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The spectrum of Poincaré recurrence

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Abstract. We investigate the relationship between Poincaré recurrence and topological entropy of a dynamical system (X, f). For $0 \le \alpha \le \beta \le \infty$, let $D(\alpha, \beta)$ be the set of x with lower and upper recurrence rates α and β, respectively. Under the assumptions that the system is not minimal and that the map f is positively expansive and satisfies the specification condition, we show that for any open subset $\emptyset \ne U \subseteq X$, $D(\alpha, \beta) \cap U$ has the full topological entropy of X. This extends a result of Feng and Wu [The Hausdorff dimension of recurrence sets in symbolic spaces. *Nonlinearity* **14** (2001), 81–85] for symbolic spaces.

1. *Introduction*

Let X be a compact metric space and let f be a continuous transformation on X. We call the pair (X, f) a *dynamical system*. The *orbit* of a point $x \in X$ is the set of iterates of x under f, i.e. the sequence $\{f^k(x)\}_{k=0}^{\infty}$. Since X is compact, under some measure-preserving property most orbits are expected to return to the neighborhoods of their starting points. Indeed, the famous Poincaré recurrence theorem says that if μ is an f-invariant probability measure on X and A is a measurable subset with positive measure, then for almost all points x in A, the orbit of x will return to A infinitely many times.

To study quantitatively the recurrence behavior of the dynamical system, we define, for any point x in a subset A of X, the *first (Poincaré) return time* of x to A as

$$\tau_A(x) = \inf\{i > 0 : f^i(x) \in A\}.$$

Let $\{A_n\}$ be a sequence of sets shrinking to the point x; it is natural to consider the exponential rate of increase of the quantity $\tau_{A_n}(x)$. In 1993, Ornstein and Weiss [**20**] first analyzed this rate for a partition under the action of f. Let ξ be a finite partition of

X and let $\xi_{-n}(x)$ be the intersection of all the elements of $\xi, f^{-1}(\xi), \ldots, f^{-n+1}(\xi)$ that contain x. It was proved that if μ is f-ergodic, then with probability one,

$$\lim_{n\to\infty} \frac{1}{n} \log \tau_{\xi_{-n}(x)}(x) = h_\mu(f, \xi),$$

where $h_\mu(f, \xi)$ is the measure-theoretic entropy of f with respect to the partition ξ. This remarkable theorem points out that the chaotic behavior of a system, which is described by $h_\mu(f, \xi)$, is reflected in the local recurrence rate of the generic orbits. The theorem generated a great deal of interest, and further substantial investigations can be found in the literature. For example, in [**26**] the partition $\xi_{-n}(x)$ was replaced by balls of radius r centered at x; it was shown that if f is a piecewise $C^{1+\alpha}$ monotonic interval map, then for an f-ergodic measure μ with positive entropy,

$$\lim_{r\to 0} \frac{\log \tau_{B(x,r)}(x)}{-\log r} = \dim_{\mathrm{H}}(\mu) \quad \mu\text{-a.e.} \tag{1}$$

Other results in this direction can be found in [**2, 3**]. Here $\dim_{\mathrm{H}}(\mu)$ denotes the Hausdorff dimension of μ, which equals $\lim_{r\to 0} \log(\mu(B(x, r))/\log r$ (the local dimension of μ) for μ-almost all x (see [**8**]).

The expression in (1) resembles the local dimension of a measure in the theory of multifractal formalism. Indeed, Feng and Wu [**15**] undertook the first investigation of this on the full shift spaces of finite symbols. Let $\Sigma^{\mathbb{N}}$ be the canonical infinite product space of finite symbols with the shift map S. Given any Gibbs measure μ on $(\Sigma^{\mathbb{N}}, S)$ associated with a Hölder continuous potential, it is well known that for the level sets

$$K_\alpha := \left\{x \in \Sigma^{\mathbb{N}} : -\lim_{n\to\infty} \frac{1}{n} \log \mu([x]_n) = \alpha\right\}$$

(where $[x]_n$ is the standard cylinder set), $\dim_{\mathrm{H}}(K_\alpha)$ is (a multiple of) the entropy of a certain Gibbs measure μ_α induced by μ and the potential [**21**]; moreover, the dimension spectrum $\Psi(\alpha) = \dim_{\mathrm{H}}(K_\alpha)$ is a strictly concave function. For the local recurrence, it was shown that for any $0 \le \alpha \le \beta \le \infty$,

$$\dim_{\mathrm{H}}\left\{x \in \Sigma^{\mathbb{N}} : \varliminf_{n\to\infty} \frac{1}{n} \log \tau_{[x]_n}(x) = \alpha,\ \varlimsup_{n\to\infty} \frac{1}{n} \log \tau_{[x]_n}(x) = \beta\right\} = \dim_{\mathrm{H}}(\Sigma^{\mathbb{N}}). \tag{2}$$

This unusual conclusion is significantly different from the local dimensions, as the dimension spectrum is a constant. The result was also extended in [**27**] to conformal repellers, and in [**19**] to self-conformal sets satisfying the strong open set condition.

In this paper, we will consider recurrence by making use of the Bowen metric on the dynamical system. Recall that the nth Bowen metric on (X, f) is defined by

$$d_n(x, y) := \max\{d(f^i(x), f^i(y)) : i = 0, 1, \ldots, n-1\} \quad \text{for all } x, y \in X. \tag{3}$$

This is a basic notion in the study of topological entropy [**6, 21, 32**]. It is also useful in connection with measure-theoretic entropy $h_\mu(f)$ [**7, 16**]; in [**7**], Brin and Katok showed that if μ is an f-ergodic measure on X, then

$$h_\mu(f) = -\lim_{\epsilon\to 0} \varliminf_{n\to\infty} \frac{1}{n} \log \mu(B_n(x, \epsilon)) \quad \mu\text{-a.e.},$$

where $B_n(x, \epsilon)$ is the nth Bowen ball of radius ϵ centered at x. We define the local (Poincaré) recurrence rates $\underline{\tau}(x)$ and $\overline{\tau}(x)$ by

$$\lim_{\epsilon\to 0} \varliminf_{n\to\infty} \frac{1}{n} \log \tau_{B_n(x,\epsilon)}(x) \quad \text{and} \quad \lim_{\epsilon\to 0} \varlimsup_{n\to\infty} \frac{1}{n} \log \tau_{B_n(x,\epsilon)}(x), \tag{4}$$

respectively. Based on the results by Ornstein and Weiss [**20**] and Katok [**16**], in [**29**] one of the authors proved that for any f-invariant probability measure μ on X with $h_\mu(f) < \infty$,

$$\overline{\tau}(x) = \underline{\tau}(x) = h_\mu(f, x) \quad \mu\text{-a.e.},$$

where $h_\mu(f, x)$ $(= -\lim_{\epsilon\to 0} \varliminf_{n\to\infty}(1/n) \log \mu(B_n(x, \epsilon))$ a.e.) is the local entropy of μ at x. (See also [**10**] for a different approach to proving the same result.) This reveals the connection between the local recurrence rates and the entropy of the measure in terms of the Bowen metric.

Our goal in this paper is to consider the spectrum structure due to the recurrence, as in (2); our set-up for the recurrence is to use Bowen balls on the dynamical systems and to consider the *topological entropy*. We assume that the map f is *positively expansive* and satisfies the *specification condition* (SPEC). The exact definitions are given in §2 and §3. Roughly speaking, a map f is positively expansive if it increases the distance between any two distinct points in an iteration; f has the SPEC if one can always find a single orbit to interpolate between different pieces of orbits, up to a pre-assigned error. There are well-known examples of dynamical systems with the SPEC—for example, the canonical symbolic space with the shift map or, more generally, the mixing subshifts of finite type [**9**]. There are also some expanding dynamical systems that are factors of subshifts of finite type [**25**] and hence have the SPEC if the systems are mixing [**30**] (e.g. the mixing conformal repellers). On the other hand, there are dynamical systems with the SPEC that cannot be embedded into a symbolic space. We shall discuss this in more detail in §3.

For $0 \le \alpha \le \beta \le \infty$, let

$$D(\alpha, \beta) = \{x \in X : \underline{\tau}(x) = \alpha,\ \overline{\tau}(x) = \beta\}$$

be *level sets for Poincaré recurrence*, and let $h_t(f, A)$ be the *topological entropy* of a subset $A \subseteq X$. Our main theorem is the following.

THEOREM 1.1. *Let (X, f) be a non-minimal dynamical system which is positively expansive and satisfies the specification condition. Then, for any $0 \le \alpha \le \beta \le \infty$ and for any non-empty open set U of X, we have*

$$h_t(f, U \cap D(\alpha, \beta)) = h_t(f, X).$$

By a non-minimal dynamical system, we mean $\overline{\{f^k(x)\}_k} \ne X$ for some $x \in X$. The theorem says that $D(\alpha, \beta)$ always has the full topological entropy $h_t(f, X)$ (as in (2)). It differs from the multifractal theory for local entropy of a measure μ. In some sense this is attributable to the fact that for a given Gibbs measure μ associated with a potential φ, the condition $h_\mu(f, x) = \alpha$ is equivalent to $\lim_{n\to\infty}(1/n) \sum_{i=0}^{n-1} \varphi(f^{i+j}(x)) = \alpha'$ (for some α') for all j; this places a restriction on the collection of possible pieces of orbits $\{f^j(x), \ldots, f^{j+n-1}(x)\}$ for all $j \in \mathbb{N}$, whereas $\underline{\tau}(x) = \alpha$ (and also $\overline{\tau}(x) = \beta$) imposes a much weaker restriction on those pieces. We will see later that such limits may only depend on the orbit of x in a sequence of positions with 'zero density' in $\mathbb{N}$ (see Propositions 5.2 and 5.4). Hence the freedom of choosing orbits in other places implies that the set $D(\alpha, \beta)$ is as large as X in the sense of topological entropy.

The proof of the theorem relies on constructing sufficiently many x that have the desired recurrence rates α and β. The main idea, inspired by [**13, 31**], is to use the Moran fractal. A Moran fractal is a Cantor-type set defined by iteration; it has a tractable structure and has been used extensively for estimating entropies and dimensions (see e.g. [**1, 11, 17, 18, 21, 22**]). In our context, we will make use of a special dynamically defined (by the Bowen metric) Moran fractal (Definition 4.1). There are two major steps in the construction; in the first step, we prove the following.

PROPOSITION 1.2. *Suppose (X, f) is a positively expansive dynamical system which satisfies the SPEC. Then there exists a dynamically defined Moran fractal $F \subset X$ such that $h_t(f, F) = h_t(f, X)$.*

This proposition is proved via Theorem 4.3 and Propositions 4.4 and 4.6. Note that in this step there is no restriction on recurrence rates for the elements in F. Our next step, actually the more elaborate step, is to modify the elements $y \in F$ to form another Moran fractal F' having approximately the same entropy, while at the same time each $x \in F'$ has the required recurrence rates. The technique is to keep adding the previous segment inductively to an appropriate position in the later part of the orbit sequence of y to ensure recurrence, and meanwhile use the SPEC to 'shadow' the sequence by another approximating orbit sequence of x.

We organize the paper as follows. In §2 we define topological entropy and positively expansive maps, and present some preliminary properties. We introduce the specification condition in §3, and set up the shadowing maps of the orbits. Proposition 1.2 is then proved in §4, and the proof of Theorem 1.1 is in §5. The insertion of the segments into the orbit sequences of F to obtain the prescribed recurrence rates is a rather complicated procedure; we divide the proof into three subsections to clarify the construction of $D(\alpha, \beta)$. Some of the technical lemmas are deferred to Appendix A.

2. *Preliminaries*

Throughout this paper we assume that X is a compact metric space and that $f : X \to X$ is a continuous map. We call such (X, f) a dynamical system and we let d_n denote the Bowen metric as defined in (3). We adopt the following definition of topological entropy [**21**].

Definition 2.1. Let (X, f) be a dynamical system. Given any subset $Z \subseteq X$, we define, for $\epsilon > 0$, $s > 0$ and $N > 0$,

$$m(Z; s, N, \epsilon) = \inf_{\Gamma} \sum_i e^{-sn_i},$$

where $\Gamma = \{B_{n_i}(x_i, \epsilon)\}_i$ is any collection of n_ith Bowen balls, with $\min_i n_i > N$, that covers Z. Let

$$m(Z; s, \epsilon) = \lim_{N\to\infty} m(Z; s, N, \epsilon) = \sup_{N>0} m(Z; s, N, \epsilon).$$

The critical value $h_t(f, Z; \epsilon)$ is defined as

$$m(Z; s, \epsilon) = \begin{cases} +\infty, & s < h_t(f, Z; \epsilon), \\ 0, & s > h_t(f, Z; \epsilon). \end{cases}$$

The topological entropy of Z is then defined as $h_t(f, Z) = \lim_{\epsilon \to 0} h_t(f, Z; \epsilon)$. In particular, if $Z = X$, then we write $h_t(f)$ for simplicity.

