldots, q-1\}$. The following rearrangement lemma plays an important role in the proof of the main theorems.

Lemma 3.1. *For each $0 \leqslant i \leqslant q-1$, there exists j_k (depending on i) such that*

$$i + j_k p = d_{ij_k} q + k, \quad k = 0, 1, \ldots, q-1.$$

Moreover

(i) $\{j_0, j_1, \ldots, j_{q-1}\} = \{0, 1, \ldots, q-1\}$;
(ii) $\{d_{ij_k}, d_{ij_{k+1}}, \ldots, d_{ij_{k+p-1}}\} \equiv \{0, 1, \ldots, p-1\} \pmod p$ *for each* $0 \leqslant k \leqslant q-p$;
(iii) $d_{ij_k} \equiv d_{ij_{k+p}} \pmod p$ *for* $0 \leqslant k \leqslant q-p-1$.

Proof. (i) follows from the fact that the j_k's are all distinct, and the same for (ii). To prove (iii), note that the assumption implies $(j_{k+p} - j_k - 1)p = (d_{ij_{k+p}} - d_{ij_k})q$. Hence we have either $d_{ij_{k+p}} = d_{ij_k}$ or $|d_{ij_{k+p}} - d_{ij_k}| = p$. □

Proof of Theorem 1.1 with rational *a*. In view of the results in Section 2, it suffices to prove the existence of a positive lower frame bound. We see that $a = 1$ is just a special case of Theorem 1.2. Now let $a = p/q < 1$ with p, q co-prime. Similar to the above, we have

$$S(f) = \sum_{k\in\mathbb{Z}}\sum_{j=0}^{q-1}\int_0^1 \left|F(x; jp/q, kp)\right|^2 dx = \sum_{i,j=0}^{q-1}\int_{i/q}^{(i+1)/q}\sum_k \left|F(x; jp/q, kp)\right|^2 dx.$$

Fix i, j and let $i + jp = d_{ij}q + r_{ij}$ as above. Let $x \in [i/q, (i+1)/q)$, and write

$$x + \frac{jp}{q} = x - \frac{i}{q} + \frac{i+jp}{q} = x - \frac{i}{q} + d_{ij} + \frac{r_{ij}}{q}.$$

Then $x + jp/q - d_{ij} \in [r_{ij}/q, (r_{ij}+1)/q)$. By (2.2),

$$Z_\phi(x + jp/q, t) = Z_\phi(x + jp/q - d_{ij}, t)e^{2\pi i d_{ij}} = H\left(e^{-2\pi i t}\right)e^{2\pi i d_{ij}}.$$

Hence, by Lemma 3.1,

$$\begin{aligned}
S(f) &= \sum_{i=0}^{q-1}\int_{i/q}^{(i+1)/q} dx \sum_{j=0}^{q-1}\sum_k \left|\int_0^1 Z_f(x,t)\overline{H}\left(e^{-2\pi i t}\right)e^{-2\pi i\cdot(kp+d_{ij})t}\,dt\right|^2 \\
&\geqslant \sum_{i=0}^{q-1}\int_{i/q}^{(i+1)/q} dx \sum_{\ell=0}^{p-1}\sum_{k\in\mathbb{Z}} \left|\int_0^1 Z_f(x,t)\overline{H}\left(e^{-2\pi i t}\right)e^{-2\pi i\cdot(kp+\ell)t}\,dt\right|^2 \\
&= \sum_{i=0}^{q-1}\int_{i/q}^{(i+1)/q} dx \int_0^1 \left|Z_f(x,t)\overline{H}\left(e^{-2\pi i t}\right)\right|^2 dt \\
&= \int_Q \left|Z_f(x,t)\overline{H}\left(e^{2\pi i t}\right)\right|^2 \geqslant A^2\|f\|^2.
\end{aligned}$$

Hence $(\phi, a, 1)$ is a frame. □

Proof of Theorem 1.3 with rational *a*. Again, we need only prove the existence of a positive lower frame bound. Without loss of generality, we assume that $c = \max\{c, 1-c\}$ ($\geqslant 1/2$). That $E = [0, c) \cup [N + c, N + 1)$ implies

$$Z_\phi(x,t) = \begin{cases} H(e^{-2\pi i t}), & x \in [0, c); \\ e^{-2\pi i N t}H(e^{-2\pi i t}), & x \in [c, 1). \end{cases}$$

Let $a = p/q$ with p, q co-prime. There exists a positive integer β such that $\beta/q \leqslant c < (\beta+1)/q$ and thus $p \leqslant \beta$.

