The Journal of Fourier Analysis and Applications
Volume 7, Issue 2, 2001

L_p Solutions of Refinement Equations

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Communicated by Chris Heil

ABSTRACT. In the recent characterizations of the L_p solution of the refinement equation in terms of the "p-norm joint spectral radius," there are problems in choosing the initial function for iteration [3, 23], or in addition, requiring stability of the refinable function [13, 17]. In this article we overcome these difficulties and give a more complete characterization of this nature. The criterion is constructive and can be implemented. It can be used to describe the regularity of the solution without assuming stability. This has significant advantages over the previous work. The corresponding results for vector refinement equations are also discussed.

1. Introduction

A refinement equation is a functional equation of the form

$$\phi(x) = \sum\nolimits_{\alpha\in\mathbb{Z}} a(\alpha)\phi(2x-\alpha)\,, \tag{1.1}$$

where a is a finitely supported sequence on $\mathbb{Z}$, called the refinement mask. The study of refinement equations plays an important role in wavelet analysis.

Throughout this article we assume that $\sum_{\alpha\in\mathbb{Z}} a(\alpha) = 2$. Under this condition, it is well known (see [1] and [4]) that the refinement equation (1.1) has a unique compactly supported distributional solution ϕ subject to $\hat{\phi}(0) = 1$, where $\hat{\phi}$ denotes the Fourier transform of ϕ. This solution is called the normalized solution of (1.1).

The question of existence of continuous solutions and L_p solutions ($1 \le p < \infty$) of refinement equations has attracted much attention from mathematicians in approximation theory and wavelet analysis. Micchelli and Prautzsch [24] first provided necessary and sufficient conditions for the existence of continuous solutions of refinement equations. In [4], Daubechies and Lagarias used the joint spectral radius of two matrices in their study of refinement equations. Colella and Heil [3] characterized the existence of continuous solutions in terms of the joint spectral radius of two matrices associated with the mask.

Math Subject Classifications. 42C40, 41A25, 39B12.
Keywords and Phrases. refinement equations, refinable functions, joint spectral radii, wavelets.
Acknowledgements and Notes. First author; research was supported in part by NSERC Canada under Grant # OGP 121336; second author; research was supported in part by the Research Grants Council of Hong Kong, and third author; research was supported in part by the Research Grants Council of Hong Kong.

ISSN 1069-5869

In [23], Lau and Wang gave a characterization of the existence of compactly supported L_p solutions ($1 \leq p < \infty$) of refinement equations. Suppose the mask a is supported on $[0, N]$, where N is a positive integer. Let A_0 and A_1 be the matrices given by

$$A_0 := \big(a(2i-j)\big)_{0 \leq i,j \leq N-1} \quad \text{and} \quad A_1 := \big(a(2i-j+1)\big)_{0 \leq i,j \leq N-1} .$$

For $J = (j_1, \ldots, j_k)$, where $j_i = 0$ or 1, $i = 1, 2, \ldots, k$, we define

$$A_J := A_{j_1} \ldots A_{j_k} .$$

The length of J is denoted by $|J|$. The result of Lau and Wang [23, Theorem 1.3] can be stated as follows: The refinement equation (1.1) has a nonzero compactly supported L_p-solution ($1 \leq p < \infty$) if and only if there exists a 2-eigenvector v of $(A_0 + A_1)$ such that

$$\frac{1}{2^l} \sum_{|J|=l} \|A_J (A_0 - I) v\|^p \to 0 \quad \text{as } l \to \infty . \tag{1.2}$$

It is possible that 2 is a multiple eigenvalue of $A_0 + A_1$. The following is such an example. Let a be the mask given by its symbol

$$\tilde{a}(z) := \sum\nolimits_{j \in \mathbb{Z}} a(j) z^j = 1 - z + z^2 + z^3 - z^4 + z^5 .$$

In this case, $N = 5$ and

$$A_0 + A_1 = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 2 & 0 & 0 & 1 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} .$$

Clearly, 2 is a double eigenvalue of $A_0 + A_1$. The corresponding eigenspace has dimension 2 and a basis consisting of $(0, 0, 1, 0, 0)^T$ and $(1, 2, 0, 2, 1)^T$. A 2-eigenvector v of $(A_0 + A_1)$ has the form

$$v = \lambda v_1 + \mu v_2 ,$$

where $\lambda, \mu \in \mathbb{C}$. In this case, it is impossible to check (1.2) for all 2-eigenvectors of $(A_0 + A_1)$. Therefore, the above theorem is not applicable to this example. The same phenomenon occurs in the work of Micchelli and Prautzsch [24]. This difficulty was also recognized by Colella and Heil [3].

The purpose of this article is to give a complete characterization for the refinement equation (1.1) to have nontrivial continuous solutions or L_p solutions ($1 \leq p < \infty$) strictly in terms of the mask. It solves the problem mentioned in the preceding paragraph. We will also provide a complete characterization for the regularity of the solutions in terms of the mask. All our results are obtained without any assumption on the stability of ϕ.

The main tool in our study is the joint spectral radius of a finite collection of matrices. The uniform joint spectral radius was introduced by Rota and Strang in [25]. The mean spectral radius was introduced by Wang [29] in his study of L_1 refinable functions. The concept of the p-norm joint spectral radius was defined by Jia in [13] and was used implicitly by Lau and Wang [23] independently. Let us recall from [13] the definition of the p-norm joint spectral radius.