The quantity $m(Z; s, N, \epsilon)$ is non-decreasing with respect to N; hence the limit $m(Z; s, \epsilon)$ exists. The critical value $h_t(f, Z; \epsilon)$ is non-increasing with respect to ϵ, so the limit as $\epsilon \to 0$ exists and the notion of topological entropy is well defined.

We say two metrics d and d' on X are *uniformly (topologically) equivalent* if the identity maps are uniformly continuous with respect to the two metrics. It is easy to show that the topological entropy $h_t(f, Z)$ and the recurrence exponents $\underline{\tau}(\cdot)$, $\overline{\tau}(\cdot)$ are independent of the uniformly equivalent metric used.

Definition 2.2. Let (X, f) be a dynamical system. The map f is said to be positively expansive if there exists $\gamma > 0$ such that for any $x \neq y \in X$,

$$d(f^n(x), f^n(y)) > \gamma \quad \text{for some } n > 0. \tag{5}$$

In this case, we call (X, f) a positively expansive system. (In what follows we will fix the expansive constant γ.)

This class of maps was introduced by Williams [**33**] and Eisenberg [**12**]. It is known [**23**] that *for a compact metric space* X, $f : X \to X$ *being positively expansive is equivalent to the existence of a compatible metric* d' *on* X *and constants* $\eta > 0$, $\lambda > 1$ *such that*

$$d'(f(x), f(y)) \geq \lambda d'(x, y) \quad \text{for all } d'(x, y) < \eta. \tag{6}$$

For any finite cover $\mathcal{C}$ of X, define diam $\mathcal{C}$ to be the maximum of the diameters of the members of $\mathcal{C}$. Also let $\mathcal{C}_{-n} = \bigvee_{i=0}^{n-1} f^{-i}(\mathcal{C})$ be the family of intersections of members of $f^{-i}(\mathcal{C})$. Using the same proof as [**32**, Theorem 5.23] for expansive homeomorphisms, we have the following.

LEMMA 2.3. *Let* (X, f) *be a positively expansive system. For any finite cover* $\mathcal{C}$ *of* X *with* diam $\mathcal{C} \leq \gamma$, $\lim_{n\to\infty}$ diam $\mathcal{C}_{-n} = 0$.

LEMMA 2.4. *Let* (X, f) *be a positively expansive system. Then for any* $\delta < \gamma/4$ *and any* $\epsilon > 0$, *there exists* $N > 0$ *(which depends on* δ, ϵ*) such that*

$$d_{n+N}(x, y) \leq \delta \Rightarrow d_n(x, y) < \epsilon \quad \textit{for all } n > 0. \tag{7}$$

Proof. This lemma has appeared elsewhere in the literature, but we provide a proof here for the reader's convenience.

Choose $x_1, \ldots, x_k$ such that $\{B(x_i, \gamma/2 - 2\delta)\}_{i=1}^k$ is a cover of X. Let $\mathcal{C} = \{B(x_i, \gamma/2) : 1 \leq i \leq k\}$; this is also a finite cover of X, with diam $\mathcal{C} \leq \gamma$. Hence, by Lemma 2.3, there is $N > 0$ such that diam $\mathcal{C}_{-N} < \epsilon$. Note that $\mathcal{C}$ has Lebesgue number 2δ. (To see this, take $x \in B(x_i, \gamma/2 - 2\delta)$ for some $i \leq k$; then $d(x, y) < 2\delta$ gives $d(y, x_i) < \gamma/2$. Hence $x, y \in B(x_i, \gamma/2)$.) Now $d_{n+N}(x, y) \leq \delta$ implies $f^i x$ and $f^i y$ belong to the same element of $\mathcal{C}$ for $0 \leq i \leq n + N - 1$. Consequently $f^i(x)$ and $f^i(y)$ belong to the same element of $\mathcal{C}_{-N}$ for $0 \leq i \leq n - 1$, and $d_n(x, y) <$ diam $\mathcal{C}_{-N} < \epsilon$. □

A set $E \subseteq X$ is called (n, ϵ)-*separated* if for any $x \neq y$ in E, $d_n(x, y) \geq \epsilon$. E is called *maximal* if it attains the maximal cardinality. We use $s_n(\epsilon)$ to denote the cardinality of a maximal (n, ϵ)-separated set in X. It is known [**32**, Theorem 7.9] that *for any continuous map f on X*,

$$h_t(f) = \lim_{\epsilon\to\infty} \underline{\lim}_{n\to\infty} \frac{1}{n} \log s_n(\epsilon) = \lim_{\epsilon\to\infty} \overline{\lim}_{n\to\infty} \frac{1}{n} \log s_n(\epsilon). \tag{8}$$

THEOREM 2.5. *Let (X, f) be a positively expansive system. Then for any $\delta < \gamma/4$,*

$$h_t(f) = \lim_{n\to\infty} \frac{1}{n} \log s_n(\delta).$$

Proof. Let $0 < \epsilon < \delta$ and let N be such that (7) holds. Then we have

$$s_{n+N}(\delta) \geq s_n(\epsilon) \geq s_n(\delta),$$

and the conclusion follows from (8). □

To conclude this section, we show that the recurrence exponents $\underline{\tau}(\cdot)$, $\overline{\tau}(\cdot)$ defined in (4) can be simplified if we assume the positively expansive property on f. Moreover, we can put it in a slightly more general form which will be used later (in Proposition 5.2 and Lemma 5.7).

PROPOSITION 2.6. *Let (X, f) be a positively expansive system and let γ be an associated expansive constant. Let $\{p_n\}_{n=0}^{\infty}$ be a strictly increasing sequence such that $\lim_{n\to\infty} p_{n+1}/p_n = 1$. Then for any $\delta < \gamma/4$, we have*

$$\underline{\tau}(x) = \underline{\lim}_{n\to\infty} \frac{1}{p_n} \log \tau_{B_{p_n}(x,\delta)}(x),$$

and similarly for $\overline{\tau}(x)$.

Proof. It suffices to consider the case $p_n = n$. The general case follows by observing that for any sequence $\{a_n\}_{n=0}^{\infty}$ in $\mathbb{R}$, $\underline{\lim}_{n\to\infty} a_n/n = \underline{\lim}_{n\to\infty} a_{p_n}/p_n$ if $\lim_{n\to\infty} p_{n+1}/p_n = 1$ and $\lim_{n\to\infty} p_n = \infty$.

Let $0 < \epsilon < \delta$ and let N be such that (7) holds. Then we have $B_{n+N}(x, \delta) \subseteq B_n(x, \epsilon) \subseteq B_n(x, \delta)$, and so

$$\tau_{B_{n+N}(x,\delta)}(x) \geq \tau_{B_n(x,\epsilon)}(x) \geq \tau_{B_n(x,\delta)}(x).$$

The desired equality follows from the definition of $\underline{\tau}(\cdot)$, $\overline{\tau}(\cdot)$. □

COROLLARY 2.7. *Let $\Sigma^{\mathbb{N}}$ be the canonical infinite product space of finite symbols with the shift map S. Let $\{p_n\}_{n=0}^{\infty}$ be a strictly increasing sequence such that $\lim_{n\to\infty} p_{n+1}/p_n = 1$. Then*

$$\underline{\tau}(x) = \underline{\lim}_{n\to\infty} \frac{1}{p_n} \log \tau_{[x]_{p_n}}(x),$$

and similarly for $\overline{\tau}(x)$. (Here $[x]_n$ is the cylinder set determined by the first n coordinates of x.)

The proof follows easily upon noticing that for given $\epsilon > 0$, $B_n(x, \epsilon)$ coincides with $[x]_{n+N}$ for some N, and then applying the above proposition.

3. *The specification condition*

The notion of *specification*, introduced by Bowen [**5**], says that one can always find a single orbit to interpolate between different pieces of orbits.

Definition 3.1. Let (X, f) be a dynamical system; f is said to satisfy the condition of specification (SPEC) if for any $\epsilon > 0$, there exists an integer $m(\epsilon)$ such that for arbitrary finite intervals of integers $I_j = [a_j, a_j + n_j - 1] \cap \mathbb{N}$, $0 \le j < k$, with

$$\mathrm{dist}(I_i, I_j) > m(\epsilon) \quad \text{for all } i \neq j$$

and for any $y_0, \ldots, y_{k-1}$ in X, there exists a point $x \in X$ satisfying

$$d(f^p(x), f^{p-a_j}(y_j)) < \epsilon \quad \text{for all } p \in I_j \text{ and } 0 \le j < k. \tag{9}$$

(We follow the tradition not requiring x to be periodic, which is slightly different from the original definition in [**5**].) For $y \in X$, we let $y|_l = \{y, \ldots, f^{l-1}(y)\}$ denote the orbit of y of length ℓ. In the case where $\mathrm{dist}(I_i, I_j) = m + 1$, a point x satisfying (9) is said to ϵ*-shadow* the pieces $y_0|_{n_0}, \ldots, y_{k-1}|_{n_{k-1}}$ (*with gaps* $m(\epsilon)$). The constant $m(\epsilon)$ can be interpreted as the time for switching over from one orbit to the other, up to an approximation of ϵ; we fix $m(\epsilon)$ for future use.

There are important classes of positively expansive maps that satisfy the SPEC. The most basic example is the shift map on the full symbolic space; another example is X, a subshift space of finite type with shift map mixing on X [**9**]. A map f on X is said to be *(Ruelle) expanding* if it is open and positively expansive ([**25**, p. 143]; see also [**24**, pp. 3–4] for examples of positively expansive maps that are not open). An expanding map has Markov partitions of arbitrarily small size; hence the system is a factor of a subshift of finite type [**25**], and has the SPEC if the system is mixing. It follows that topologically mixing *conformal repellers* (see e.g. [**21**]) also satisfy the SPEC. The requirement of existence of a finite Markov partition is stronger than the SPEC. Indeed, consider the one-parameter transformations $\{T_\beta\}_{\beta>1}$ on $[0, 1]$ defined by taking $T_\beta(x) = \beta x \pmod 1$. Clearly each map is positively expansive. The following results are known for T_β [**4, 28**]: (i) the set of β such that T_β admits a finite Markov partition is at most countable; and (ii) the set of β such that T_β satisfies the SPEC has Hausdorff dimension 1 and Lebesgue measure 0.

The following *orbit specification lemma* [**30**] (stated in a slightly different way in our context) provides a method of finding a single orbit to approximate an infinite sequence $\{y_i\}_{i=0}^{\infty}$ simultaneously.

LEMMA 3.2. *Let (X, f) be a dynamical system that satisfies the SPEC. Let $\{\epsilon_k\}_{k=0}^{\infty} \searrow 0$ be a sequence of positive numbers, and let $m_k = m(\epsilon_k - \epsilon_{k+1})$. Then for any increasing sequence of disjoint finite intervals of integers $I_k = [a_k, a_k + n_k - 1]$, with $a_{k+1} - (a_k + n_k) \ge m_k$, and any sequence $\{y_k\}_{k=0}^{\infty} \in X$, there exists $x \in X$ such that*

$$d(f^j(x), f^{j-a_k}(y_k)) \le \epsilon_k \quad \textit{for } j \in I_k,\ k \in \mathbb{N}. \tag{10}$$

(In the case $a_0 = 0$ and $a_{k+1} - (a_k + n_k) = m_k$, a point x satisfying (10) is said to $\{\epsilon_k\}$-*shadow the sequence of pieces* $\{y_k|_{n_k}\}_{k=0}^{\infty}$ with gaps $\{m_k\}_{k=0}^{\infty}$.)

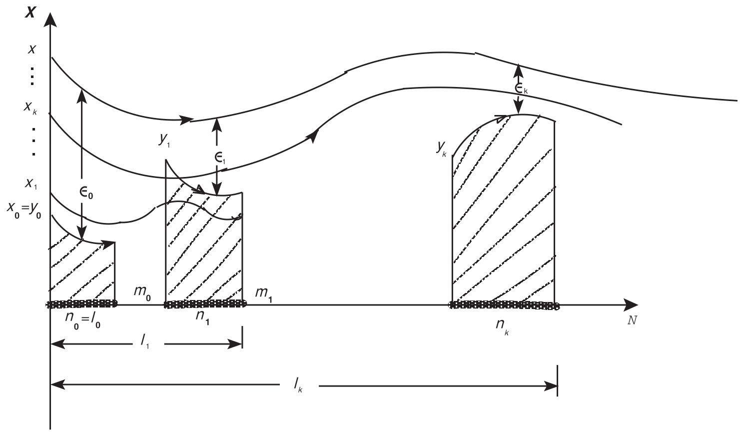

FIGURE 1. Approximation by means of the orbit specification lemma.