Note that for $x \in [i/q, (i+1)/q)$ and for $0 \leqslant s < p$, we can find j_s and d_{ij_s} such that $x + j_s p/q - d_{ij_s} \in [s/q, (s+1)/q) \subset [0, c)$ and $\{d_{ij_0}, \ldots, d_{ij_{p-1}}\} \equiv \{0, \ldots, p-1\} \pmod p$ (by Lemma 3.1). Similar to the argument in the proof of the last theorem, we have

$$\begin{aligned}
S(f) &= \sum_{i=0}^{q-1} \int_{i/q}^{(i+1)/q} dx \sum_{s=0}^{q-1} \sum_{k\in\mathbb{Z}} \left|F(x; j_s p/q - d_{ij_s}, kp + d_{ij_s})\right|^2 \\
&\geqslant \sum_{i=0}^{q-1} \int_{i/q}^{(i+1)/q} dx \sum_{s=0}^{p-1} \sum_{k\in\mathbb{Z}} \left| \int_0^1 Z_f(x,t)\overline{H}\bigl(e^{-2\pi it}\bigr) e^{-2\pi i(kp+d_{ij_s})t}\, dt \right|^2 \\
&= \sum_{i=0}^{q-1} \int_{i/q}^{(i+1)/q} dx \int_0^1 \left|Z_f(x,t)\overline{H}\bigl(e^{-2\pi it}\bigr)\right|^2 dt \\
&= \int_Q \left|Z_f(x,t)\overline{H}\bigl(e^{-2\pi it}\bigr)\right|^2 \geqslant A^2\|f\|^2,
\end{aligned}$$

and the theorem follows. □

In the following we prove Example 1.4, which shows that the condition $a \leqslant \max\{c, 1-c\}$ in Theorem 1.3 cannot be omitted.

Proof of Example 1.4. Let $E = [0, \frac{1}{2}) \cup [N + \frac{1}{2}, N+1)$ where $N \equiv 1 \pmod 3$. For $a = \frac{3}{4}$, we have, as the first identity in the proof of the last theorem,

$$S(f) = \sum_{i=0}^{3} \sum_{j=0}^{3} \int_{i/4}^{(i+1)/4} dx \sum_{k\in\mathbb{Z}} \left|F(x; j3/4 - d_{ij}, 3k + d_{ij})\right|^2,$$

where $i + j3 = d_{ij}4 + r_{ij}$ for $0 \leqslant i, j \leqslant 3$. We give a calculation of the sum of $S(f)$ for $i = 0$. Note that

$$Z_\phi(x,t) = \begin{cases} H(e^{-2\pi it}), & x \in [0, 1/2); \\ e^{-2\pi iNt} H(e^{-2\pi it}), & x \in [1/2, 1). \end{cases}$$

Since $d_{00} = d_{01} = 0$, $d_{02} = 1$, $d_{03} = 2$, we have

$$\begin{aligned}
&\sum_{k\in\mathbb{Z}} \int_0^{1/4} dx \sum_{j=0}^{3} \left|F(x; j3/4 - d_{0j}, 3k + d_{0j})\right|^2 \\
&\quad = \sum_{k\in\mathbb{Z}} \int_0^{1/4} dx \left(\left| \int_0^1 Z_f(x,t)\overline{H}\bigl(e^{-2\pi it}\bigr) e^{-2\pi i(3k)t}\, dt \right|^2 \right.
\end{aligned}$$

$$
\begin{aligned}
&+\left|\int_0^1 Z_f(x,t)\overline{H}\big(e^{-2\pi it}\big)e^{-2\pi i(3k-N)t}\,dt\right|^2 \\
&+\left|\int_0^1 Z_f(x,t)\overline{H}\big(e^{-2\pi it}\big)e^{-2\pi i(3k+1-N)t}\,dt\right|^2 \\
&+\left|\int_0^1 Z_f(x,t)\overline{H}\big(e^{-2\pi it}\big)e^{-2\pi i(3k+2)t}\,dt\right|^2\Bigg) \\
=\;&2\sum_{k\in\mathbb{Z}}\int_0^{1/4}dx\Bigg(\left|\int_0^1 Z_f(x,t)\overline{H}\big(e^{-2\pi it}\big)e^{-2\pi i(3k)t}\,dt\right|^2 \\
&+\left|\int_0^1 Z_f(x,t)\overline{H}\big(e^{-2\pi it}\big)e^{-2\pi i(3k+2)t}\,dt\right|^2\Bigg).
\end{aligned}
$$