Let V be a *finite-dimensional* vector space equipped with a vector norm $\|\cdot\|$. For a linear operator A on V, define $\|A\| := \max_{\|v\|=1}\big\{\|Av\|\big\}$. Let $\mathcal{A}$ be a finite collection of linear operators on V. For a positive integer n we denote by $\mathcal{A}^n$ the nth Cartesian power of $\mathcal{A}$:

$$\mathcal{A}^n = \big\{(A_1, \ldots, A_n) : A_1, \ldots, A_n \in \mathcal{A}\big\} .$$

For $1 \le p < \infty$, let

$$\|\mathcal{A}^n\|_p := \left(\sum_{(A_1,\ldots,A_n)\in\mathcal{A}^n} \|A_1 \ldots A_n\|^p \right)^{1/p} ,$$

and for $p = \infty$, define

$$\|\mathcal{A}^n\|_\infty := \max\left\{\|A_1 \ldots A_n\| : (A_1, \ldots, A_n) \in \mathcal{A}^n\right\} .$$

For $1 \le p \le \infty$, the **p-norm joint spectral radius** of $\mathcal{A}$ is defined to be

$$\rho_p(\mathcal{A}) := \lim_{n\to\infty} \|\mathcal{A}^n\|_p^{1/n} .$$

It is easily seen that this limit indeed exists, and

$$\lim_{n\to\infty} \|\mathcal{A}^n\|_p^{1/n} = \inf_{n\ge1} \|\mathcal{A}^n\|_p^{1/n} .$$

Clearly, $\rho_p(\mathcal{A})$ is independent of the choice of the vector norm on V. If $\mathcal{A}$ consists of a single linear operator A, then $\rho_p(\mathcal{A}) = \rho(A)$, where $\rho(A)$ denotes the spectral radius of A, which is independent of p.

The above definition of joint spectral radius also applies to a finite collection of square matrices of the same size. Indeed, an $s \times s$ matrix can be viewed as a linear operator on $\mathbb{C}^s$. Thus, if $\mathcal{A}$ is a finite collection of $s \times s$ matrices, the joint spectral radius $\rho_p(\mathcal{A})$ is well defined for $1 \le p \le \infty$. For the computation and the estimation of the p-norm joint spectral radius, see the examples and the discussion in Section 4. In particular in [21] and [31], when p is an even integer, the p-norm joint spectral radius can be computed exactly and explicitly in terms of the spectral radius of a finite matrix.

Before proceeding further, we introduce some notation. As usual, let $\mathbb{N}$ denote the set of positive integers, and $\mathbb{Z}$ the set of integers. By $\ell(\mathbb{Z})$ we denote the linear space of all sequences on $\mathbb{Z}$, and by $\ell_0(\mathbb{Z})$ we denote the linear space of all finitely supported sequences on $\mathbb{Z}$. For $v \in \ell_0(\mathbb{Z})$ and $k \in \mathbb{Z}$, define $\nabla_k v := v - v(\cdot - k)$. When $k = 1$, ∇_k will be abbreviated as ∇. We also define $\Delta v := -v(\cdot + 1) + 2v - v(\cdot - 1)$. For $a \in \ell_0(\mathbb{Z})$, its **symbol** is defined by

$$\tilde{a}(z) := \sum\nolimits_{j\in\mathbb{Z}} a(j)z^j, \qquad z \in \mathbb{C} \setminus \{0\} .$$

Let a be an element of $\ell_0(\mathbb{Z})$. For $\varepsilon \in \{0, 1\}$, let A_ε be the linear operator on $\ell_0(\mathbb{Z})$ given by

$$A_\varepsilon v(\alpha) = \sum\nolimits_{\beta\in\mathbb{Z}} a(\varepsilon + 2\alpha - \beta)v(\beta), \qquad \alpha \in \mathbb{Z}, \quad v \in \ell_0(\mathbb{Z}) . \tag{1.3}$$

By $V(v)$ we denote the minimal common invariant subspace of A_0 and A_1 generated by v. For a detailed procedure of finding $V(v)$, see Section 3. We write T_a for A_0 and call it the **transition operator** associated with a.

For a bounded subset K of $\mathbb{R}$, we denote by $\ell(K)$ the linear space of all sequences supported on $K \cap \mathbb{Z}$. Suppose the mask a is supported on the interval $[N_1, N_2]$, where N_1 and N_2 are integers such that $N_1 < N_2$. Then $\ell([N_1, N_2])$ is invariant under the transition operator T_a. Also, if v is an eigenvector of T_a corresponding to a nonzero eigenvalue σ, then v must be supported on $[N_1, N_2]$. To see this, let $n := \max\{\alpha : v(\alpha) \ne 0\}$. Suppose $n > N_2$. By $T_a v = \sigma v$ we have

$$\sigma v(n) = \sum_{\beta\in\mathbb{Z}} a(2n - \beta)v(\beta) .$$

If $v(\beta) \ne 0$, then $\beta \le n$; hence $2n - \beta \ge n > N_2$ and $a(2n - \beta) = 0$. It follows that $v(n) = 0$. This contradiction shows that $n \le N_2$. In other words, $v(\alpha) = 0$ for $\alpha > N_2$. Similarly, $v(\alpha) = 0$ for $\alpha < N_1$. Consequently, T_a has only finitely many nonzero eigenvalues.

The spectrum of a square matrix A is denoted by spec(A), and it is understood to be the *multiset* of its eigenvalues. In other words, multiplicities of eigenvalues are counted in the spectrum of a square matrix.

Let b be the mask given by $\tilde{b}(z) = \tilde{a}(z)(1+z)^m/2^m$, where m is a positive integer. Then b is supported on $[N_1, N_2+m]$. In Section 2 we will establish the following relation between the spectra of the transition operators T_a and T_b:

$$\operatorname{spec}\left(T_b|_{\ell([N_1,N_2+m])}\right) = 2^{-m}\operatorname{spec}\left(T_a|_{\ell([N_1,N_2])}\right) \cup \left\{2^{-j} : j = 0, 1, \ldots, m-1\right\} .$$

Consequently, if m is a positive integer such that $2^m > \rho(T_a|_{\ell([N_1,N_2])})$, then the above relation tells us that 1 is a simple eigenvalue of T_b.