Proof. The construction is given in [**30**] for a special case; we include a proof for completeness. Without loss of generality, assume that $a_0 = 0$. Set $l_k = a_k + n_k$, $\epsilon'_k = \epsilon_k - \epsilon_{k+1}$ and $m_k := m(\epsilon'_k)$ for $k \in \mathbb{N}$. We define a sequence $x_k \in X$ by induction (see Figure 1). Let $x_0 = y_0$. Since $a_1 - n_0 \geq m_0$, by the definition of SPEC there exists $x_1 \in X$ such that

$$d_{l_0}(x_1, x_0) < \epsilon'_0 \quad \text{and} \quad d_{n_1}(f^{a_1}(x_1), y_1) < \epsilon'_0.$$

Suppose we have already constructed x_k. Let x_{k+1} be such that

$$d_{l_k}(x_{k+1}, x_k) < \epsilon'_k \quad \text{and} \quad d_{n_{k+1}}(f^{a_{k+1}}(x_{k+1}), y_{k+1}) < \epsilon'_k. \tag{11}$$

Such a point exists due to the SPEC. The sequence $\{x_k\}_{k=0}^{\infty}$ forms a Cauchy sequence, as

$$d(x_k, x_{k+1}) < d_{l_k}(x_k, x_{k+1}) < \epsilon'_k$$

and $\sum_{k=0}^{\infty} \epsilon'_k = \epsilon_0 < \infty$. Let x be the limit point of $\{x_k\}_{k=0}^{\infty}$. It then follows from (11) that

$$d(f^j(x_k), f^{j-a_n}(y_n)) < \sum_{i=n}^{k} \epsilon'_i < \epsilon_n \quad \text{for } j \in I_n,\ k \geq n.$$

The conclusion for x follows from taking limits with respect to k. □

It is clear that the above construction implies the next result.

COROLLARY 3.3. *Let $\{x_i\}_{i=0}^{\infty}$ and $\{l_i\}_{i=0}^{\infty}$ be as in the proof of Lemma 3.2. Then x_i satisfies (11) with respect to $\{y_k\}_{k\leq i}$, and the sequence $\{\overline{B_{l_i}}(x_i, 2\epsilon_i)\}_{i=0}^{\infty}$ decreases to x.*

We will generalize the orbit specification lemma a little, and introduce some notation to be used in the rest of the paper. Let $\mathcal{X}$ be the collection of initial segments of the orbits of $x \in X$, i.e.

$$\mathcal{X} = \{x|_n : x \in X, n \geq 1\}.$$

Let $\mathcal{X}^N$ be the N-tuples of elements in $\mathcal{X}$.

(i) Fix $\epsilon > 0$ and let $m(\epsilon)$ be as in the definition of SPEC. Suppose that $(y_0|_{n_0}, \ldots, y_{N-1}|_{n_{N-1}}) \in \mathcal{X}^N$ is given, and take x to be a point which ϵ-shadows $\{y_k\}_{k<N}$ with gaps $m(\epsilon)$ (note that there may be more than one x satisfying the requirement; we fix one of these choices). Let $l = \sum_{k=0}^{N-1} n_k + (N-1)m(\epsilon)$, and define Φ_ϵ on $\mathcal{X}^N$ by

$$\Phi_\epsilon(y_0|_{n_0}, \ldots, y_{N-1}|_{n_{N-1}}) = x|_l.$$

Intuitively, Φ_ϵ glues the orbits together with error bound ϵ and gaps of length $m(\epsilon)$.

(ii) More generally, for $\{\epsilon_k\}_{k=0}^{\infty} \searrow 0$, let $\epsilon'_k = \epsilon_k - \epsilon_{k+1}$ and let $m_k = m(\epsilon'_k)$ be as in Definition 3.1. Given $(y_0|_{n_0}, \ldots, y_{N-1}|_{n_{N-1}}, \ldots) \in \mathcal{X}^{\mathbb{N}}$, take x to be a point as in Lemma 3.2 which $\{\epsilon_k\}$-shadows the sequence $\{y_k|_{n_k}\}_{k=0}^{\infty}$ with gaps m_k. We then define

$$\Phi_{\{\epsilon_k\}}(y_0|_{n_0}, \ldots, y_{N-1}|_{n_{N-1}}, \ldots) = x.$$

In this way, $\Phi_{\{\epsilon_k\}}(\cdot)$ gives a point whose orbit shadows the given sequence of finite orbits with accuracies $\{\epsilon_k\}_{k=0}^{\infty}$ and gaps $\{m_k\}_{k=0}^{\infty}$. For later use, we write $\Phi_{\{\epsilon_k\}}(y_0|_{n_0}, \ldots, y_N|_{n_N})$ for the point x_N obtained as in Lemma 3.2 for the selection of x.

(iii) Let $\{z_j\}_{j=0}^{\infty} \in \mathcal{X}^{\mathbb{N}}$ be a sequence of orbit segments. We group the sequence into blocks as $\{z_{\mathbf{j}_k}\}_{k=0}^{\infty}$ where $z_{\mathbf{j}_0} = \{z_0, \ldots, z_{j_0}\}$, $z_{\mathbf{j}_1} = \{z_{j_0+1}, \ldots, z_{j_1}\}$ and so on. We denote the new sequence by $\mathbf{z}$ for convenience. Let $x = \Phi(\mathbf{z})$ be chosen by first setting $y_k = \Phi_\epsilon(z_{\mathbf{j}_k})$ as in (i), then $\Phi_{\{\epsilon_k\}}(y_0, y_1, \ldots)$ as in (ii). We see that the orbit of x is within $\epsilon + \epsilon_0$ distance of the $\{z_j\}_j$ in corresponding places.

In order to ease the notation for later on, we introduce two 'fictitious' symbols $\{\bowtie_k\}_{k=0}^{\infty}$ and $\smile$, and write the block sequence as

$$\mathbf{z} = z_0 \smile \cdots \smile z_{j_0} \bowtie_0 z_{j_0+1} \smile \cdots \smile z_{j_1} \bowtie_1 \cdots \tag{12}$$

where the z_j connected by $\smile$ means that we will ϵ-shadow each such block by (i), and the $\{\bowtie_k\}_k$ means that we will $\{\epsilon_k\}$-shadow the resulting sequence. We also assign a length $m(\epsilon)$ for $\smile$ and a length m_k for $\bowtie_k$ with respect to $\epsilon'_k = \epsilon_k - \epsilon_{k+1}$. For the $\{\epsilon'_k\}$ with a certain convergence rate, we can estimate the length of m_k (see Lemma 4.5).

4. *Dynamically defined Moran fractals*

In this section we introduce a dynamically defined Moran fractal, which is a Cantor-type set (cf. [**21**, Ch. 5]). The entropy of this set is trackable, and our main purpose is to use it to obtain the lower bound of the topological entropy $h_t(f, \cdot)$. The fractal we define is a generalization of the constructions used in [**13, 14**] for symbolic spaces and in [**31**] for dynamical systems satisfying the SPEC.

Let Q be a subset of $\mathbb{N}^\infty$. A word $i_0 \ldots i_n$ is called *Q-admissible* if $i_0 \ldots i_n i_{n+1} \ldots$ belongs to Q for some $i_{n+1} i_{n+2} \ldots$. Let Q_n denote the Q-admissible words of length $n+1$. We assume that $\#Q_n$ is finite for each n.

Definition 4.1. (cf. [**21**, Ch. 5]) Let (X, d) be a compact metric space, and let f be a continuous transformation on X. A dynamically defined Moran fractal F of X modeled by Q is defined by

$$F = \bigcap_{n=0}^{\infty} \bigcup_{\substack{(i_0 \ldots i_n) \\ Q\text{-admissible}}} \Delta_{i_0 \ldots i_n},$$

where the $\Delta_{i_0\ldots i_n}$ are closed subsets (called basic sets) in the nth level which satisfy:

(C1) $\Delta_{i_0\ldots i_n j} \subseteq \Delta_{i_0\ldots i_n}$ for all $i_0 \ldots i_n j$ that are Q-admissible;

(C2) $\lim_{n\to\infty} \operatorname{diam} \Delta_{i_0\ldots i_n} = 0$;

(C3) (dynamical separation condition) there exist $\delta > 0$ and $\{l_n\}_{n=0}^{\infty} \uparrow \infty$ such that, for large n, $d_{l_n}(\Delta_{i_0\ldots i_n}, \Delta_{j_0\ldots j_n}) \geq \delta$ for any $(i_0 \ldots i_n) \neq (j_0 \ldots j_n)$.

The set Q gives the number of basic sets in each step. In the following, we will show that Q, together with $\{l_n\}_{n=0}^{\infty}$, provides an estimation of the topological entropy of the limit set (Theorem 4.3). This is due to the intuitive fact that the topological entropy of a set represents the exponential growth of the number of orbit segments that can be distinguished in the Bowen metric d_n (with a certain accuracy). First we need a relationship between topological entropy and measure-theoretical entropy. Define the *lower local entropy of a probability measure* μ at $x \in X$ as

$$\underline{h}_\mu(f, x) = -\lim_{\epsilon\to 0} \overline{\lim_{n\to\infty}} \frac{1}{n} \log \mu(B_n(x, \epsilon)).$$

The upper local entropy $\overline{h}_\mu(f, x)$ is defined similarly.

PROPOSITION 4.2. *Let (X, f) be a dynamical system, and let μ be a probability measure on X. Then for any Borel subset Z of X with positive measure, we have:*

(i) *if $\underline{h}_\mu(f, x) \geq s$ for μ-almost all $x \in Z$, then $h_t(f, Z) \geq s$;*

(ii) *if $\overline{h}_\mu(f, x) \leq s$ for every $x \in Z$, then $h_t(f, Z) \leq s$.*

We omit the proof as it is analogous to the Hausdorff dimension case by using Definition 2.1 (see [**21**, Theorem 7.4], [**29**]). Let $\operatorname{diam}|_{d_n}(E)$ be the diameter of a set E in the Bowen metric d_n.

THEOREM 4.3. *Let F be a Moran fractal defined by $Q = \prod_{k=0}^{\infty}\{1, \ldots, c_k\}$, as in Definition 4.1. Assume that l_n in (C3) satisfies $\lim_{n\to\infty} l_{n+1}/l_n = 1$. Then for any open set U which has non-empty intersection with F, we have*

$$h_t(f, F \cap U) \geq \underline{\lim_{n\to\infty}} \frac{1}{l_n}(\log \#Q_n) = \underline{\lim_{n\to\infty}} \frac{1}{l_n} \log \prod_{i=0}^{n} c_i. \tag{13}$$

Moreover, if f is positively expansive, then the equality in (13) holds provided that $r = \overline{\lim}_{n\to\infty} \sup_{i_0\ldots i_n}\{\operatorname{diam}|_{d_{l_n}} \Delta_{i_0\ldots i_n}\} < \gamma/4$, where γ is an expansive constant as in (5).

Proof. For any $i_0 \ldots i_n \in Q_n$, define the probability $p(i_0 \ldots i_n) = (\#Q_n)^{-1}$. By using the Kolmogorov consistency theorem, there exists a probability measure μ on F such that

$$\mu(\Delta_{i_0\ldots i_n}) = p(i_0 \ldots i_n) > 0. \tag{14}$$

Then for any open set U that has non-empty intersection with F, we have $\mu(F \cap U) > 0$. Suppose $n > l_0$ and let $k \geq 0$ be such that $l_k < n \leq l_{k+1}$. Observe that for any $x \in F$ and $\epsilon < \delta/2$ (where δ is as in (C3)), at most one basic set $\Delta_{i_0\ldots i_k}$ intersects $B_n(x, \epsilon)$. Thus

$$\mu(B_n(x, \epsilon)) = \mu(B_n(x, \epsilon) \cap F) \leq \mu(\Delta_{i_0\ldots i_k}) = \left(\prod_{i=0}^{k} c_i\right)^{-1}.$$

This gives (13) by using (i) of Proposition 4.2.

For the second part of the theorem, we have by Lemma 2.4 that for any $0 < \epsilon < r$, there is N_0 such that

$$\operatorname{diam}|_{d_{l_k - N_0}} \Delta_{i_0 \ldots i_k} < \epsilon \quad \text{for any } i_0 \ldots i_k \in Q_k \text{ and } l_k \geq N_0.$$

Hence for $x \in F$ and $l_k < n \leq l_{k+1}$, there is some $i_0 \ldots i_{k+1}$ such that $\Delta_{i_0 \ldots i_{k+1}} \subseteq B_{n-N_0}(x, \epsilon)$. Using this and (14), we get the upper bound of $h_t(f, F \cap U)$ by applying (ii) of Proposition 4.2. □

In the following, we construct a dynamically defined Moran fractal when the system satisfies the SPEC. We will adopt the notations and construction of $\Phi(\mathbf{z})$ in part (iii) near the end of §3.