Let $P=\overline{\operatorname{span}}\{e^{2\pi i3kt},e^{2\pi i(3k+2)t}\}_{k\in\mathbb{Z}}$ in $L^2[0,1]$. For any nonzero function $h\in P^{\perp}$, we let $h(t)/\overline{H}(e^{-2\pi it})=\sum_{n\in\mathbb{Z}}b_ne^{-2\pi int}$. Define

$$
f^*(x+n)=\begin{cases} b_n, & x\in[0,1/4);\\ 0, & x\in[1/4,1)\end{cases}
$$

for $n\in\mathbb{Z}$. Then by the definition of the Zak transform

$$
Z_{f^*}(x,t)=\begin{cases} h(t)/\overline{H}(e^{-2\pi it}), & x\in[0,1/4);\\ 0, & x\in[1/4,1).\end{cases}
$$

Hence $S(f^*)=\sum_{m,n\in\mathbb{Z}}|\langle f^*,E_mT_{n3/4}\phi\rangle|^2=0$. This implies that $(\phi,3/4,1)$ is not a frame. □

4. Proof for irrational translations

We need a more notation. For fixed a, let

$$
\mathcal{A}_k=\big\{na-\lfloor na\rfloor\colon\ \lfloor na\rfloor=k \text{ and } n\in\mathbb{Z}\big\},\quad k\in\mathbb{Z}
$$

where $\lfloor r\rfloor$ is the largest integer $\leqslant r$. Clearly $\#\mathcal{A}_k\leqslant 1+a^{-1}$ for each $k\in\mathbb{Z}$. Write

$$
\alpha_{*k}=\min\{\alpha\colon\ \alpha\in\mathcal{A}_k\},\qquad \alpha_k^*=\max\{\alpha\colon\ \alpha\in\mathcal{A}_k\},
$$

and $\alpha_k=\alpha_{*k}=\alpha_k^*$ if $\#\mathcal{A}_k=1$.

Lemma 4.1. *Let $1/2 < a < 1$ be an irrational, then $\#\mathcal{A}_k \in \{1,2\}$, and*

(i) *if $\#\mathcal{A}_k = 2$, then $0 \leqslant \alpha_{*k} < 1-a$, $a \leqslant \alpha_k^* < 1$, and $k=0$ is the only integer such that $\alpha_{*0} = 0$, $\alpha_0^* = a$;*
(ii) *if $\#\mathcal{A}_k = 1$, then $1-a \leqslant \alpha_k < a$, and $k=-1$ is the only integer such that $\alpha_k = 1-a$.*

Proof. We only check the two special cases in (i) and (ii), the rest is clear. Note that $\mathcal{A}_0 = \{0, a\}$ and $\mathcal{A}_{-1} = \{1-a\}$. In (i), if $\alpha_{*k} = 0$, then there exists $n \in \mathbb{Z}$ such that $na = k$, since a is an irrational, this forces $k = 0$. Similarly for $\alpha_k^* = a$. In (ii), if $\alpha_k = 1-a$, by definition there exists $n \in \mathbb{Z}$ such that $na = k + \alpha_k = k + 1 - a$. This implies that $n = -1$ because a is irrational, and thus $k = -1$. $\square$

Lemma 4.2. *Let $0 < a < 1$ be an irrational.*

(i) *If $1/2 < a < 1$ and $\#\mathcal{A}_k = \#\mathcal{A}_{k+1} = \cdots = \#\mathcal{A}_{k+m}$ for some $k \in \mathbb{Z}$, then $m \leqslant \max\{\frac{2a-1}{1-a}, \frac{1-a}{2a-1}\}$;*
(ii) *if $1/2 < a < 2/3$ and $\#\mathcal{A}_k = 1$, then $\#\mathcal{A}_{k-1} = \#\mathcal{A}_{k+1} = 2$;*
(iii) *if $2/3 < a < 1$ and $\#\mathcal{A}_k = 2$, then $\#\mathcal{A}_{k-1} = \#\mathcal{A}_{k+1} = 1$.*

Proof. For (i), we let m be the largest with the property in the hypothesis. When $\#\mathcal{A}_k = 1$, by the definition of $\mathcal{A}_k$ and Lemma 4.1, there exists n such that $na = k + \alpha_k$ with $\alpha_k \leqslant a$, and $(n+m)a = (k+m) + \alpha_{k+m}$ with $\alpha_{k+m} \geqslant 1-a$. This yields $m \leqslant \frac{2a-1}{1-a}$. The proof for $\#\mathcal{A}_k = 2$ is similar.