In Section 3 we will give the following characterization for the existence of L_p solutions and continuous solutions of refinement equations.

Theorem 1.

Let m be a positive integer such that the transition operator T_b induced by $\tilde{b}(z) := \tilde{a}(z)(1+z)^m/2^m$ has 1 as its simple eigenvalue. Let v be an eigenvector of T_b corresponding to eigenvalue 1. Then (1.1) *has a nontrivial compactly supported L_p solution ϕ (continuous solution in the case $p=\infty$) if and only if*

$$\rho_p\left(A_0|_{V(\nabla v)}, A_1|_{V(\nabla v)}\right) < 2^{1/p} .$$

One sufficient condition for the existence of an L_p solution is the L_p convergence of the subdivision scheme with a stable initial function (the hat or box function). When a is supported in $[N_1, N_2]$, this convergence requires the sum rule: $\sum a(2\alpha) = \sum a(2\alpha+1) = 1$. Then the L_p convergence can be characterized [13, 28] by

$$\rho_p\left(A_0|_U, A_1|_U\right) < 2^{1/p} .$$

Here U is the subspace

$$U := \left\{u \in \ell([N_1, N_2-1]) : \sum_{\alpha=N_1}^{N_2-1} u(\alpha) = 0\right\} , \tag{1.4}$$

which is invariant under both A_0 and A_1. Usually, this subspace contains $V(\nabla v)$ in Theorem 1, see Examples 3, 4, and 5, even if the shifts of the solution are stable. However, [16, Lemma 5.2] shows that $\rho_p(A_0|_W, A_1|_W)$ are often equal for different subspaces W (see [16] for more details). In such a case, different invariant subspaces will produce the same result. Since $V(\nabla v)$ is smaller, the computation involving $V(\nabla v)$ will be simpler.

The key point of our approach is an appropriate choice of the initial nonstable function in the subdivision scheme. In our study, the initial function will be a finite linear combination of shifts of the B-spline B_1 or B_2 with the coefficients chosen appropriately. In [23], the vector of the coefficients is chosen as the 2-eigenvector of $(A_0 + A_1)$. In this article we generalize their method and obtain sharper results.

We shall use the generalized Lipschitz space to measure the regularity of a given function. Let us recall from [6] the definition of the generalized Lipschitz space. For $t \in \mathbb{R}$, the difference operator ∇_t is defined by $\nabla_t f = f - f(\cdot - t)$, where f is a function from $\mathbb{R}$ to $\mathbb{C}$. Let k be a positive integer. The kth **modulus of smoothness** of $f \in L_p(\mathbb{R})$ $(1 \le p \le \infty)$ is defined by

$$\omega_k(f,h)_p := \sup_{|t|\le h}\left\|\nabla_t^k f\right\|_p , \qquad h \ge 0 .$$

When $k=1$, $\omega_1(f,h)_p$ reduces to the **modulus of continuity** $\omega(f,h)_p$. For $\nu > 0$, let k be an integer greater than ν. The **generalized Lipschitz space** $\mathrm{Lip}^*(\nu, L_p(\mathbb{R}))$ consists of those functions $f \in L_p(\mathbb{R})$ for which

$$\omega_k(f,h)_p \le Ch^\nu \qquad \forall h > 0 ,$$

where C is a positive constant independent of h.

The optimal regularity of a function $f \in L_p(\mathbb{R})$ in the L_p norm is described by its **critical exponent** $\nu_p(f)$ defined by

$$\nu_p(f) := \sup\left\{ \nu : f \in \mathrm{Lip}^*\left(\nu, L_p(\mathbb{R})\right) \right\} .$$

Let ϕ be the normalized solution of the refinement equation (1.1) associated with a finitely supported mask a. If $\nu_p(\phi) > k$ for some positive integer k, then the shifts of ϕ reproduce all polynomials of degree at most k (see [1, p. 158]). Consequently, $1, 1/2, \ldots, 1/2^k$ are eigenvalues of the transition operator T_a (see [15]). However, since a is finitely supported, T_a has only finitely many nonzero eigenvalues. This shows that $\nu_p(\phi) < \infty$.

In Section 4, we will give the following characterization for the L_p regularity of the normalized solution ϕ of the refinement equation (1.1).

Theorem 2.

Suppose the conditions of Theorem 1 are satisfied. Let k be the smallest positive integer such that $k > 1/p - \log_2 \rho_p\big(A_0|_{V(\nabla^k v)}, A_1|_{V(\nabla^k v)}\big)$. Then

$$\nu_p(\phi) = 1/p - \log_2 \rho_p\left(A_0|_{V(\nabla^k v)}, A_1|_{V(\nabla^k v)}\right) .$$

All our results on existence and regularity of L_p solutions without assuming stability can be extended to vector refinement equations. For more details, see Section 5.

2. The Subdivision and Transition Operators

In this section we investigate the spectral properties of the subdivision and transition operators.

Let a be an element in $\ell_0(\mathbb{Z})$. The **subdivision operator** S_a is the linear operator on $\ell(\mathbb{Z})$ defined by

$$S_a u(\alpha) := \sum\nolimits_{\beta\in\mathbb{Z}} a(\alpha - 2\beta)u(\beta), \qquad \alpha \in \mathbb{Z} ,$$

where $u \in \ell(\mathbb{Z})$. The **transition operator** T_a is the linear operator on $\ell_0(\mathbb{Z})$ defined by

$$T_a v(\alpha) := \sum\nolimits_{\beta\in\mathbb{Z}} a(2\alpha - \beta)v(\beta), \qquad \alpha \in \mathbb{Z} ,$$

where $v \in \ell_0(\mathbb{Z})$.