Construction M. Let $C_k = \{z_1^k, \ldots, z_{c_k}^k\}$, $k \in \mathbb{N}$, be subsets of X with cardinalities c_k. Let $Q = \prod_{k=0}^{\infty} \{1, \ldots, c_k\}^{N_k}$, where $\{N_k\}_{k=0}^{\infty}$ is an arbitrary sequence of numbers in $\mathbb{N}$. For each $\mathbf{i} \in Q$, we define a point $x_{\mathbf{i}}$ as follows. Write $\mathbf{i}$ as

$$\mathbf{i} = \mathbf{i}_0 \ldots \mathbf{i}_k \ldots \quad \text{with } |\mathbf{i}_k| = N_k.$$

Let

$$h(\mathbf{i}) = z^0_{\mathbf{i}_0(0)}|_{n_0} \smile z^0_{\mathbf{i}_0(1)}|_{n_0} \cdots \smile z^0_{\mathbf{i}_0(N_0-1)}|_{n_0} \bowtie_0 z^1_{\mathbf{i}_1(0)}|_{n_1} \smile \cdots . \tag{15}$$

be the block sequence as in (12) ($\mathbf{i}_k(j)$ denotes the jth coordinate of $\mathbf{i}_k$), and let $x_{\mathbf{i}} = \Phi(h(\mathbf{i}))$.

For the orbit $\{f^j(x_{\mathbf{i}})\}_j$, we see from the SPEC and the orbit specification lemma that for the gaps in the ϵ-shadowing corresponding to the part $\smile$, the length of the orbit is m, and for the gaps in the ϵ_k-shadowing corresponding to $\bowtie_k$, the length is m_k. For convenience of counting, we use the convention of assigning a length m for $\smile$ and m_k for $\bowtie_k$, as described in the last paragraph of §3.

PROPOSITION 4.4. *Let $F = \{x_i = \Phi(h(\mathbf{i})) : \mathbf{i} \in Q\}$. If we assume that the points in C_k are δ-separated in the Bowen metric d_{n_k}, with $\delta > 2(\epsilon + \epsilon_0)$ and $\lim_{k\to\infty}(n_{k+1} + m_{k+1})/N_k = 0$, then F is a Moran fractal modeled by Q.*

Moreover, if (X, f) is a positively expansive system and $\{n_k\}_{k=0}^{\infty}$ is strictly increasing, then for any open set U that has non-empty intersection with F,

$$\underline{\lim}_{k\to\infty} \frac{1}{n_k} \log c_k \leq h_t(f, F \cap U) \leq \overline{\lim}_{k\to\infty} \frac{1}{n_k} \log c_k.$$

Proof. Let $\mathbf{i} = \mathbf{i}_0 \mathbf{i}_1 \ldots \in Q$ with $|\mathbf{i}_i| = N_i$. For $k \in \mathbb{N}$, we let

$$y_{\mathbf{i}_k} = \Phi_\epsilon(z^k_{\mathbf{i}_k(0)}|_{n_k}, \ldots, z^k_{\mathbf{i}_k(N_k-1)}|_{n_k})$$

and then let

$$x_{\mathbf{i}_0 \cdots \mathbf{i}_k} = \Phi_{\{\epsilon_k\}}(y_{\mathbf{i}_0}, y_{\mathbf{i}_1}, \ldots, y_{\mathbf{i}_k}).$$

Define $t_k = \sum_{i=0}^{k} N_i - 1$; this is the index such that $\mathbf{i}_0 \ldots \mathbf{i}_k = i_0 \ldots i_{t_k}$. Let l_n be the length of the sequence in (15) up to the nth block; then

$$l_n = \begin{cases} \displaystyle\sum_{i=0}^{k} (n_i' + m_i) - m_k & \text{if } n = t_k \text{ for some } k, \\ \displaystyle\sum_{i=0}^{k} (n_i' + m_i) + (n - t_k)(m + n_{k+1}) - m & \text{if } t_k < n < t_{k+1}, \end{cases}$$

where $n_i' = N_i(n_i + m) - m$. For any $i_0 \dots i_n \in Q_n$, let

$$\Delta_{i_0 \dots i_n} = \begin{cases} \overline{B_{l_{t_k}}(x_{i_0 \dots i_{t_k}}, 2\epsilon_k)} & \text{if } n = t_k, \\ \bigcup_{i_{n+1}, \dots, i_{t_k}} \Delta_{i_0 \cdots i_{t_k}} & \text{if } t_{k-1} < n < t_k. \end{cases}$$

By using Lemma 3.2 and Corollary 3.3, we can easily check that F and $\{\Delta_{i_0 \dots i_n}\}$ satisfy the requirement of Definition 4.1. The last statement in the proposition follows from Theorem 4.3, by observing that $\log \# Q_n = \sum_{i=0}^{k} N_i \log c_i + (n - t_k) \log c_{k+1}$ and

$$\lim_{n \to \infty} \frac{1}{l_n} \left(\sum_{i=0}^{k} N_i n_i + (n - t_k) n_{k+1} \right) = 1,$$

where k is such that $t_k \le n < t_{k+1}$. □

As an application of the theorem, we show that for dynamical systems which are positively expansive and which satisfy the SPEC, there are plenty of Moran fractals by Construction M with full topological entropy. Moreover, we have good control of the m_k's, i.e. the lengths of the $\bowtie_k$ gaps. We need the following lemma.

LEMMA 4.5. *Let (X, f) be a positively expansive system (with d as in (6)) which satisfies the specification condition. For $0 < \epsilon < \eta$, $p > 1$ and $k \in \mathbb{N}$,*

$$m\left(\frac{\epsilon}{k^p}\right) \le p \frac{\log k}{\log \lambda} + 2 + m(\epsilon).$$

Proof. Let $N = [p \log k / \log \lambda] + 2$ (where $[a]$ is the integer part of a). By invoking the expansive property of d in (6) repeatedly, we deduce from $d_N(x, y) < \epsilon$ that

$$\lambda^{N-1} d(x, y) \le d(f^{N-1}(x), f^{N-1}(y)) < d_N(x, y) < \epsilon,$$

i.e. $d(x, y) < k^{-p}\epsilon$. For any $I_j = [a_j, a_j + n_j]$ $(j \le n)$ with $\operatorname{dist}(I_i, I_j) > m(\epsilon) + N$, we let $I_j' = [a_j, a_j + n_j + N]$ $(j \le n)$. Then any point that ϵ-shadows y_j in I_j' must $k^{-p}\epsilon$-shadow y_j in I_j. The conclusion follows from the definition of $m(\cdot)$. □

PROPOSITION 4.6. *Let (X, f) be a positively expansive system which satisfies the specification condition. There exists a subset $F \subseteq X$ by Construction M, with $m_k \le O(k)$, such that $h_t(f, F) = h_t(f)$.*

Proof. We may assume the metric d on X satisfies (6). Take $\epsilon < \frac{1}{5} \min\{\eta, \gamma/4\}$, where η is from (6) and γ is an expansive constant as in (5). Let $p > 1$. By Lemma 4.5, there is K such that

$$m\left(\frac{\epsilon}{k^p}\right) \le k + m(\epsilon) \quad \text{for } k \ge K.$$

Let $\epsilon_k = \sum_{j=k+K}^{\infty} \epsilon / j^p$. As k increases, ϵ_k decreases to zero since $\sum_{k=1}^{\infty} \epsilon / k^p < \infty$. By taking K to be large, we can assume $\epsilon_0 < \frac{1}{4}\epsilon$. Moreover, $m_k = m(\epsilon_k - \epsilon_{k+1})$ satisfies

$$m_k = m\left(\frac{\epsilon}{(k+K)^p}\right) \le k_0 + k \quad \text{where } k_0 = K + m(\epsilon).$$

Let $\{n_k\}_{k=0}^{\infty}$ and $\{N_k\}_{k=0}^{\infty}$ be strictly increasing sequences with $\lim_{k\to\infty}(n_{k+1}/N_k) = 0$. We take C_k to be any maximal (n_k, δ)-separated set with $2(\epsilon + \epsilon_0) < \delta < \gamma/4$. By Theorem 2.5,

$$\lim_{k\to\infty} \frac{1}{n_k} \log \# C_k = h_t(f).$$

Hence the set F defined by Construction M using ϵ, $\{\epsilon_k\}$, $\{n_k\}$, $\{N_k\}$ and C_k above has entropy $h_t(f)$ by Proposition 4.4. □

5. *The proof of the main theorem*

In this section we give a proof of our main result, Theorem 1.1. The technique is to construct a Moran fractal, arbitrarily close to having full entropy, such that each of its elements has the pre-assigned upper and lower exponential recurrence rates. Such a fractal is derived from the one in Proposition 4.6. The construction for obtaining the right rates is rather complicated notation-wise. We will therefore present it in three subsections, which deal with three cases of increasing complexity: the full symbolic space, the general symbolic space, and finally the dynamical system (X, f) with the SPEC. In order to make the main idea of the proof clearer, we leave some of the not-so-essential and tedious lemmas to Appendix A.

5.1. *The full symbolic space.*

Let $\{p_k\}_{k=0}^{\infty}$ and $\{q_k\}_{k=0}^{\infty}$ be two strictly increasing sequences with $p_{k+1} < q_k$. We define two other sequences $\{p'_k\}_{k=0}^{\infty}$ and $\{q'_k\}_{k=0}^{\infty}$ as follows: set $p'_{-1} = 1$, $q_{-1} = 0$, and let

$$p'_k = p_k + \sum_{q_i < p_k} (p'_i + 1), \quad q'_k = q_k + \sum_{i<k} (p'_i + 1). \tag{16}$$

It is easy to see that $p_i < q_j$ if and only if $p'_i < q'_j$.

LEMMA 5.1. *For any $0 \le \alpha \le \beta \le \infty$, there exist $\{p_k\}_{k=0}^{\infty}$ and $\{q_k\}_{k=0}^{\infty}$ with $p_{k+1} < q_k$ such that the sequences $\{p'_k\}_{k=0}^{\infty}$, $\{q'_k\}_{k=0}^{\infty}$ defined by (16) satisfy (for each k)*

$$p'_k < q_k - q_{k-1}, \tag{17}$$

$$\lim_{k\to\infty} \sum_{i=-1}^{k} p'_i / q_k = 0, \tag{18}$$

$$\lim_{k\to\infty} p'_k / p'_{k-1} = 1, \tag{19}$$

$$\underline{\lim}_{k\to\infty} (\log q'_k)/p'_k = \alpha, \quad \overline{\lim}_{k\to\infty} (\log q'_k)/p'_k = \beta. \tag{20}$$

This elementary proof is given in Appendix A. (We can take, for example, $p_k = k$ and $q_k = C[e^{\alpha p_k}]$ for some $C > 1$ when $0 < \alpha = \beta < \infty$.) Now let $\Sigma^{\mathbb{N}}$ be an infinite product of finite symbols with the shift map S. It is well known that $(\Sigma^{\mathbb{N}}, S)$ is a positively expanding dynamical system and that the canonical cylinders coincide with the Bowen balls. Moreover, $(\Sigma^{\mathbb{N}}, S)$ has the SPEC, and the words can be joined together directly. We want to construct a set of $\mathbf{y} \in \Sigma^{\mathbb{N}}$ so that the recurrence rates satisfy

$$\underline{\tau}(\mathbf{y}) = \alpha \quad \text{and} \quad \overline{\tau}(\mathbf{y}) = \beta.$$

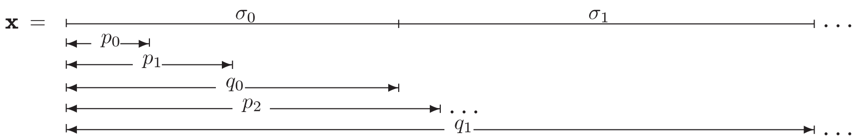

FIGURE 2. The positions of the p_k and q_k.

The idea is similar to that in [**15**], and involves adding in the initial pieces of the given sequence $\mathbf{y}$ recursively. In our construction, however, the places to insert such pieces are fixed beforehand; this is needed in the general case.

We proceed inductively with the construction . Let $\mathcal{A}$ be a *proper* subset of Σ. Let $\{p_k\}_{k=0}^{\infty}$ and $\{q_k\}_{k=0}^{\infty}$ be as in Lemma 5.1. Choose $\mathbf{x} = \mathbf{x}(0)\mathbf{x}(1)\ldots \in \mathcal{A}^{\mathbb{N}}$ (where $\mathbf{x}(i)$ denotes the ith coordinate of $\mathbf{x}$) and write it as

$$\mathbf{x} = \sigma_0\sigma_1 \ldots \sigma_k \ldots \quad \text{with } |\sigma_0 \ldots \sigma_k| = q_k$$

(see Figure 2) and also as

$$\mathbf{x} = \tau_0\tau_1 \ldots \tau_k \ldots \quad \text{with } |\tau_0 \ldots \tau_k| = p_k.$$

Our construction is to recursively insert the $\tau_0 \ldots \tau_k$ of $\mathbf{x}$ between the σ_k and σ_{k+1} as follows. Let $\{p'_k\}_{k=0}^{\infty}$ and $\{q'_k\}_{k=0}^{\infty}$ be defined as in (16). We choose $a \in \mathcal{A}$, $b \in \mathcal{A}^c$ and $w_0 = bb$ as 'markers' for the recurrence. Let $\mathbf{x}_0 = w_0\sigma_0$. Suppose we have constructed w_i and $\mathbf{x}_i$ with $|w_i| = p'_{i-1} + 1$ and $|\mathbf{x}_i| = q'_i$ for $i \leq k$. Inductively, let $\mathbf{x}_{k+1} = \mathbf{x}_k w_{k+1}\sigma_{k+1}$, where

$$w_{k+1} = \begin{cases} (\mathbf{x}_k)|_{p'_k}\, a & \text{if } p_k \leq q_i < p_{k+1} \text{ for some } i, \\ (\mathbf{x}_k)|_{p'_k}\, b & \text{otherwise.} \end{cases} \tag{21}$$

Hence $|w_{k+1}| = p'_k + 1$ and $|\mathbf{x}_{k+1}| = q'_{k+1}$. It follows that p'_k and q'_k satisfy the equations in (16).