For (ii), there exists $n \in \mathbb{Z}$ such that $na = k + \alpha_k$. Let $(n+1)a = k+1+x$, then $x = a - 1 + \alpha_k \leqslant 2a - 1 < 1 - a$ (by Lemma 4.1(ii) and $a < 2/3$). On the other hand, $x > a - 1 + 1 - a = 0$ if $k \neq -1$ (by Lemma 4.1(ii) again). Hence $0 < x < 1-a$ and $\#\mathcal{A}_{k+1} = 2$ if $k \neq -1$. Similarly, let $(n-1)a = k - 1 + x$ we have $x = 1 - a + \alpha_k \geqslant 2 - 2a > a$ and $x < 1 - a + a = 1$, then $\#\mathcal{A}_{k-1} = 2$. For the exceptional case $k = -1$, we have $\mathcal{A}_{-2} = \{2-3a, 2-2a\}$ and $\mathcal{A}_0 = \{0, a\}$. Hence (ii) holds.

The proof of (iii) is similar to (ii). We will show the case that $\#\mathcal{A}_{k+1} = 1$. Note that the assumption implies that there exists $n \in \mathbb{Z}$ such that $na = k + \alpha_k^*$. Let $(n+1)a = (k+1) + x$, then $x = a - 1 + \alpha_k^*$. It follows that $1 - a < 2a - 1 \leqslant x < a$ (by Lemma 4.1(i) and $a > 2/3$), and thus $\mathcal{A}_{k+1} = \{x\}$ $(= \{\alpha_{k+1}\})$. $\square$

Proof for Theorem 1.1 with irrational *a*. In view of Proposition 2.3, we need only prove the sufficiency. We separate the proof into three cases:

Case 1: Assume $0 < a < 1/2$. Then $\#\mathcal{A}_k \geqslant 2$, and $\alpha_{*k} \in [0, 1/2)$, $\alpha_k^* \in (1/2, 1)$.

$$
\begin{aligned}
S(f) &= \sum_{k\in\mathbb{Z}} \sum_{\alpha\in\mathcal{A}_k} \int_0^1 \left|F(x;\alpha,k)\right|^2 dx \\
&\geqslant \sum_k \left(\int_0^{1/2} \left|F(x;\alpha_{*k},k)\right|^2 dx + \int_{1/2}^1 \left|F\left(x;\alpha_k^*,k\right)\right|^2 dx \right)
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_k \int_0^{1/2} \left| \int_0^1 Z_f \overline{H}\left(e^{-2\pi i t}\right) e^{-2\pi i k t}\, dt \right|^2 dx \\
&\quad + \sum_k \int_{1/2}^1 \left| \int_0^1 Z_f \overline{H}\left(e^{-2\pi i t}\right) e^{-2\pi i (k+1) t}\, dt \right|^2 dx \\
&= \int_Q \left| Z_f(x,t) \overline{H}\left(e^{-2\pi i t}\right) \right|^2 \geqslant A^2 \| f \|^2.
\end{aligned}
$$

Hence the sufficiency follows.

Case 2: $1/2 < a < 2/3$. We consider the integral of x on $[0, 1-a]$, $[1-a, a]$ and $[a, 1]$ separately. By the first identity above,

$$
\begin{aligned}
\sum_{k\in\mathbb{Z}} \sum_{\alpha\in\mathcal{A}_k} \int_0^{1-a} \left| F(x;\alpha,k) \right|^2 dx &\geqslant \int_0^{1-a} \sum_{k\in\mathbb{Z}} \left| F(x;\alpha_{*k},k) \right|^2 dx \\
&= \int_0^{1-a} dx \sum_{k\in\mathbb{Z}} \left| \int_0^1 Z_f(x,t) \overline{H}\left(e^{-2\pi i t}\right) e^{-2\pi i k t}\, dt \right|^2 \\
&= \int_0^{1-a} dx \int_0^1 \left| Z_f(x,t) \overline{H}\left(e^{-2\pi i t}\right) \right|^2 dt.
\end{aligned}
$$

A similar estimation for $[a, 1]$ holds by replacing α_{*k} with α_k^*.

For the integral on $[1-a, a]$, let $\{k_l\}_{l\in\mathbb{Z}}$ be the subsequence of all k such that $\#\mathcal{A}_k = 1$. By Lemma 4.2(ii), $k_{l+1} - k_l \geqslant 2$. We split the sum $\sum_{k\in\mathbb{Z}}$ into $\sum_{k_\ell}$, $\sum_{k_\ell - 1}$ and $\sum_{k\notin\{k_\ell, k_\ell - 1\}}$. We drop away the first sum, and consider the second sum. By noting that $\#\mathcal{A}_{k_l - 1} = 2$, we have the estimation

$$
\begin{aligned}
&\sum_{k_\ell - 1} \sum_{\alpha\in\mathcal{A}_{k_l - 1}} \int_{1-a}^{a} \left| F(x;\alpha,k_\ell - 1) \right|^2 dx \\
&\quad \geqslant \sum_{k_\ell - 1} \int_{1-a}^{a} \left| F(x;\alpha_{*(k_\ell - 1)}, k_\ell - 1) \right|^2 dx + \sum_{k_\ell - 1} \int_{1-a}^{a} \left| F\left(x;\alpha^*_{k_\ell - 1}, k_\ell - 1\right) \right|^2 dx \\
&\quad = \sum_{k_\ell - 1} \int_{1-a}^{a} \left| F(x;0,k_\ell - 1) \right|^2 dx + \sum_{k_\ell} \int_{1-a}^{a} \left| F(x;0,k_\ell) \right|^2 dx.
\end{aligned}
$$