We introduce a bilinear form on the pair of the linear spaces $\ell_0(\mathbb{Z})$ and $\ell(\mathbb{Z})$ as follows:

$$\langle u, v\rangle := \sum\nolimits_{\alpha\in\mathbb{Z}} u(\alpha)v(-\alpha), \qquad u \in \ell(\mathbb{Z}),\ v \in \ell_0(\mathbb{Z}) . \tag{2.1}$$

For $u \in \ell(\mathbb{Z})$ and $v \in \ell_0(\mathbb{Z})$ we have

$$\langle S_a u, v\rangle = \sum_{\alpha\in\mathbb{Z}}\sum_{\beta\in\mathbb{Z}} u(\beta)a(\alpha - 2\beta)v(-\alpha) = \sum_{\beta\in\mathbb{Z}} u(\beta)\left(T_a v\right)(-\beta) = \langle u, T_a v\rangle .$$

Hence, S_a is the algebraic adjoint of T_a with respect to the bilinear form given in (2.1). It was proved by Jia, Riemenschneider, and Zhou in [15] that S_a and T_a have the same nonzero eigenvalues with the same multiplicities.

The following result was essentially established by Deslauriers and Dubuc in [5]. Also, see [19] for discussions on spectral properties of the transition operator associated with multivariate refinement equations.

Lemma 1.

Let a be an element in $\ell([N_1, N_2])$, where N_1 and N_2 are integers such that $N_1 < N_2$. Let b be given by $\tilde{b}(z) = \tilde{a}(z)(1+z)^m/2^m$, where m is a positive integer. Then

$$\operatorname{spec}\left(T_b|_{\ell([N_1,N_2+m])}\right) = 2^{-m}\operatorname{spec}\left(T_a|_{\ell([N_1,N_2])}\right) \cup \left\{2^{-j} : j = 0, 1, \ldots, m-1\right\}. \tag{2.2}$$

Proof. It suffices to establish (2.2) for $m = 1$, since the general case can be easily proved by induction on m.

Let b be the sequence given by $\tilde{b}(z) = \tilde{a}(z)(1+z)/2$. Then $b(\alpha) = \left[a(\alpha) + a(\alpha - 1)\right]/2$ for all $\alpha \in \mathbb{Z}$. Since $\sum_{\alpha\in\mathbb{Z}} a(\alpha) = 2$, it follows that

$$\sum_{\alpha\in\mathbb{Z}} b(2\alpha) = \sum_{\alpha\in\mathbb{Z}} b(2\alpha - 1) = 1. \tag{2.3}$$

For $j \in \mathbb{Z}$, we denote by δ_j the element in $\ell_0(\mathbb{Z})$ given by $\delta_j(j) = 1$ and $\delta_j(\alpha) = 0$ for all $\alpha \in \mathbb{Z} \setminus \{j\}$. Clearly, $\{\delta_{N_1}, \ldots, \delta_{N_2}\}$ is a basis for $\ell([N_1, N_2])$. Suppose

$$T_a\delta_j = \sum_{k=N_1}^{N_2} \lambda_{kj}\delta_k, \qquad j = N_1, \ldots, N_2. \tag{2.4}$$

Note that b is supported on $[N_1, N_2 + 1]$. Let

$$V := \left\{v \in \ell([N_1, N_2+1]) : \sum_{\alpha\in\mathbb{Z}} v(\alpha) = 0\right\}.$$

It is easily seen that $\{\nabla\delta_{N_1}, \ldots, \nabla\delta_{N_2}\}$ is a basis for V. For $j = N_1, \ldots, N_2$ and $\alpha \in \mathbb{Z}$, we have

$$\begin{aligned} T_b(\nabla\delta_j)(\alpha) &= \sum_{\beta\in\mathbb{Z}} b(2\alpha - \beta)\left[\delta_j(\beta) - \delta_j(\beta - 1)\right] \\ &= \frac{1}{2}\sum_{\beta\in\mathbb{Z}} \left[a(2\alpha - \beta) + a(2\alpha - \beta - 1)\right]\left[\delta_j(\beta) - \delta_j(\beta - 1)\right] \\ &= \frac{1}{2}\left[\sum_{\beta\in\mathbb{Z}} a(2\alpha - \beta)\delta_j(\beta) - \sum_{\beta\in\mathbb{Z}} a(2\alpha - \beta - 1)\delta_j(\beta - 1)\right] \\ &= \frac{1}{2}\left[\sum_{k=N_1}^{N_2} \lambda_{kj}\delta_k(\alpha) - \sum_{k=N_1}^{N_2} \lambda_{kj}\delta_k(\alpha - 1)\right]. \end{aligned}$$

Consequently,

$$T_b\left(\nabla\delta_j\right) = \frac{1}{2}\sum_{k=N_1}^{N_2} \lambda_{kj}\left(\nabla\delta_k\right), \qquad j = N_1, \ldots, N_2. \tag{2.5}$$

We observe that b is supported on $[N_1, N_2 + 1]$ and $\{\delta_{N_1}, \nabla\delta_{N_1}, \ldots, \nabla\delta_{N_2}\}$ is a basis for $\ell([N_1, N_2 + 1])$. In light of (2.3), we have

$$\sum_{\alpha\in\mathbb{Z}} \left(T_b\delta_{N_1}\right)(\alpha) = \sum_{\alpha\in\mathbb{Z}} b\left(2\alpha - N_1\right) = 1.$$