Finally, we let

$$\Lambda(\mathbf{x}) = \bigcap_{k=0}^{\infty}[\mathbf{x}_k] = w_0\sigma_0 w_1 \ldots \sigma_k w_{k+1} \ldots$$

and let $\Lambda(\mathcal{A}^{\mathbb{N}})$ denote all the $\Lambda(x)$, with $x \in \mathcal{A}^{\mathbb{N}}$. We define the *first return time* for the shift map S on $\Sigma^{\mathbb{N}}$ with respect to $\mathbf{y}|_{p'_k}$ as

$$R_k(\mathbf{y}) = \min\{i > 0 : \; S^i(\mathbf{y})|_{p'_k} = \mathbf{y}|_{p'_k}\} \quad \text{for all } \mathbf{y} \in \Sigma^+, \; k \geq 0.$$

As an illustration, we shall use the $\mathbf{x} = \sigma_0\sigma_2 \ldots = \tau_0\tau_1 \ldots$ in Figure 2 to explain the construction and the first return time. According to the construction,

$$\Lambda(\mathbf{x}) = (bb)\sigma_0 \underset{q'_0}{\vdots} (bb\tau_0 b)\sigma_1 \underset{q'_1}{\vdots} (bb\tau_0\tau_1 a)\sigma_2 \underset{q'_2}{\vdots} (bb\sigma_0 bb\tau_0 b\sigma_1|_{p_2-q_0} b)\sigma_3 \underset{q'_3}{\vdots} \cdots$$

(the symbols within parentheses are the inserted segments). Letting $\mathbf{y} = \Lambda(\mathbf{x})$, we have

$$\mathbf{y}|_{p'_0} = (bb)\tau_0 = (bb)\sigma_0|_{p_0},$$
$$\mathbf{y}|_{p'_1} = (bb)\tau_0\tau_1 = (bb)\sigma_0|_{p_1},$$
$$\mathbf{y}|_{p'_2} = (bb)\tau_0\tau_1\tau_2|_{q_0-p_1}(bb\tau_0 b)S^{q_0-p_1}(\tau_2) = (bb)\sigma_0(bb\tau_0 b)\sigma_1|_{p_2-q_0}.$$

(Note that from Figure 2, $\sigma_0\sigma_1|_{p_2-q_0} = \tau_0\tau_1\tau_2$.) Write $R_i = R_i(\mathbf{y})$. It is clear that $R_0 \le R_1 \le R_2$ and $R_i \le q_i'$ for $i = 0, 1, 2$. Since b does not appear in σ_0, we have $R_0 = q_0'$. Next, observe that $R_1 \ne R_0$ since τ_1 does not contain b and hence $(bb\tau_0 b)$ is not a prefix of $\mathbf{y}|_{p_1'}$. Then we must have $R_1 = q_1'$, because otherwise $\mathbf{y}|_2 = bb$ would appear in $b\tau_0 b\sigma_1$ or b would be the end symbol of σ_1, contradicting the fact that τ_0, σ_1 do not contain b. We have $R_2 \ne R_1$, since the subword $(bb\tau_0 b)$ of $\mathbf{y}|_{p_2'}$ does not appear in σ_2. Notice that $\tau_0\tau_1 a\sigma_2$ is composed of symbols from $\mathcal{A}$; this forces $R_2 = q_2'$, since otherwise $\mathbf{y}|_2 = bb$ would appear in $b\tau_0\tau_1 a\sigma_2$ or b would be the suffix of σ_2.

PROPOSITION 5.2. *$R_k(\mathbf{y}) = q_k'$ for $\mathbf{y} \in \Lambda(\mathcal{A}^{\mathbb{N}})$. Consequently,*

$$\underline{\tau}(\mathbf{y}) = \alpha \quad \textit{and} \quad \overline{\tau}(\mathbf{y}) = \beta \quad \textit{for all } \mathbf{y} \in \Lambda(\mathcal{A}^{\mathbb{N}}). \tag{22}$$

Proof. We express $\mathbf{y}$ as

$$\mathbf{y} = \underbrace{w_0\sigma_0 w_1\sigma_1 w_2 \ldots \sigma_k}_{|\longleftarrow ----- q_k' ------\rightarrow|}\, w_{k+1} \ldots .$$

Note that $|w_{k+1}| = p_k' + 1$. It is clear that if $n = q_k'$, then

$$S^n(\mathbf{y})|_{p_k'} = w_{k+1}|_{p_k'} = y|_{p_k'}.$$

The proof of q_k' being smallest follows from a direct check of the construction and an inductive argument; it is omitted here, but the details can be found in [**29**] (the idea is the same as in the above illustration). We obtain (22) by applying Corollary 2.7 to $\mathbf{y}$ and using $R_k(\mathbf{y}) = q_k'$ together with Lemma 5.1. □

Proof of Theorem 1.1 in the case of the full symbolic space. The result in this case was already known in [**15**]. We provide this new proof in order to motivate the more complicated case of a dynamical system in §5.3.

Assume $0, 1 \in \Sigma$. Let $n \in \mathbb{N}$ be fixed and let $\mathcal{A} = \{0a_0a_1 \ldots a_{n-1}0 : a_j \in \Sigma\}$. The set $\mathcal{A}$ consists of words of length $n+2$ which begin and end with 0. (Here $\mathcal{A} \subset \Sigma^{n+2}$ instead of Σ, but this does not affect the construction of Λ and the recurrence behavior.) Consider $\mathcal{A}^{\mathbb{N}}$; this is a dynamically defined Moran fractal with $Q = \prod_0^\infty \{1, 2, \ldots, \#\mathcal{A}\}$ and $l_k = (n+2)k$, and the kth level sets are cylinders $[\tau]$ with $\tau \in \mathcal{A}^k$. Theorem 4.3 then yields

$$h_t(S, \mathcal{A}^{\mathbb{N}}) = \frac{1}{n+2}\log \#\mathcal{A} = \frac{n}{n+2} h_t(S),$$

where $h_t(S) = \log \#\Sigma$. Hence for any $\epsilon > 0$, $h_t(S, \mathcal{A}^{\mathbb{N}}) > h_t(S) - \epsilon$ for large n.

Next, we let

$$a = 00 \ldots 0 \in \mathcal{A} \quad \text{and} \quad b = 11 \ldots 1 \in \mathcal{A}^c$$

be words made up of 0 or 1, with length $n+2$. We change the definition of $\{p_k'\}$ and $\{q_k'\}$ accordingly: set $p_{-1}' = |b| = n+2$, $q_{-1} = 0$ and let

$$p_k' = p_k + \sum_{q_i < p_k} (p_i' + |b|), \quad q_k' = q_k + \sum_{i<k} (p_i' + |b|).$$

Given $0 \le \alpha \le \beta \le \infty$, let $\{p_k\}$ and $\{q_k\}$ satisfy Lemma 5.1. We have by Proposition 5.2 that $\Lambda(\mathcal{A}^{\mathbb{N}}) \subseteq D(\alpha, \beta)$ and hence $h_t(S, D(\alpha, \beta)) \ge h_t(S, \Lambda(\mathcal{A}^{\mathbb{N}}))$.

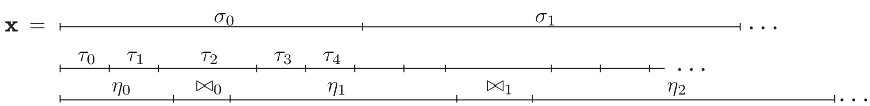

FIGURE 3. The positions of σ_k, τ_k, η_k and $\bowtie_k$.

Finally, we estimate the entropy of $\Lambda(\mathcal{A}^{\mathbb{N}})$. We see that $\Lambda(\mathcal{A}^{\mathbb{N}})$ is also a dynamically defined Moran fractal with $Q' = Q$ and $l'_k = l_k + \Sigma_{q_i < l_k} p'_i$, and the kth level sets are cylinders of length l'_k that intersect $\Lambda(\mathcal{A}^{\mathbb{N}})$. Hence

$$h_t(S, \Lambda(\mathcal{A}^{\mathbb{N}})) \geq \underline{\lim}_{k\to\infty} \frac{1}{l'_k} \log \#Q'_n \geq \underline{\lim}_{k\to\infty} \frac{l_k}{l'_k} h_t(S, \mathcal{A}^{\mathbb{N}}).$$

Since $\lim_{k\to\infty} \sum_{i\leq k} p'_i/q_k = 0$, it follows that $\underline{\lim}_{k\to\infty} l_k/l'_k = 1$ and

$$h_t(S, D(\alpha, \beta)) \geq h_t(S, \Lambda(\mathcal{A}^{\mathbb{N}})) > h_t(S) - \epsilon.$$

As $\epsilon > 0$ is arbitrary, this shows $h_t(S, D(\alpha, \beta)) = h_t(S)$ and the theorem is proved. □

5.2. *The general symbolic space.* We generalize the construction in §5.1 to some Moran fractals in the symbolic space; the procedure is similar to that in Construction M. This construction will be extended to the more elaborate case of dynamical systems in §5.3. Let $\mathcal{A}$ be a proper subset of Σ and let $\mathcal{A}_*$ denote the set of finite words admissible in $\mathcal{A}^{\mathbb{N}}$. We will consider those $\mathbf{x}$ of $\Sigma^{\mathbb{N}}$ that can be expressed as

$$\mathbf{x} = \underbrace{\eta_0 \bowtie_0 \eta_1 \bowtie_1 \ldots \eta_{k-1} \bowtie_{k-1}}_{\ell_k} \ \ldots, \tag{23}$$

where η_k belongs to $\mathcal{A}_*$ with $|\eta_k| = n'_k$ and $|\eta_0 \bowtie_0 \ldots \eta_{k-1} \bowtie_{k-1}| = \ell_k$. We also let $|\bowtie_k| = m_k$. (Note that $\bowtie_k$ may not belong to $\mathcal{A}_*$. The expression above is an extension of the one in §5.1 where $\bowtie_k$'s do not appear; moreover, the expression resembles the sequence in (15) without the $\smile$'s.) It is clear that $n'_k + \ell_k < \ell_{k+1}$.

As in the previous section, we write $\mathbf{x} \in \Sigma^{\mathbb{N}}$ as

$$\mathbf{x} = \sigma_0\sigma_1 \ldots \sigma_k \ldots \quad \text{with } |\sigma_0 \ldots \sigma_k| = q_k,$$
$$\text{and} \quad \mathbf{x} = \tau_0\tau_1 \ldots \tau_k \ldots \quad \text{with } |\tau_0 \ldots \tau_k| = p_k.$$

The specific choice of the above sequences comes from Lemma 5.3. For example, we can take $n'_k = O([e^{k^4}])$ and $m_k = k$ so that n'_k dominates the length of ℓ_{k+1}; let $\{p_k\}$ be the set of numbers (of an arithmetic progression) satisfying (29), and choose q_k to satisfy Lemma 5.3 (e.g. $q_k \approx O([e^{\alpha p_k}])$ in the case where $0 < \alpha = \beta < \infty$). Comparing with (23) (see Figure 3), we see that it could happen that $\sigma_k, \tau_k \notin \mathcal{A}_*$; hence Proposition 5.2 does not follow directly. Thus we need to modify the previous construction slightly to accommodate the extra $\bowtie_k$'s.

Take $a \in \mathcal{A}$, $b \in \mathcal{A}^c$ and $w_0 = bb$ as before. Assume σ_0 does not contain w_0 as a segment (otherwise adjust $w_0 = bb\ldots b$ to be longer). We use induction to construct sequences $\{\mathbf{x}_k\}$, $\{w_k\}$, $\{p'_k\}$, $\{q'_k\}$, $\{\sigma'_k\}$ and $\{r_j\}$ for $l_j \leq q_k$ (which is used as

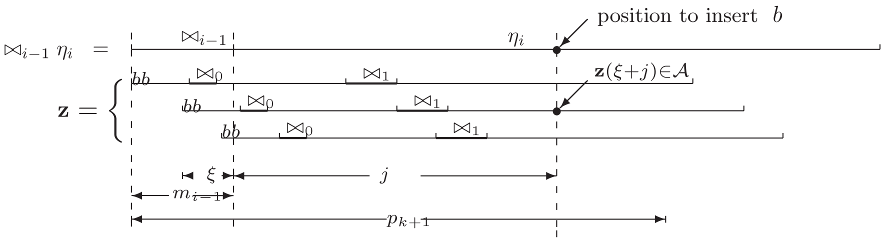

FIGURE 4. The positions in $\mathbf{z}$ and the insertion of b.

an indicator function). We take $\ell_0 < q_0$. Let $p'_{-1} = 1$ and $q_{-1} = 0$, and set $\sigma'_0 = \sigma_0$, $\mathbf{x}_0 = w_0\sigma_0$ and $r_i = \infty$ for $\ell_i \le q_0$. Suppose we have already constructed the sequences up to the kth step, such that

$$p'_k = p_k + \sum_{q_j < p_k} (p'_j + 1) + \sum_{r_j < p_k} 1, \tag{24}$$

$$q'_k = q_k + \sum_{q_j < q_k} (p'_j + 1) + \sum_{r_j < q_k} 1. \tag{25}$$

We proceed to the $(k + 1)$th step as follows. Let

$$\sigma_{k+1} = \eta_s^T \bowtie_s \eta_{s+1} \ldots \bowtie_{t-1} \eta_t^T, \tag{26}$$

where η_s^T and η_t^T are truncated segments from η_s and η_t, respectively.