Note that the last equality follows from $x + \alpha_{*(k_\ell - 1)} \in [0, 1]$ and $x + \alpha^*_{k_\ell - 1} \in [1, 2]$ for $x \in [1-a, a]$. Similarly for the third sum, we have

$$\sum_{k\notin\{k_\ell,k_\ell-1\}}\sum_{\alpha\in\mathcal{A}_k}\int_{1-a}^{a}\left|F(x;\alpha,k)\right|^2dx\geqslant\sum_{k\notin\{n_\ell,n_\ell-1\}}\int_{1-a}^{a}\left|F(x;\alpha_{*k},k)\right|^2dx$$
$$=\sum_{k\notin\{n_\ell,n_\ell-1\}}\int_{1-a}^{a}\left|F(x;0,k)\right|^2dx.$$

Summing up the above, we have

$$\sum_{k\in\mathbb{Z}}\sum_{\alpha\in\mathcal{A}_k}\int_{1-a}^{a}\left|F(x;\alpha,k)\right|^2dx\geqslant\int_{1-a}^{a}\sum_{k\in\mathbb{Z}}\left|F(x;0,k)\right|^2dx$$
$$=\int_{1-a}^{a}dx\int_0^1\left|Z_f(x,t)\overline{H}\left(e^{-2\pi it}\right)\right|^2dt.$$

Hence

$$S(f)\geqslant\int_0^1\int_0^1\left|Z_f(x,t)\overline{H}\left(e^{-2\pi it}\right)\right|^2dt\geqslant A^2\|f\|^2.$$

Case 3: $2/3<a<1$. We will need another technical lemma that strengthens Lemma 4.2(iii). □

Lemma 4.3. *Let a be an irrational number with $\frac{q+1}{q+2}<a<\frac{q+2}{q+3}$ where $q\in\mathbb{N}$ and $\#\mathcal{A}_k=2$.*

(i) *If $a\leqslant\alpha_k^*<(q+2)(1-a)$, then for $1\leqslant s\leqslant q$, $\#\mathcal{A}_{k+s}=1$ with $(s+1)a-s\leqslant\alpha_{k+s}<(q+2-s)(1-a)$, and $\#\mathcal{A}_{k+q+1}=2$;*
(ii) *if $(q+2)(1-a)\leqslant\alpha_k^*<1$, then for $1\leqslant s\leqslant q+1$, $\#\mathcal{A}_{k+s}=1$ with $(q+2-s)(1-a)\leqslant\alpha_{k+s}<sa-s+1$, and $\#\mathcal{A}_{k+q+2}=2$.*

Proof. We consider case (i). By the definition of $\mathcal{A}_k$ there exists $n\in\mathbb{Z}$ such that $na=k+\alpha_k^*$, write $(n+1)a=k+1+x$. By the assumption on α_k^*, $x=\alpha_k^*-(1-a)$. It follows that

$$1-a<2a-1\leqslant x<(q+1)(1-a)<a.$$

This says that $\mathcal{A}_{k+1}=\{x\}=\{\alpha_{k+1}\}$ and α_{k+1} satisfies $(n+1)a=k+1+\alpha_{k+1}$. Inductively, for $1\leqslant s\leqslant q$, we have $(n+s)a=k+s+\alpha_{k+s}$, then

$$1-a\leqslant(s+1)a-s\leqslant\alpha_{k+s}<(q+2-s)(1-a)\leqslant a,\quad 1\leqslant s\leqslant q,$$

which implies that $\mathcal{A}_{k+s}=\{\alpha_{k+s}\}$ for $1\leqslant s\leqslant q$. Finally for $s=q+1$, let $(n+q+1)a=k+q+1+x$, then by assumption that $a<(q+2)/(q+3)$, we have $x=\alpha_{*(k+q+1)}$ with $0<(q+2)a-(q+1)\leqslant\alpha_{*(k+q+1)}<1-a$. Thus $\#\mathcal{A}_{k+q+1}=2$.