Hence, $T_b\delta_{N_1} - \delta_{N_1}$ lies in V. In other words, there exists an element $y \in V$ such that

$$T_b\delta_{N_1} = \delta_{N_1} + y. \tag{2.6}$$

Combining (2.5) and (2.6), we see that the matrix representation of $T_b|_{\ell([N_1,N_2+1])}$ with respect to the basis $\{\delta_{N_1}, \nabla\delta_{N_1}, \ldots, \nabla\delta_{N_2}\}$ is

$$\begin{bmatrix} 1 & 0 \\ E & F \end{bmatrix},$$

where $F = (\lambda_{jk}/2)_{N_1 \le j,k \le N_2}$. This together with (2.4) gives

$$\operatorname{spec}\left(T_b|_{\ell([N_1,N_2+1])}\right) = \{1\} \cup \operatorname{spec}(F) = \{1\} \cup 2^{-1} \operatorname{spec}\left(T_a|_{\ell([N_1,N_2])}\right) .$$

This establishes (2.2) for the case $m = 1$, as desired. □

Let m be a positive integer such that 1 is a simple eigenvalue of $T_b|_{\ell([N_1,N_2+m])}$. By Lemma 1, this is true if $2^m > \rho(T_a|_{\ell([N_1,N_2])})$. Let v be an eigenvector of $T_b|_{\ell([N_1,N_2+m])}$ corresponding to eigenvalue 1. Let B be the matrix representation of $T_b|_{\ell([N_1,N_2+m])}$ with respect to the basis $\{\delta_{N_1}, \ldots, \delta_{N_2+m}\}$. Then $(v(N_1), \ldots, v(N_2+m))^T$ is a right eigenvector of B associated with eigenvalue 1. But (2.3) tells us that $(1, \ldots, 1)$ is a left eigenvector of B associated with eigenvalue 1. Hence,

$$\sum\nolimits_{\alpha \in \mathbb{Z}} v(\alpha) = v(N_1) + \cdots + v(N_2+m) \neq 0 .$$

As an application of Lemma 1, we construct in what follows a family of refinable functions $\phi_{m,n}$ associated with masks $a_{m,n}$ such that $\phi_{m,n}$ has the same regularity as that of the cardinal B-spline of order m, but $\rho(T_{a_{m,n}}) \ge 2^n$.

For $m = 1, 2, \ldots$, let B_m denote the cardinal B-spline of order m. Precisely, B_1 is the characteristic function of the interval $[0, 1)$, and for $m \ge 2$,

$$B_m(x) = B_{m-1} * B_1(x) = \int_0^1 B_{m-1}(x-t)\,dt, \qquad x \in \mathbb{R} .$$

In particular, $B_2(x) = \max\{0, 1 - |x-1|\}$, $x \in \mathbb{R}$. Obviously, B_m is continuous for $m \ge 2$. The B-spline B_m is supported on $[0, m]$ and is nonnegative. The shifts of B_m forms a partition of unity, that is,

$$\sum\nolimits_{j \in \mathbb{Z}} B_m(x-j) = 1, \qquad x \in \mathbb{R} .$$

Moreover, B_m is refinable:

$$B_m(x) = 2^{1-m} \sum_{j=0}^{m} \binom{m}{j} B_m(2x-j), \qquad x \in \mathbb{R} .$$

Example 1. Let $a_{m,n}$ be the sequence given by

$$\tilde{a}_{m,n}(z) = 2^{1-m}\left(1+z^3\right)^m \left(1 - z + z^2\right)^n ,$$

where m is a positive integer and n is a nonnegative integer. Let $\phi_{m,n}$ be the normalized distributional solution of the refinement equation associated with the mask $a_{m,n}$. Then $\phi_{m,n}$ and B_m have the same regularity but

$$\rho\left(T_{a_{m,n}}\right) \ge 2^n . \tag{2.7}$$

Proof. We observe that

$$2\left(\frac{1+z}{2}\right)^m = \tilde{a}_{m,n}(z)\left(\frac{1+z+z^2}{1+z^2+z^4}\right)^{m+n} .$$

The left-hand side of the above equality is the symbol of the refinement mask of the B-spline B_m. Thus, from the proof of [13, Theorem 5.3] we see that $\phi_{m,n}$ is a linear combination of B_m, $B_m(\cdot - 1), \ldots, B_m(\cdot - 2m - 2n)$. Therefore, $\phi_{m,n}$ and B_m have the same regularity. In particular, $\phi_{m,n} \in C^{m-2}(\mathbb{R})$ for $m \ge 2$.

In order to prove (2.7), we claim that 1 is an eigenvalue of the subdivision operator $S_{a_{m+n,0}}$ of multiplicity at least 2. To see this, we choose two elements u_1 and u_2 in $\ell(\mathbb{Z})$ as follows. Let u_1 be given by $u_1(j) = 1$ for all $j \in \mathbb{Z}$, and let u_2 be given by $u_2(3j) = 1$, $u_2(3j+1) = u_2(3j+2) = 0$ for all $j \in \mathbb{Z}$. Note that, for $k \in \mathbb{Z}$, $a_{m+n,0}(3k+1) = 0$, $a_{m+n,0}(3k+2) = 0$, and

$$a_{m+n,0}(3k) = 2^{1-m-n}\binom{m+n}{k} .$$

Thus, a simple computation yields

$$S_{a_{m+n,0}}u_1 = u_1 \quad \text{and} \quad S_{a_{m+n,0}}u_2 = u_2 .$$

This justifies our claim. Consequently, 1 is an eigenvalue of the transition operator $T_{a_{m+n,0}}$ of multiplicity at least 2. Moreover, we have

$$\tilde{a}_{m+n,0}(z) = 2^{1-m-n}\left(1+z^3\right)^{m+n} = \tilde{a}_{m,n}(z)(1+z)^n/2^n .$$

In light of (2.2) we obtain

$$\operatorname{spec}\left(T_{a_{m+n,0}}|_{\ell([0,3m+3n])}\right) = 2^{-n}\operatorname{spec}\left(T_{a_{m,n}}|_{\ell([0,3m+2n])}\right) \cup \left\{2^{-j} : j = 0, 1, \ldots, n-1\right\} .$$

Since 1 is an eigenvalue of $T_{a_{m+n,0}}$ of multiplicity at least 2, it follows from the above relation that 2^n belongs to $\operatorname{spec}(T_{a_{m,n}}|_{\ell([0,3m+2n])})$. Hence, (2.7) is true. □

3. Characterization of L_p Solutions

In this section we give a characterization for the existence of L_p solutions of the refinement equation (1.1) in terms of the mask.