(i) We will modify the above η_i in σ_{k+1} to η'_i so that the new piece $\bowtie_i \eta'_i$ does not contain $\mathbf{z} = \mathbf{x}_k|_{p'_{k+1}}$ as a segment. (Note that p'_{k+1} is defined since $p_{k+1} < q_k$ and r_i with $l_i \le q_k$ has been determined in the kth step.)

Observe that $\mathbf{z}|_2 = bb$ does not appear in the η_i; hence the only situation in which σ_{k+1} may contain $\mathbf{z}$ is as shown in Figure 4 for some $l \le m_{i-1}$. Thus, we consider η_i with $s + 1 \le i \le t$. If b appears in $S^{m_{i-1}}(\mathbf{z})$ (not counting b in the $\bowtie_k$'s in $\mathbf{z}$), we let $\eta'_i = \eta_i$ and $r_i = \infty$; otherwise, we insert b into η_i at position j to form η'_i, i.e.

$$\eta'_i = \eta_i(0) \ldots \eta_i(j - 1)b\eta_i(j) \ldots \eta_i(n'_i - 1), \tag{27}$$

where $j \in (m_{i-1}, p_{k+1})$ is such that

$$\mathbf{z}(\xi + j) \in \mathcal{A} \quad \text{for any } 1 \le \xi \le m_{i-1}. \tag{28}$$

Let $r_i = \ell_i + j$. The existence of such j follows from the choice of the sequences $\{p_i\}$, $\{q_i\}$, $\{n'_i\}$ and $\{\ell_i\}$ (Lemma A.2 in Appendix A). Intuitively, the symbols of $S^{m_{i-1}}(\mathbf{z})$ belong to $\mathcal{A}$ except for those in the $\bowtie_i$. Since the total length of such $\bowtie_i$'s is very small compared with that of $S^{m_{i-1}}(\mathbf{z})$, there must exist some j such that (28) holds.

(ii) We define $\sigma'_{k+1} = \sigma_{k+1}$ and $r_s = \infty$ if $\sigma_{k+1} = \eta_s^T$ in (26); otherwise, let

$$\sigma'_{k+1} = {\eta_s}^T \bowtie_s \eta'_{s+1} \ldots \bowtie_{t-1} {\eta'_t}^T,$$

using (i). Note that no segment of σ'_{k+1} equals $\mathbf{x}_k|_{p'_{k+1}}$. We then define

$$\mathbf{x}_{k+1} = \mathbf{x}_k w_{k+1} \sigma'_{k+1},$$

where

$$w_{k+1} = \begin{cases} (\mathbf{x}_k)|_{p'_k} a & \text{if } p_k \le q_i < p_{k+1} \text{ for some } i, \\ (\mathbf{x}_k)|_{p'_k} b & \text{otherwise} \end{cases}$$

is as in (21).

Finally, we define, for $\mathbf{x}$ in (23),

$$\Lambda'(\mathbf{x}) = \bigcap_{k=0}^{\infty} [\mathbf{x}_k] = w_0 \sigma'_0 w_1 \ldots \sigma'_k w_{k+1} \ldots .$$

As in §5.1, we need to choose $\{p_k\}$ and $\{q_k\}$ to have the desired properties.

LEMMA 5.3. *For any $0 \le \alpha \le \beta \le \infty$, there exist $\{p_k\}_{k=0}^{\infty}$ and $\{q_k\}_{k=0}^{\infty}$ with $p_{k+1} < q_k$,*

$$\ell_i < p_k < \ell_i + n'_i \quad \textit{for some } i, \tag{29}$$

$$\textit{and} \quad \ell_j + 4p_k < q_k < \ell_j + n'_j - 4p_k \quad \textit{for some } j; \tag{30}$$

the sequences $\{p'_k\}_{k=0}^{\infty}$, $\{q'_k\}_{k=0}^{\infty}$ defined by (24) and (25) satisfy $p'_k < 3p_k$ and (18)–(20).

The proof is in Appendix A, modified from Lemma 5.1.

Note that $p'_k < 3p_k$, and (30) implies (17).

PROPOSITION 5.4. *Let F be the set of $\mathbf{x}$ in (23). Then, for any $\mathbf{y} \in F' = \Lambda'(F)$, we have $R_k(\mathbf{y}) = q'_k$ and*

$$\underline{\tau}(\mathbf{y}) = \alpha, \quad \overline{\tau}(\mathbf{y}) = \beta.$$

The proof of this proposition is again deferred to Appendix A. We note that, by using Theorem 4.3, it can be shown that the set F is a Moran fractal with the same entropy as $\mathcal{A}^{\mathbb{N}}$ (which is $\log \#\mathcal{A}$) if $\lim_{n\to\infty} \sum_{l_i<n} m_i/n = 0$; the same is true for F' if, moreover, $\lim_{k\to\infty} \sum_{i\le k} p'_i/q_k = 0$.

5.3. *The dynamical system case.* Let (X, f) be a dynamical system with the SPEC, and let F be the Moran fractal as in Proposition 4.6, consisting of $x_{\mathbf{i}} = \Phi(h(\mathbf{i}))$ with

$$h(\mathbf{i}) = \eta_0^0 \bowtie_0 \eta_1^0 \bowtie_1 \ldots \bowtie_{k-1} \eta_k^0 \ldots \tag{31}$$

and $\eta_k^0 = z_{k1}|_{n_k} \smile z_{k2}|_{n_k} \cdots \smile z_{kN_k}|_{n_k}$. We shall convert each $x_{\mathbf{i}}$ into another $x'_{\mathbf{i}}$ which satisfies

$$\underline{\tau}(x'_{\mathbf{i}}) = \alpha, \quad \overline{\tau}(x'_{\mathbf{i}}) = \beta.$$

This is done by adapting the main idea in the previous section to change $h(\mathbf{i}) \in \mathcal{X}^{\mathbb{N}}$ into another sequence in $G(h(\mathbf{i})) \in \mathcal{X}^{\mathbb{N}}$ that has the specified recurrence rate; the SPEC (i.e., Φ) is then used to push the sequence down into X and to obtain F' (see item (iii) near the end of §3).

We let $\epsilon > 0$ and $\epsilon_0 < \epsilon/4$. For fixed $n \in \mathbb{N}$, let $\{n_k\}$, $\{N_k\}$ be the two sequences in the Moran set F as given in Proposition 4.6; for example, we can take

$$n_k = n(k+k_0)^4 \quad \text{and} \quad N_k = [e^{(k+k_0)^4}], \tag{32}$$

with $m_k \le k + k_0$. Let $u^*, v^* \in X$ be such that $d(u^*, \overline{O_f(v^*)}) > 8\epsilon$ ($\epsilon \ll \gamma/4$, where γ is the expansive constant in (5)).

The construction of $G(h(\mathbf{i}))$ is as follows. Recall that the symbol $\smile$ represents the ϵ-shadow and is assigned a length $m = m(\epsilon)$. First we change $h(\mathbf{i})$ slightly. For each η_k^0, define η_k by breaking each $z_{kj}|_{n_k}$ into segments of orbits of equal length n and then inserting $\smile u^* \smile$ in between, so that

$$\begin{aligned}\eta_k = z_{k1}|_n \smile u^* \smile f^n(z_{k1})|_n \smile u^* \smile \cdots \smile u^* \smile f^{n_k-n}(z_{k1})|_n \smile u^* \\ \smile \cdots z_{kN_k}|_n \smile u^* \smile f^n(z_{kN_k})|_n \smile u^* \smile \cdots \smile u^* \smile f^{n_k-n}(z_{kN_k})|_n. \quad (33)\end{aligned}$$

In this way, $\smile u^* \smile$ appears periodically in η_k. Let

$$G_1(h(\mathbf{i})) = \eta_0 \bowtie_0 \eta_1 \ldots \bowtie_{k-1} \eta_k \ldots.$$

With this adjustment, $v^*|_s$ (for $s > n + 2m$) will not appear in $\Phi_\epsilon(\eta_i)$ because $d(u^*, \overline{O_f(v^*)}) > 8\epsilon$. (By $\Phi_\epsilon(\eta_i)$ we mean any orbit of length $|\eta_i|$ that ϵ-shadows the pieces in η_i with $\smile$ omitted.) Let $n_k' = |\eta_k|$, $m_k = |\bowtie_k|$ and $\ell_k = |\eta_0 \bowtie_0 \ldots \eta_{k-1} \bowtie_{k-1}|$. Then, by (32), we have $n_k' = O(k^4 e^{k^4}) \gg m_k$.

Next, we define new elements a, b and w_0 like those in §5.2:

(i) $a = \smile u^* \smile u^* \smile \cdots u^* \smile$ with $|a| > n + 4(2m+1)$;

(ii) $b = \smile u^* \smile v^*|_{\hat{s}} \smile u^* \smile$ with $|a| = |b|$;

(iii) $w_0 = u^* \smile v^*|_{\hat{r}} \smile u^* \smile$ with $\hat{r} > \max\{k_0, |a|\}$, where k_0 is as in (32).

Let $q_{-1} = 0$, $p''_{-1} = |w_0| - |v| - m$ and

$$p_k'' = p_k + \sum_{q_j < p_k} (p_j'' + |v| + m) + \sum_{r_j' < p_k} |v|, \quad (34)$$

$$q_k'' = q_k + \sum_{q_j < q_k} (p_j'' + |v| + m) + \sum_{r_j' < q_k} |v|, \quad (35)$$

where r_j' is to be constructed as in §5.2, counting the position at which we insert the segment b. (The term m in the first sum of both (34) and (35) corresponds to the extra $\smile$ in (36) below.)

Write $\mathbf{y} = \sigma_0\sigma_1 \ldots \sigma_k \ldots$ with $|\sigma_0 \ldots \sigma_k| = q_k$. In the following, we define $\{\sigma_k'\}$ and $\{\mathbf{y}_k\}$.

(i) For

$$\sigma_{k+1} = \eta_s^T \bowtie_s \eta_{s+1} \ldots \bowtie_{t-1} \eta_t^T$$

as in (26), we modify it to σ_{k+1}' so that no segment of σ_{k+1}' contains $\mathbf{z} = \mathbf{y}_k|_{p''_{k+1}}$ (analogous to what was done in §5.2). For $s + 1 \le i \le t$, since u^* appears in every $n + 2m + 1$ symbols of η_i, if a segment of $O_f(v^*)$ with length $n + 2m + 1$ already appears in $S^{m_{i-1}}(\mathbf{z})$ (not counting the $\smile$ and $\bowtie_t$ in $\mathbf{z}$), then, by observing $d(u^*, \overline{O_f(v^*)}) > 8\epsilon$, the requirement is fulfilled; in this case we can just take $\eta_i' = \eta_i$ and set $r_i' = \infty$. Otherwise, we let

$$\eta_i' = \eta_i(0) \ldots \eta_i(j-1) b \eta_i(j) \ldots \eta_i(n_i' - 1)$$

and $r_i' = \ell_i + j$, where j is such that u^* appears in every $n + 2m + 1$ symbols of

$$\mathbf{z}(\xi + j) \ldots \mathbf{z}(\xi + j + |v| - 1) \quad \text{for each } \xi \le m_{i-1}.$$

(ii) We define $\mathbf{y}_{k+1} = \mathbf{y}_k w_{k+1} \sigma'_{k+1}$, where

$$w_{k+1} = \begin{cases} \smile (\mathbf{y}_k)|_{p''_k}\, a & \text{if } p_k \le q_i < p_{k+1} \text{ for some } i, \\ \smile (\mathbf{y}_k)|_{p''_k} b & \text{otherwise.} \end{cases} \tag{36}$$

Note that the above $\mathbf{y}_k$ is not necessarily a segment of an orbit. Hence, to complete the construction, we need to make a further refinement:

$$w'_{k+1} = \begin{cases} \smile \Phi(\mathbf{y}_k)|_{p''_k}\, a & \text{if } p_k \le q_i < p_{k+1} \text{ for some } i, \\ \smile \Phi(\mathbf{y}_k)|_{p''_k} b & \text{otherwise,} \end{cases}$$

where Φ is the shadowing function defined in (iii) at the end of §3. (We make the convention that once η'_i is determined, so are $\Phi_\epsilon(\eta'_i)$ and $\Phi_{\{\epsilon_i\}}(\eta'_0 \bowtie_0 \ldots \bowtie_{i-1} \eta'_i)$.) Now let

$$G(h(\mathbf{i})) = w_0 \sigma'_0 w'_1 \sigma'_1 w'_2 \sigma'_2 \ldots w'_k \sigma'_k \ldots, \tag{37}$$

and let $x'_{\mathbf{i}} = \Phi(G(h(\mathbf{i})))$. We collect all the $x'_{\mathbf{i}}$ with $\mathbf{i} \in Q$ in Construction M and denote this set by F'. We will see, in Lemma 5.8, that F' is a Moran fractal.