The proof of the second case is similar with some obvious modification. □

Continuation of the proof of Theorem 1.1. We consider Case 3 that $2/3 < a < 1$ is an irrational number. Similar to the proof of Case 2, it is easy to show that

$$\sum_{k\in\mathbb{Z}}\sum_{\alpha\in\mathcal{A}_k}\int_c^d \left|F(x;\alpha,k)\right|^2 dx \geqslant \int_c^d dx\int_0^1 \left|Z_f(x,t)\overline{H}\left(e^{-2\pi it}\right)\right|^2 dt \tag{4.1}$$

holds for both $c=0$, $d=1-a$ and $c=a$, $d=1$. To estimate the above integrals from $1-a$ to a, we fix a in an open interval $(\frac{q+1}{q+2},\frac{q+2}{q+3})$, where $q\geqslant 1$. By Lemma 4.2(iii), there exists an increasing sequence of integers $\{k_l\}_{l\in\mathbb{Z}}$ such that $\#\mathcal{A}_k=2$ if and only if $k\in\{k_l\}_{l\in\mathbb{Z}}$. Note that $k_{l+1}-k_l\geqslant 2$ for $l\in\mathbb{Z}$. We estimate the integral for each $l\in Z$ through the following two cases:

(i) $a\leqslant\alpha^*_{k_l}<(q+2)(1-a)$: then $k_{\ell+1}=k_\ell+q+1$ by Lemma 4.3(i). We divide $[1-a,1]$ as

$$I_j=\left[j(1-a),(j+1)(1-a)\right],\quad 1\leqslant j\leqslant q,\quad\text{and}\quad I_{q+1}=\left[(q+1)(1-a),1\right).$$

For $1\leqslant j\leqslant q+1$, we have the following estimation:

$$\begin{aligned}S_j &:= \sum_{k=k_l}^{k_l+q}\int_{I_j}\sum_{\alpha\in\mathcal{A}_k}\left|F(x;\alpha,k)\right|^2 dx\\ &\geqslant \int_{I_j}\left(\left|F(x;\alpha_{*k_\ell},k_\ell)\right|^2+\left|F\left(x;\alpha^*_{k_l},k_\ell\right)\right|^2\right)dx\\ &\quad+\sum_{k=k_l+1}^{k_l+j-1}\int_{I_j}\left|\int_0^1 F(x;\alpha_k,k)\right|^2 dx+\sum_{k=k_l+j+1}^{k_l+q}\int_{I_j}\left|\int_0^1 F(x;\alpha_k,k)\right|^2 dx.\end{aligned}$$

(The first two terms after the inequality follow from $\#\mathcal{A}_{k_\ell}=2$; we drop away the term $k=k_\ell+j$ in the last sum. Note also that for $j=q+1$, the last sum is 0 by convention.) Observe that in the above integrands, $x+\alpha_{*k_\ell}\in[0,1]$, $x+\alpha^*_{k_\ell}\in[1,2]$ by our choice of $\alpha^*_{k_l}$ in this paragraph. Also it follows from Lemma 4.3(i) that on the third row, $\alpha_k+j(1-a)\geqslant 1$ in the first sum, so that $x+\alpha_k\in[1,2]$, and in the second sum, $\alpha_k+(j+1)(1-a)\leqslant 1$, so that $x+\alpha_k\in[0,1]$. Hence we have (see the proof of Case 2),

$$\begin{aligned}S_j &\geqslant \int_{I_j}\left|F(x;0,k_\ell)\right|^2 dx+\int_{I_j}\left|F(x;0,k_l+1)\right|^2 dx\\ &\quad+\sum_{k=k_l+1}^{k_l+j-1}\int_{I_j}\left|F(x;0,k+1)\right|^2 dx+\sum_{k=k_l+j+1}^{k_l+q}\int_{I_j}\left|F(x;0,k)\right|^2 dx\\ &=\sum_{k=k_l}^{k_l+q}\int_{I_j}\left|F(x;0,k)\right|^2 dx.\end{aligned}$$

Therefore

$$\sum_{k=k_l}^{k_l+q} \int_{1-a}^{a} \sum_{\alpha \in \mathcal{A}_k} \left|F(x;\alpha,k)\right|^2 dx \geqslant \sum_{k=k_l}^{k_l+q} \int_{1-a}^{a} \left|F(x;0,k)\right|^2 dx.$$

(ii) $(q+2)(1-a) \leqslant \alpha_{k_l}^* < 1$: then $k_{l+1} = k_l + q + 2$ by Lemma 4.3(ii). We adopt the same proof as in (i) to consider

$$\sum_{k=k_l}^{k_l+q+1} \int_{I_j} \sum_{\alpha \in \mathcal{A}_k} \left|F(x;0,\alpha,k)\right|^2 dx$$

where $I_j = [(j+1)a - j, ja - j + 1]$ for $1 \leqslant j \leqslant q$ and $I_{q+1} = [1-a, (q+1)a - q]$ and obtain the desired inequality as in case (i), which implies (4.1) by summing all the $\ell \in \mathbb{Z}$. This completes the proof of Theorem 1.1. □