We first introduce some notations. For $1 \le p < \infty$, the ℓ_p norm of an element v in $\ell_0(\mathbb{Z})$ is defined by

$$\|v\|_p := \left(\sum\nolimits_{\alpha\in\mathbb{Z}} |v(\alpha)|^p\right)^{1/p} .$$

The ℓ_∞ norm of v is defined by $\|v\|_\infty := \sup\{|v(\alpha)| : \alpha \in \mathbb{Z}\}$.

Let $\mathcal{A} := \{A_0, A_1\}$, where A_0 and A_1 are the linear operators on $\ell_0(\mathbb{Z})$ defined in (1.3). Given $v \in \ell_0(\mathbb{Z})$, we define, for $1 \le p < \infty$,

$$\left\|\mathcal{A}^n v\right\|_p := \left(\sum\nolimits_{\varepsilon_1,\ldots,\varepsilon_n\in\{0,1\}} \left\|A_{\varepsilon_n}\cdots A_{\varepsilon_1}v\right\|_p^p\right)^{1/p} ,$$

and for $p = \infty$,

$$\left\|\mathcal{A}^n v\right\|_\infty := \max\left\{\left\|A_{\varepsilon_n}\cdots A_{\varepsilon_1}v\right\|_\infty : \varepsilon_1, \ldots, \varepsilon_n \in \{0,1\}\right\} .$$

It was proved in [9] that there exist two positive constants C_1 and C_2 such that

$$C_1\left\|\mathcal{A}^n v\right\|_p \le \left\|\mathcal{A}^n|_{V(v)}\right\|_p \le C_2\left\|\mathcal{A}^n v\right\|_p \qquad \forall n \in \mathbb{N} , \tag{3.1}$$

where $V(v)$ denotes the minimal common invariant subspace of A_0 and A_1 generated by v.

If the mask a and a sequence v are supported on $[N_1, N_2]$, then $V(v)$ is contained in the space $\ell([N_1, N_2])$, which is invariant under A_0 and A_1. Therefore, the dimension of $V(v)$ is at most $N_2 - N_1 + 1$. Consequently,

$$V(v) = \operatorname{span}\left\{A_{\varepsilon_1}\cdots A_{\varepsilon_n}v : n = 0, 1, \ldots, N_2 - N_1, \varepsilon_1, \ldots, \varepsilon_n \in \{0,1\}\right\} . \tag{3.2}$$

To verify our assertion, we set $E_0 := \{v\}$ and $E_n := \{A_{\varepsilon_1} \dots A_{\varepsilon_n} v : \varepsilon_1, \dots, \varepsilon_n \in \{0, 1\}\}$ for $n = 1, \dots, N_2 - N_1$. If $E_n \subseteq \operatorname{span}\{E_0 \cup \dots \cup E_{n-1}\}$ for some $n \le N_2 - N_1$, then (3.2) is true. Otherwise, there exists some $u_n \in E_n \setminus \operatorname{span}\{E_0 \cup \dots \cup E_{n-1}\}$ for each $n = 1, \dots, N_2 - N_1$. Then $v, u_1, \dots, u_{N_2-N_1}$ will be linearly independent. In this case, $V(v) = \operatorname{span}\{v, u_1, \dots, u_{N_2-N_1}\}$ and (3.2) is still valid.

In order to study L_p solutions of the refinement equation we shall employ the following iteration scheme. Let Q_a be the linear operator on $L_p(\mathbb{R})$ given by

$$Q_a f(x) := \sum_{\alpha \in \mathbb{Z}} a(\alpha) f(2x - \alpha), \qquad f \in L_p(\mathbb{R}) . \tag{3.3}$$

Let ϕ_0 be an initial function in $L_p(\mathbb{R})$. For $n = 1, 2, \dots$, let $\phi_n := Q_a^n \phi_0$. If $(\phi_n)_{n=1,2,\dots}$ converges to some ϕ in the L_p norm ($1 \le p \le \infty$), then the limit ϕ is a solution of (1.1). Usually, ϕ_0 is chosen to be the hat function B_2. With this choice of ϕ_0, if $(\phi_n)_{n=1,2,\dots}$ converges in the L_p norm, then we say that the cascade algorithm (or the subdivision scheme) associated with mask a is L_p convergent ($1 \le p \le \infty$). Let A_0 and A_1 be the linear operators given in (1.3), and let U be the linear space defined in (1.4). It was proved in [13] that the cascade algorithm associated with mask a is L_p convergent if and only if $\rho_p(A_0|_U, A_1|_U) < 2^{1/p}$. However, if $a_{m,n}$ is the mask given in Example 1, then (2.7) tells us that the cascade algorithm associated with $a_{m,n}$ does not converge in the L_∞ norm, even though $\phi_{m,n}$ is continuous for $m \ge 2$. Therefore, the L_p convergence of the cascade algorithm is a sufficient but not a necessary condition for the existence of L_p solutions. Thus, the key to our investigation is an appropriate choice of the initial function ϕ_0. In our study, the initial function will be a finite linear combination of shifts of the B-spline B_1 or B_2 with the coefficients chosen appropriately.