The main theorem follows from the following lemmas. As before, we need conditions on $\{p_k\}$ and $\{q_k\}$ to ensure the recurrence property. The proof is analogous to the arguments for Lemmas 5.1 and 5.3 and hence is omitted.

LEMMA 5.5. *For any $0 \le \alpha \le \beta \le \infty$, there exist $\{p_k\}_{k=0}^\infty$ and $\{q_k\}_{k=0}^\infty$, with $p_{k+1} < q_k$, which satisfy (for each k)*

$$2M < p_k - p_{k-1},$$

$$\ell_i + 2M < p_k < \ell_i + n'_i - 2M \quad \textit{for some } i, \tag{38}$$

$$\textit{and} \quad \ell_j + 4p_k < q_k < \ell_j + n'_j - 4p_k \quad \textit{for some } j, \tag{39}$$

where $M = n + 2m + 1$; the sequences $\{p''_k\}_{k=0}^\infty$, $\{q''_k\}_{k=0}^\infty$ defined by (34) and (35) satisfy $p'_k < 3p_k$ and (18)–(20).

The selection of p_k, q_k is after the construction of $G_1(\cdot)$. We may take, for example, $p_k = O(3kM)$ so that each p_k is the end position of a piece of length n in (33), and then omit some items and relabel the sequence so that (38) holds. Take q_k (which depends on α, β) as in the proof of Lemma 5.3 (see Appendix A); then adjust the sequence to satisfy (39). We may assume that each q_k avoids the places of $\smile$. Hence the construction will not disturb the $\smile$ in the process of cutting and inserting the recurrence pieces.

We now consider the recurrence behavior of the $G(h(\mathbf{i}))$ in (37). Let S be the (left) shift map on $X^{\mathbb{N}}$. For $x, x' \in X^{\mathbb{N}}$, we let

$$\mathbf{d}_n(x, x') := \max\{d(x(i), x'(i)) : i = 0, 1, \ldots, n-1\},$$

where d is the metric on X. Note that $\mathbf{d}_n$ is a semi-metric on $X^{\mathbb{N}}$.

LEMMA 5.6. *Let $z = G(h(\mathbf{i}))$ as in (37). Then for any $k \in \mathbb{N}$, we have*

$$\mathbf{d}_{p''_k}(z, S^i(z)) > 5\epsilon \quad \textit{for all } m < i < q''_k,$$

$$\mathbf{d}_{p''_k}(z, S^i(z)) \le \epsilon + \epsilon_0 \quad \textit{for } i = q''_k + m.$$

Proof. Let $z' = w_0\sigma_0' w_1\sigma_1' w_2\sigma_2' \ldots w_k\sigma_k' \ldots \in X^{\mathbb{N}}$ be the sequence corresponding to z. By the same argument as in Proposition 5.4, we obtain that for any $m < i < q_k''$, there exists some $j < p_k''$ such that

$$\begin{aligned} \text{either} \;\; & z'(j) = u^*, \quad (S^i(z'))\,(j) \in O_f(v^*), \\ \text{or} \;\; & z'(j) \in O_f(v^*), \quad (S^i(z'))\,(j) = u^*. \end{aligned}$$

Recall that $d(u^*, \overline{O_f(v^*)}) > 8\epsilon$. This implies

$$\mathbf{d}_{p_k''}(z', S^i(z')) > 8\epsilon \quad \text{for all } m < i < q_k''. \tag{40}$$

From the definition of map Φ, we see that

$$\mathbf{d}_{|w_k|}(w_k, w_k') \le \epsilon + \epsilon_0.$$

(Here we assume that w_k' also $(\epsilon + \epsilon_0)$-shadows $\smile$ and $\bowtie_k$ of w_k in the corresponding positions. This will not affect the following argument as (40) is deduced from $d(u^*, \overline{O_f(v^*)}) > 8\epsilon$.) Hence

$$\mathbf{d}_l(z, z') \le \epsilon + \epsilon_0 \quad \text{for any } l \in \mathbb{N} \tag{41}$$

and the first statement of the lemma follows by (40). Equation (41) implies the second statement by observing that $\mathbf{d}_{p_k''}(z', S^i(z')) = 0$ for $i = q_k'' + m$. □

As a consequence of Lemma 5.6, we have the following.

LEMMA 5.7. *F' is a subset of $D(\alpha, \beta)$.*

Proof. Let $x_{\mathbf{i}}' \in F'$. It cannot be a point with period less than $m + 1$, owing to the insertion of u^* and $v^*|_{\hat{r}}, v^*|_{\hat{s}}$. Thus there exists $0 < \epsilon' < \epsilon$ such that

$$\tau_{B(x_{\mathbf{i}}', \epsilon')}(x_{\mathbf{i}}') = \min\{j > 0 : \; f^j(x_{\mathbf{i}}') \in B(x_{\mathbf{i}}', \epsilon')\} > m.$$

Consider $\tau_{B(x_{\mathbf{i}}', \epsilon')}(x_{\mathbf{i}}')$. By the definition of Φ, the orbit of $x_{\mathbf{i}}'$ is within $\epsilon + \epsilon_0$ $(< 5\epsilon/4)$ distance of the pieces (excluding the $\smile$ and $\bowtie_k$) of $G(h(\mathbf{i}))$ in the corresponding places. Hence we have by Lemma 5.6 that

$$\tau_{B_{p_k''}(x_{\mathbf{i}}', \epsilon')}(x_{\mathbf{i}}') \ge q_k'' \quad \text{for all } k \in \mathbb{N}.$$

Similarly, we obtain from Lemma 5.6 that

$$\tau_{B_{p_k''}(x_{\mathbf{i}}', 4\epsilon)}(x_{\mathbf{i}}') \le q_k'' + m \quad \text{for all } k \in \mathbb{N}.$$

Combining the two estimates, we conclude by Lemma 5.5 and Proposition 2.6 that

$$\underline{\tau}(x_{\mathbf{i}}') = \alpha \quad \text{and} \quad \overline{\tau}(x_{\mathbf{i}}') = \beta,$$

which proves the lemma. □

Our last step is to estimate the topological entropy of F'.

LEMMA 5.8. *F' is a dynamically defined Moran fractal and*

$$h_t(f, F') \ge \frac{n}{n + 2m + 1} h_t(f, F).$$

Proof. For the sth element of $\{z_{ij}\}_{ij}$ in $h(\mathbf{i})$ (counting lexically) in (31), we let l_s, l'_s and l''_s be its positions in $h(\mathbf{i})$, $G_1(h(\mathbf{i}))$ and $G_1(h(\mathbf{i}))$, respectively. Then

$$l'_s = \begin{cases} \sum_{i=0}^{k}(n'_i + m_i) - m_k & \text{if } s = t_k \text{ for some } k, \\ \sum_{i=0}^{k}(n'_i + m_i) + c\left(\frac{n_{k+1}(s - t_k)}{n} - 1\right) & \text{if } t_k < s < t_{k+1}, \end{cases}$$

where

$$t_k = \sum_{i=0}^{k} N_i - 1, \quad n'_i = c\left(\frac{n_i N_i}{n} - 1\right) \quad \text{and} \quad c = n + 2m + 1;$$

$$l''_s = l'_s + \sum_{q_k < l'_s} p''_k \quad \text{for all } s \in \mathbb{N}.$$

It is clear that $l_s < l'_s \le (n + 2m + 1/n)l_s$. Moreover, since $\lim_{k\to\infty} \sum_{i=1}^{k} p''_i/q_k = 0$, we have $\lim_{s\to\infty} l'_s/l''_s = 1$. It follows that $\underline{\lim}_{s\to\infty} l_s/l''_s \ge n/(n + 2m + 1)$.

Next, take $\mathbf{i} = \mathbf{i}_0\mathbf{i}_1 \ldots \in Q$ with $|\mathbf{i}_i| = N_i$, and write

$$G(h(\mathbf{i})) := \widetilde{\eta}_0 \bowtie_0 \widetilde{\eta}_1 \bowtie_1 \ldots \bowtie_{k-1} \widetilde{\eta}_k \ldots,$$

where each $\widetilde{\eta}_i$ is a tuple of elements of $\mathcal{X}$ connected by $\smile$. For $k \in \mathbb{N}$, we let $y_{\mathbf{i}_k} = \Phi_\epsilon(\widetilde{\eta}_k)$ and

$$x'_{\mathbf{i}_0\ldots\mathbf{i}_k} = \Phi_{\{\epsilon_i\}}(y_{\mathbf{i}_0}, \ldots, y_{\mathbf{i}_k}).$$

For any $i_0 \ldots i_s \in Q_s$, we let

$$\Delta_{i_0\ldots i_s} = \begin{cases} \overline{B_{l''_{t_k}}(x'_{i_0\ldots i_{t_k}}, 2\epsilon_k)} & \text{if } s = t_k, \\ \bigcup_{i_{s+1},\ldots,i_{t_k}} \Delta_{i_0\ldots i_{t_k}} & \text{if } t_{k-1} < s < t_k. \end{cases}$$

Then it is easy to check that F' is a dynamically defined Moran fractal modeled by Q with the level sets $\Delta_{i_0\ldots i_s}$ defined above. Moreover, the set F' has entropy

$$h_t(f, F') = \underline{\lim}_{s\to\infty} \frac{1}{l''_s} \log \#Q_s.$$

Consequently, we have

$$h_t(f, F') \ge \underline{\lim}_{s\to\infty} \frac{l_s}{l''_s} h_t(f, F) \ge \frac{n}{n + 2m + 1} h_t(f, F).$$

□

Proof of Theorem 1.1. Let $0 \le \alpha \le \beta \le \infty$. Let F be a Moran fractal as in Proposition 4.6. Choose $\{p_k\}_k$, $\{q_k\}_k$ as in Lemma 5.5, and let F' be the reconstruction of F in the above. We have by Lemma 5.7 that

$$h_t(f, D(\alpha, \beta)) \ge h_t(f, F').$$

For any $\vartheta > 0$, by taking n to be large compared with m, Lemma 5.8 implies

$$h_t(f, F') \geq h_t(f, F) - \vartheta = h_t(f) - \vartheta.$$

It follows that $h_t(f, D(\alpha, \beta)) \geq h_t(f) - \vartheta$. This proves $h_t(f, D(\alpha, \beta)) = h_t(f)$, since $\vartheta > 0$ is arbitrary.

For any non-empty open set $U \subseteq X$, we choose z^* with $B_t(z^*, 5\epsilon) \subseteq U$ for some $t \in \mathbb{N}$. If we replace w_0 in the construction of F' by

$$z^*|_t \smile u^* \smile v^*|_{\hat{r}} \smile u^* \smile \quad \text{for some large } \hat{r},$$

then the new Moran fractal denoted by F'' will be a subset of $U \cap D(\alpha, \beta)$. The conclusion that $h_t(f, U \cap D(\alpha, \beta)) = h_t(f)$ follows from the same argument as for $D(\alpha, \beta)$ in the previous paragraph. □

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A. *Appendix*

We begin with the following lemma.

LEMMA A.1. *Let $0 \leq \alpha \leq \beta \leq \infty$ and let $C > 1$. For any strictly increasing sequence $\{p_k\}_{k=0}^{\infty}$ with $\lim_{k\to\infty} p_k/p_{k-1} = 1$. there exists $\{q_k\}_{k=0}^{\infty}$ such that*

$$C \cdot p_k < q_k - q_{k-1}, \tag{A.1}$$

$$e^{\sqrt{p_k}} < q_k < e^{p_k^2} \quad \textit{for large } k, \tag{A.2}$$

$$\textit{and} \quad \varliminf_{k\to\infty} (\log q_k)/p_k = \alpha, \quad \varlimsup_{k\to\infty} (\log q_k)/p_k = \beta. \tag{A.3}$$

The proof is the same as in [**15**, p. 84] for the case $p_k = k$. Basically, the condition $0 \leq \alpha \leq \beta \leq \infty$ is divided into eight cases according to whether each relation is $<$ or $=$. For example, when $0 < \alpha = \beta < \infty$, one can take $q_k = [e^{\alpha k}]$ (where $[a]$ denotes the integrer part of a); if $0 < \alpha < \beta < \infty$, one can take $q_k = \sum_{i=1}^{k} u_i$, where

$$u_k = \begin{cases} [e^{\alpha k}] & \text{if } k \in \mathcal{K}, \\ [e^{\beta k}] & \text{otherwise} \end{cases}$$

and $\mathcal{K} = \{k \in \mathbb{N} : \sum_{j=1}^{2i-1} 2^{4^j} \leq k < \sum_{j=1}^{2i} 2^{4^j}$ for some integer $i > 0\}$.