Proof of Theorem 1.3 for irrational translations. As $E = [0,c) \cup [N+c, N+1)$, we have

$$Z_\phi(x,t) = \begin{cases} H(e^{-2\pi i t}), & x \in [0,c); \\ e^{-2\pi i N t} H(e^{-2\pi i t}), & x \in [c,1). \end{cases}$$

If $\alpha \leqslant c$, let $I_k^{(i)}(\alpha) = \int_{E_i} |F(x;\alpha,k)|^2\,dx$ where $E_1 = [0, c-\alpha)$, $E_2 = [c-\alpha, 1-\alpha)$ and $E_3 = [1-\alpha, 1)$. Then it is a direct calculation to verify (using $x+\alpha \in [0,c)$ or $x+\alpha \in [c,1)$ for all $x \in E_i$) that

$$\begin{aligned} I_k^{(1)}(\alpha) &= \int_0^{c-\alpha} dx \left| \int_0^1 Z_f(x,t)\overline{H}\big(e^{-2\pi i t}\big) e^{-2\pi i k t}\,dt \right|^2, \\ I_k^{(2)}(\alpha) &= \int_{c-\alpha}^{1-\alpha} dx \left| \int_0^1 Z_f(x,t)\overline{H}\big(e^{-2\pi i t}\big) e^{-2\pi i (k-N) t}\,dt \right|^2, \\ I_k^{(3)}(\alpha) &= \int_{1-\alpha}^{1} dx \left| \int_0^1 Z_f(x,t)\overline{H}\big(e^{-2\pi i t}\big) e^{-2\pi i (k+1) t}\,dt \right|^2. \end{aligned} \tag{4.2}$$

If $\alpha > c$, we let $J_k^{(i)}(\alpha) = \int_{E_i} |F(x;\alpha,k)|^2\,dx$ with $E_1 = [0, 1-\alpha)$, $E_2 = [1-\alpha, 1-\alpha+c)$ and $E_3 = [1-\alpha+c, 1)$. Then we have

$$\begin{aligned} J_k^{(1)}(\alpha) &= \int_0^{1-\alpha} dx \left| \int_0^1 Z_f(x,t)\overline{H}\big(e^{-2\pi i t}\big) e^{-2\pi i (k-N) t}\,dt \right|^2, \\ J_k^{(2)}(\alpha) &= \int_{1-\alpha}^{1-\alpha+c} dx \left| \int_0^1 Z_f(x,t)\overline{H}\big(e^{-2\pi i t}\big) e^{-2\pi i (k+1) t}\,dt \right|^2, \end{aligned}$$

$$J_k^{(3)}(\alpha) = \int_{1-\alpha+c}^{1} dx \left| \int_0^1 Z_f(x,t)\overline{H}\left(e^{-2\pi it}\right)e^{-2\pi i(k+1-N)t}\,dt \right|^2. \tag{4.3}$$

Now consider

$$S(f) = \sum_{k\in\mathbb{Z}} \sum_{\alpha\in\mathcal{A}_k} \int_0^1 \left|F(x;\alpha,k)\right|^2 dx.$$

We make the estimates by decomposing the integral to the sum of the above forms according to the relation of c and α and match up the $e^{2\pi ikt}$ on the right. Without loss of generality we assume $c \geqslant 1/2$ (the proof for $c < 1/2$ is similar). Note that by the hypothesis of the theorem, $a \leqslant c$.

Case 1: $\#\mathcal{A}_{k-1}, \#\mathcal{A}_k \geqslant 2$. This implies $0 < a < 2/3$ by Lemma 4.2, and hence $0 \leqslant \alpha_{*i} < \min\{1-a, a\} \leqslant c$ for $i = k-1$ and k. Let $\alpha'_{k-1} = \max\{\alpha \in \mathcal{A}_{k-1}\colon\ \alpha \leqslant c\}$. Then

$$I_k^{(1)}(\alpha_{*k}) = \int_0^{c-\alpha_{*k}} dx \left| \int_0^1 Z_f(x,t)\overline{H}\left(e^{-2\pi it}\right)e^{-2\pi ikt}\,dt \right|^2$$

and

$$I_{k-1}^{(3)}\left(\alpha'_{k-1}\right) = \int_{1-\alpha'_{k-1}}^{1} dx \left| \int_0^1 Z_f(x,t)\overline{H}\left(e^{-2\pi it}\right)e^{-2\pi ikt}\,dt \right|^2.$$