Iterating (3.3) n times gives

$$Q_a^n \phi_0(x) = \sum_{\alpha \in \mathbb{Z}} a_n(\alpha) \phi_0(2^n x - \alpha), \qquad n = 1, 2, \dots .$$

In particular, $a_1 = a$. *Consequently, for* $n > 1$ we have

$$\begin{aligned} Q_a^n \phi_0(x) &= Q_a^{n-1} \left(Q_a \phi_0 \right)(x) = \sum_{\beta \in \mathbb{Z}} a_{n-1}(\beta) \left(Q_a \phi_0 \right) \left(2^{n-1} x - \beta \right) \\ &= \sum_{\beta \in \mathbb{Z}} \sum_{\alpha \in \mathbb{Z}} a_{n-1}(\beta) a(\alpha) \phi_0 \left(2^n x - 2\beta - \alpha \right) \\ &= \sum_{\alpha \in \mathbb{Z}} \left[\sum_{\beta \in \mathbb{Z}} a_{n-1}(\beta) a(\alpha - 2\beta) \right] \phi_0 \left(2^n x - \alpha \right) . \end{aligned}$$

This establishes the following iteration relation for a_n ($n = 1, 2, \dots$):

$$a_1 = a \qquad \text{and} \qquad a_n(\alpha) = \sum_{\beta \in \mathbb{Z}} a_{n-1}(\beta) a(\alpha - 2\beta), \qquad \alpha \in \mathbb{Z} . \tag{3.4}$$

The convolution of $u \in \ell_0(\mathbb{Z})$ and $v \in \ell_0(\mathbb{Z})$ is defined by

$$u * v(\alpha) := \sum_{\beta \in \mathbb{Z}} u(\alpha - \beta) v(\beta), \qquad \alpha \in \mathbb{Z} .$$

Lemma 2.

If $\alpha = \varepsilon_1 + 2\varepsilon_2 + \dots + 2^{n-1}\varepsilon_n + 2^n \gamma$, *where* $\varepsilon_1, \dots, \varepsilon_n \in \{0, 1\}$ *and* $\gamma \in \mathbb{Z}$, *then*

(i) $a_n * v(\alpha) = A_{\varepsilon_n} \dots A_{\varepsilon_1} v(\gamma)$, *and*

(ii) $\|a_n * v\|_p = \|\mathcal{A}^n v\|_p$.

Proof. Clearly, (ii) follows from (i) immediately. The proof of (i) proceeds by induction on n. For $n = 1$ and $\alpha = \varepsilon_1 + 2\gamma$, we have

$$a_1 * v(\alpha) = \sum_{\beta\in\mathbb{Z}} a(\varepsilon_1 + 2\gamma - \beta)\, v(\beta) = \left(A_{\varepsilon_1} v\right)(\gamma) .$$

Suppose $n > 1$ and (i) has been verified for $n-1$. For $\alpha = \varepsilon_1 + 2\alpha_1$, where $\varepsilon_1 \in \{0, 1\}$ and $\alpha_1 \in \mathbb{Z}$, by the iteration relation (3.4) we have

$$\sum_{\beta\in\mathbb{Z}} a_n(\alpha-\beta)v(\beta) = \sum_{\beta\in\mathbb{Z}}\sum_{\eta\in\mathbb{Z}} a_{n-1}(\eta)a(\alpha-\beta-2\eta)v(\beta)$$
$$= \sum_{\eta\in\mathbb{Z}} a_{n-1}(\alpha_1-\eta)\sum_{\beta\in\mathbb{Z}} a(\varepsilon_1+2\eta-\beta)\,v(\beta) = \left(a_{n-1} * \left(A_{\varepsilon_1}v\right)\right)(\alpha_1) .$$

Suppose $\alpha_1 = \varepsilon_2 + \cdots + 2^{n-2}\varepsilon_n + 2^{n-1}\gamma$. Then by the induction hypothesis we have

$$a_n * v(\alpha) = \left(a_{n-1} * \left(A_{\varepsilon_1}v\right)\right)(\alpha_1) = \left(A_{\varepsilon_n}\dots A_{\varepsilon_2}\right)\left(A_{\varepsilon_1}v\right)(\gamma) = A_{\varepsilon_n}\dots A_{\varepsilon_1}v(\gamma) ,$$

as desired. □

By Lemma 2, for $j \in \mathbb{Z}$ we have

$$\left\|\mathcal{A}^n\left(v(\cdot - j)\right)\right\|_p = \left\|a_n * \left(v(\cdot - j)\right)\right\|_p = \|a_n * v\|_p = \left\|\mathcal{A}^n v\right\|_p .$$

Let W be the linear span of $\{\nabla v(\cdot - j) : j \in \mathbb{Z}\}$. In other words, $w \in W$ if and only if w is a (finite) linear combination of $\nabla v(\cdot - j)$, $j \in \mathbb{Z}$. For $w \in W$, by (3.1) we see that there exists a positive constant C independent of n such that

$$\left\|\mathcal{A}^n w\right\|_p \le C \left\|\mathcal{A}^n|_{V(\nabla v)}\right\|_p \qquad \forall n \in \mathbb{N} , \tag{3.5}$$

where $V(\nabla v)$ denotes the minimal common invariant subspace of A_0 and A_1 generated by ∇v.

We will need to use the following lemma in this and the next section.

Lemma 3.