Proof of Lemma 5.1. Let $\{p_k\}_{k=0}^{\infty}$ and $\{q_k\}_{k=0}^{\infty}$ be as in Lemma A.1. To prove the lemma, it suffices to show that $\lim_{k\to\infty} p'_k/p_k = 1$ and $\lim_{k\to\infty} q'_k/q_k = 1$. Take $p_0 > 3$. We first claim that $p'_k \leq 3p_k$. This will be proved by induction on the q_i. It is clear that the statement is true for $p_k \leq q_1$, since $p'_k \leq p_k + p_0 + 5$ for such k. Now assume that the statement holds for $p_k \leq q_i$.

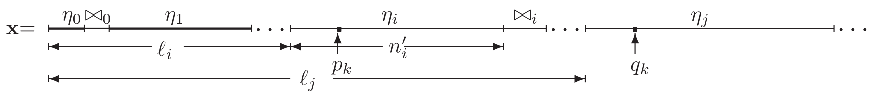

FIGURE A.1. The position of p_k, q_k in Lemma 5.3.

For any k such that $q_i < p_k \le q_{i+1}$ $(1 \le i)$, we let j be such that $p_j \le q_{i-1} < p_{j+1}$. Then

$$\begin{aligned} p'_k &= p'_j + p_k - p_j + p'_{i-1} + 1 + p'_i + 1 \\ &\le 2p_j + p_k + 3p_{i-1} + 3p_i + 2 \quad \text{by } (A.1) \\ &\le 2p_j + p_k + 2(q_i - q_{i-1}) \\ &\le 2p_j + p_k + 2(p_k - p_j) = 3p_k. \end{aligned}$$

This proves the claim. We may assume (A.2) holds for all $k \ge 0$ by taking p_0 large. Then, by using $p'_j \le 3p_j$ and $e^{\sqrt{p_j}} \le q_j$, we have

$$\sum_{i=-1}^{k} p'_i \le \sum_{i=-1}^{k} 3p_i \le 9p_k^2 \le 9 \log q_k$$

and hence $\lim_{k\to\infty} \sum_{i=-1}^{k} p'_i/q_k = 0$. Since $e^{\sqrt{p_j}} \le q_j$, $q_j < p_k$ implies $p_j \le (\log p_k)^2$. Therefore, by using $p'_j \le 3p_j$, we have

$$\begin{aligned} p_k \le p'_k &= p_k + \sum_{q_j < p_k} (p'_j + 1) \\ &\le p_k + \sum_{p_j \le (\log p_k)^2} (3p_j + 1) \\ &\le p_k + 9(\log p_k)^4, \end{aligned}$$

and so $\lim_{k\to\infty} p'_k/p_k = 1$. Similarly, we have

$$q_k \le q'_k = q_k + \sum_{i=-1}^{k-1} (p'_j + 1) \le q_k + 9p_k^2,$$

which implies $\lim_{k\to\infty} q'_k/q_k = 1$. Thus (20) follows. □

Proof of Lemma 5.3. Let $\{p_k\}_{k=0}^{\infty}$ be a strictly increasing sequence such that $\lim_{k\to\infty} p_k/p_{k-1} = 1$ and $p_k \le e^{\sqrt{p_{k-1}}}$. Take $n'_k = O([e^{k^4}])$ and $m_k = k$, so that $n'_k \gg m_k$. We may assume (29) for the sequence, since $\lim_{k\to\infty} m_k/\ell_{k+1} = 0$. Next, let $\{q_k\}_{k=0}^{\infty}$ be as in Lemma A.1 with $C = 15$. (See Figure A.1 for the position of p_k, q_k.) By taking p_0 to be large, we assume $e^{\sqrt{p_k}} < q_k < e^{p_k^2}$. Increase q_k if necessary so that (30) holds; then we find $\{q_k\}_{k=0}^{\infty}$ which satisfies (A.3), (30) and is such that

$$\begin{aligned} 6p_k &< q_k - q_{k-1}, \qquad\qquad (A.4) \\ e^{\sqrt{p_k}} &< q_k < ce^{p_k^2} \quad \text{for some } c > 0. \end{aligned}$$

Let $\{p'_k\}_{k=0}^{\infty}$, $\{q'_k\}_{k=0}^{\infty}$ be defined by (24), (25). Recall that r_j is defined to be infinity or at the beginning position in each η_j. Let $s_k = \max\{i : \ell_i < p_k\}$ and $t_k = \max\{i : \ell_i < q_k\}$. We have $p'_0 < p_0 + m_{s_0-1}$ (where m_{s_0-1} is the length of $w_0 = bb \ldots b$ if σ_0 contains bb) and

$$p'_k \le p_k + \sum_{q_j < p_k} (p'_j + 1) + s_k - s_0,$$

$$q'_k \le q_k + \sum_{q_j < q_k} (p'_j + 1) + t_k.$$

Hence, as in Lemma 5.1, we can show $p'_k \le 3p_k$ by using (A.4) and by observing that $s_j - s_i \le p_j - p_i$ for $i < j$. The arguments for showing $\lim_{k\to\infty} \sum_{i=-1}^{k} p'_i/q_k = 0$, $\lim_{k\to\infty} p'_k/p_k = 1$ and $\lim_{k\to\infty} q'_k/q_k = 1$ are also the same as for Lemma 5.1. Then (18)–(20) follow. □

LEMMA A.2. *For any i with $\ell_i \le q_{k+1}$, we have $m_{i-1} < \sqrt{p_{k+1}}$. The expression (28) holds for some j.*

Proof. Note that $q_k < e^{p_k^2}$. If p_0 is large, then for any i with $\ell_i \le q_{k+1}$ we have $m_{i-1} < \sqrt{p_{k+1}}$, and for any t with $\ell_t < p_{k+1}$ we have $m_t < (p_{k+1})^{1/4}$. The total length of $\bowtie_t$'s contained in $\mathbf{z}$ is less than $\Sigma_{\ell_t < p_{k+1}} m_t$. Hence

$$p_{k+1} - 2m_{i-1} - \sum_{\ell_t < p_{k+1}} (m_t + m_{i-1}) \ge p_{k+1} - 3\sqrt{p_{k+1}} \ge 1.$$

Then there exists some j with $m_{i-1} < j < p_{k+1} - m_{i-1}$ such that $\mathbf{z}(j+l)$ does not belong to any $\bowtie_t$ of $\mathbf{z}$. Since the segment $S^{m_{i-1}}(\mathbf{z})$ (not counting $\bowtie_t$'s in $\mathbf{z}$) does not contain the letter b, we must have that j satisfies (28). □

Proof of Proposition 5.4. Take sequences $\{p_k\}_{k=0}^{\infty}$ and $\{q_k\}_{k=0}^{\infty}$ as in Lemma 5.3. We can express $\mathbf{y} \in \Lambda'(F)$ as

$$\mathbf{y} = \underbrace{w_0\sigma'_0 w_1 \cdots w_j \overbrace{\eta^T_{t_{j-1}} \bowtie_{t_{j-1}} \eta'_{t_{j-1}+1} \cdots \bowtie_{i-1} \eta'_i \cdots \bowtie_{t_j-1} \eta'^T_{t_j}}^{\sigma'_j} w_{j+1} \cdots \sigma'_k}_{q'_k}\; w_{k+1} \cdots .$$

(Here t_j is such that q_j belongs to the piece η_{t_j}.) We will show that $R_k(\mathbf{y}) = q'_k$. First note that $|w_{k+1}| = p'_k + 1$. It is clear that if $n = q'_k$, then

$$S^n(\mathbf{y})|_{p'_k} = w_{k+1}|_{p'_k} = \mathbf{y}|_{p'_k}.$$

On the other hand, for $n < q'_k$ we have $S^n(\mathbf{y})|_{p'_k} \ne \mathbf{y}|_{p'_k}$ by the following observations.

(i) $\mathbf{y}(n) = \sigma'_j(r)$ for $j \le k$ and $r < |\sigma'_j|$ implies that $S^n(\mathbf{y})(0) = \sigma_j(r) \in \mathcal{A} \ne y(0) \in \mathcal{A}^c$ if $t_{j-1} = t_j$, or (in the case where $t_{j-1} < t_j$) that $\mathbf{y}|_{p'_j} = S^m(\bowtie_{i-1} \eta'_i)|_{p'_j}$ (or $S^m(\bowtie_{t_j-1} \eta'^T_{t_j})|_{p'_j}$) for some $t_{j-1} < i < t_j$ and $m < m_{i-1}$. This is because the first two letters of $\mathbf{y}|_{p'_j}$ are bb, but this does not appear in η'_i, $\eta^T_{t_{j-1}}$ or $\eta'^T_{t_j}$ (note that $\eta'^T_{t_j}$ does not end with b). However, by construction of η'_i, we have either that $\eta'_i = \eta_i \in \mathcal{A}_*$ and $S^{m_{i-1}}(\mathbf{y}|_{p'_j})$ contains the letter $b \in \mathcal{A}^c$, or that

$$\eta'_i(t) = b \in \mathcal{A}^c \ne \mathbf{y}(t + m_{i-1} - m) \in \mathcal{A} \quad \text{where } t = r_i - \ell_i,$$

by (27) and (28). Thus the desired contradiction is obtained.

(ii) $\mathbf{y}(n) = w_j(0)$ for $j \le k$ implies that $S^{n+n'}(\mathbf{y})|_l = S^{n'}(\mathbf{y})|_l \ne S^{n'}(w_j\sigma'_j)|_l$ for $n' = p'_{j-1}$ and $l = p'_j - p'_{j-1}$. This is because if $l = p_j - p_{j-1}$, then $\mathbf{x}_j(p'_{j-1}) \in \mathcal{A} \ne w_j(p'_{j-1}) = b \in \mathcal{A}^c$; otherwise, $\mathbf{x}_j(p'_{j-1}) \ldots \mathbf{x}_j(p'_j - 1) \notin \mathcal{A}_* \ne (a\sigma'_j)|_l \in \mathcal{A}_*$.

(iii) $\mathbf{y}(n) = w_j(p'_{j-1})$ for $j \le k$ implies that $S^n(\mathbf{y})|_2 = bb \ne w_j(p'_{j-1})\sigma_j(0)$, since $\sigma_j(0) \in \mathcal{A}$.

(iv) $\mathbf{y}(n) = w_j(i)$ for $i = 1$ or $p'_{j-1} - 1$ and $j \le k$ implies that $S^n(\mathbf{y})(1) = w_j(2) \in \mathcal{A} \ne \mathbf{y}(1) = b \in \mathcal{A}^c$, or $S^n(\mathbf{y})(0) = w_j(i) \in \mathcal{A} \ne \mathbf{y}(0) \in \mathcal{A}^c$.

(v) $\mathbf{y}(n) = w_j(i)$ for $p_{j-1} \le q_0$, $1 < i < p'_{j-1} - 1$ implies that $S^n(\mathbf{y})(0) = w_j(i) \in \mathcal{A} \ne \mathbf{y}(0) \in \mathcal{A}^c$.

(vi) $\mathbf{y}(n) = w_j(i)$ for $q_{r-1} < p_{j-1} \le q_r$, $1 < i < p'_{j-1} - 1$ and $r < j \le k$ implies that

$$\mathbf{y}|_{p'_{j-1}} = S^n(\mathbf{y})|_{p'_{j-1}} = S^i(w_j\sigma_j)|_{p'_{j-1}}. \tag{A.5}$$

If $i < q'_0$, then $\mathbf{y}(0) \in \mathcal{A}^c \ne w_j(i) \in \mathcal{A}$; and if $i > p'_{j-1} - q'_{r-1}$, then $\mathbf{y}(q'_{r-1}) \in \mathcal{A}^c \ne \sigma_j(s) \in \mathcal{A}$ for $s = i + q'_{r-1} - p'_{j-1} + 1$. Consequently, (A.5) holds for $q'_0 \le i \le p'_{j-1} - q'_{r-1}$. Note that $\mathbf{y}|_{p'_{j-1}} = w_j|_{p'_{j-1}}$. It follows that

$$\mathbf{y}|_l = S^i(\mathbf{y})|_l \quad \text{for } l = p'_{j-1} - i.$$

This implies $\mathbf{y}|_{p'_r} = S^m(\mathbf{y})|_{p'_r}$ for some $m < q'_r$. Repeat the above argument for $\mathbf{y}|_{p'_r}$. By virtue of (v), we must arrive at a contradiction in a finite number of steps.

Since $\lim_{k\to\infty} p'_k/p'_{k-1} = 1$ by (19), the proof is complete upon using $R_k(\mathbf{y}) = q'_k$ and Proposition 2.6. □

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