If $\alpha'_{k-1} = \alpha^*_{k-1}$, since $(1-\alpha'_{k-1}) - (c-\alpha_{*k}) = 1-c-(\alpha^*_{k-1}-\alpha_{*k}) = a-c \leqslant 0$, we have $[0,1] \subseteq [0, c-\alpha_{*k}) \cup [1-\alpha'_{k-1}, 1)$. Hence

$$I_{k-1}^{(3)}\left(\alpha'_{k-1}\right) + I_k^{(1)}(\alpha_{*k}) \geqslant \int_0^1 dx \left| \int_0^1 Z_f(x,t)\overline{H}\left(e^{-2\pi it}\right)e^{-2\pi ikt}\,dt \right|^2.$$

If $\alpha'_{k-1} < \alpha^*_{k-1}$, then we define $\alpha'_{k-1} = s_1 < s_2 < \cdots < s_l = \alpha^*_{k-1}$ where $s_j = s_{j-1} + a$. Note that $s_j > c$ for $2 \leqslant j \leqslant l$, then

$$J_{k-1}^{(2)}(s_j) = \int_{1-s_j}^{1-s_j+c} dx \left| \int_0^1 Z_f(x,t)\overline{H}\left(e^{-2\pi it}\right)e^{-2\pi ikt}\,dt \right|^2.$$

Let F_j denote the interval of the above integral and let $F = F_2 \cup \cdots \cup F_\ell$. Then

$$F \supseteq [1-s_\ell, 1-s_2+c] = \left[1-\alpha^*_{k-1}, 1-\alpha'_{k-1}-a+c\right].$$

We claim that

$$[0, c-\alpha_{*k}] \cup F \cup \left[1-\alpha'_{k-1}, 1\right] \supseteq [0,1].$$

Indeed it follows from a direct check that F and the first and the last intervals are overlapping intervals. Now combining the above estimates, we have

$$I_k^{(1)}(\alpha_{*k}) + \sum_{j=2}^{\ell} J_{k-1}^{(2)}(s_j) + I_{k-1}^{(3)}\left(\alpha'_{k-1}\right) \geqslant \int_0^1 dx \left| \int_0^1 Z_f(x,t)\overline{H}\left(e^{-2\pi i t}\right) e^{-2\pi i k t}\, dt \right|^2. \quad (4.4)$$

In the rest of the cases, we use the same idea of the above proof by showing that [0, 1] is a subset of the unions of some of the E_i's in (4.2) and (4.3), and (4.4) holds similarly.

Case 2: $\#\mathcal{A}_{k-1} = 2$ and $\#\mathcal{A}_k = 1$. Then $a > 1/2$ (by Lemma 4.2) and $\alpha_k \leqslant a \leqslant c$. If $\alpha^*_{k-1} \leqslant c$, then

$$[0, c-\alpha_k] \cup \left[1-\alpha^*_{k-1}, 1\right] \supseteq [0,1].$$

If $\alpha^*_{k-1} > c$, then

$$[0, c-\alpha_k] \cup \left[1-\alpha^*_{k-1}, 1-\alpha^*_{k-1}+c\right] \cup [1-\alpha_{*(k-1)}, 1] \supseteq [0,1];$$

Case 3: $\#\mathcal{A}_{k-1} = 1$, $\#\mathcal{A}_k = 2$. Then $\alpha_{k-1} \leqslant a \leqslant c$ and $\alpha_{*k} < c$. In this case we use

$$[1-\alpha_{k-1}, 1] \cup [0, c-\alpha_{*k}] \supseteq [0,1];$$

Case 4: $\#\mathcal{A}_k = \#\mathcal{A}_{k-1} = 1$. Then $a > 2/3$ (by Lemma 4.2). Also we see that $\alpha_{k-1} \leqslant a \leqslant c$. In this case we use

$$[0, c-\alpha_k] \cup [1-\alpha_{k-1}, 1] \supset [0,1].$$

To conclude we see that each k belongs to one of the above cases, we sum over all the terms in (4.2) and (4.3) through all k, then resort them according to $e^{-2\pi i k}$. Therefore from the above estimate, we have

$$S(f) \geqslant \sum_{k\in\mathbb{Z}} \int_0^1 dx \left| \int_0^1 Z_f(x,t)\overline{H}\left(e^{-2\pi i t}\right) e^{-2\pi i k t}\, dt \right|^2 = \int_Q \left| Z_f(x,t)\overline{H}\left(e^{-2\pi i t}\right)\right|^2 \geqslant A^2\|f\|^2.$$

Hence A is the lower frame bound, and the sufficiency follows. □

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