Suppose $1 \le p \le \infty$. Let $\phi \in L_1(\mathbb{R})$ be the normalized solution of the refinement equation (1.1) *and let $v(\alpha) := \phi * B_m(\alpha)$, $\alpha \in \mathbb{Z}$. Then*

(i) $a_n * (\nabla^k v)(\alpha) = \int_{\mathbb{R}} B_m(x)\nabla^k_{2^{-n}}\phi((\alpha - x)/2^n)\,dx$, *and*

(ii) $\|a_n * (\nabla^k v)\|_p \le 2^{n/p}\|\nabla^k_{2^{-n}}\phi\|_p$.

Proof. We only prove the case $k = 1$. The general case can be proved in the same way. A repeated use of the refinement equation (1.1) gives

$$\phi(x/2^n) = \sum_{\beta\in\mathbb{Z}} a_n(\beta)\phi(x-\beta), \qquad x \in \mathbb{R} .$$

Hence, for $\alpha \in \mathbb{Z}$ we have

$$\int_{\mathbb{R}} B_m(x)\phi\left((\alpha-x)/2^n\right)dx = \int_{\mathbb{R}} B_m(\alpha-x)\phi\left(x/2^n\right)dx$$
$$= \int_{\mathbb{R}} B_m(\alpha-x)\sum_{\beta\in\mathbb{Z}} a_n(\beta)\phi(x-\beta)\,dx = \sum_{\beta\in\mathbb{Z}} a_n(\beta)\int_{\mathbb{R}} B_m(\alpha-\beta-x)\phi(x)\,dx$$
$$= \sum_{\beta\in\mathbb{Z}} a_n(\beta)v(\alpha-\beta) = a_n * v(\alpha) .$$

It follows that

$$a_n*(\nabla v)(\alpha) = \int_{\mathbb{R}} B_m(x)\nabla_{2^{-n}}\phi\left((\alpha - x)/2^n\right)\,dx, \qquad \alpha \in \mathbb{Z}\,. \tag{3.6}$$

Thus, for $p = \infty$ we obtain

$$\|a_n*(\nabla v)\|_\infty \le \|\nabla_{2^{-n}}\phi\|_\infty \int_{\mathbb{R}} B_m(x)\,dx = \|\nabla_{2^{-n}}\phi\|_\infty\,.$$

Consider the case $1 \le p < \infty$. Let q be the number conjugate to p, i.e., $1/p + 1/q = 1$. Applying the Hölder inequality to (3.6), we obtain

$$\begin{aligned}|a_n*(\nabla v)(\alpha)| &\le \left(\int_{\mathbb{R}} B_m(x)\,dx\right)^{1/q} \left(\int_{\mathbb{R}} B_m(x)\left|\nabla_{2^{-n}}\phi\left((\alpha - x)/2^n\right)\right|^p dx\right)^{1/p} \\ &= \left(\int_{\mathbb{R}} B_m(\alpha - x)\left|\nabla_{2^{-n}}\phi(x/2^n)\right|^p dx\right)^{1/p}\,.\end{aligned}$$

Since the shifts of B_m form a partition of unity, it follows that

$$\|a_n*(\nabla v)\|_p = \left(\sum_{\alpha\in\mathbb{Z}} |a_n*(\nabla v)(\alpha)|^p\right)^{1/p} \le \left(\int_{\mathbb{R}} \left|\nabla_{2^{-n}}\phi(x/2^n)\right|^p dx\right)^{1/p} = 2^{n/p}\,\|\nabla_{2^{-n}}\phi\|_p\,.$$

This completes the proof of the lemma. □

Lemma 4.

Let $\tilde{b}(z) = 2^{-m}(1+z)^m\tilde{a}(z)$, *and let* v *be an element in* $\ell_0(\mathbb{Z})$ *such that* $T_b v = v$. *Then for* $\tilde{u}(z) = 2^{-k-m}(1+z)^{k+m}\tilde{v}(z)$, $k = 0, 1, \ldots,$ *we have* $T_a(\nabla^k u) = 2^{-k}\nabla^k v$.

Proof. We first consider the case $k = 0$, i.e., $\tilde{u}_0(z) = 2^{-m}(1+z)^m\tilde{v}(z)$. By the definition of b we have

$$b(\alpha) = 2^{-m}\sum_{j=0}^{m}\binom{m}{j}a(\alpha - j), \qquad \alpha \in \mathbb{Z}\,.$$

Since $T_b v = v$, it follows that

$$v(\alpha) = \sum_{\beta\in\mathbb{Z}} b(2\alpha - \beta)v(\beta) = \sum_{\beta\in\mathbb{Z}}\sum_{j=0}^{m} 2^{-m}\binom{m}{j}a(2\alpha - \beta - j)v(\beta) = T_a u_0(\alpha), \qquad \alpha \in \mathbb{Z}\,.$$

Hence, $T_a u_0 = v$. For $k > 0$, the symbol of $\nabla^k u$ is

$$2^{-k-m}(1-z)^k(1+z)^{k+m}\tilde{v}(z) = 2^{-k}\left(1 - z^2\right)^k\tilde{u}_0(z)\,.$$

It follows that $\nabla^k u = 2^{-k}\nabla_2^k u_0$. Therefore, we have

$$T_a\left(\nabla^k u\right) = 2^{-k}T_a\left(\nabla_2^k u_0\right) = 2^{-k}\nabla^k\left(T_a u_0\right) = 2^{-k}\nabla^k v\,.$$

The proof of Lemma 4 is complete. □

We are in a position to give a characterization for the existence of L_p solutions and continuous solutions of refinement equations.

Theorem 3.

Let m be a positive integer such that the transition operator T_b induced by $\tilde{b}(z) := \tilde{a}(z)(1+z)^m/2^m$ has 1 as its simple eigenvalue. Let v be an eigenvector of T_b corresponding to eigenvalue 1. Then (1.1) has a nontrivial compactly supported L_p solution ϕ (continuous solution in the case $p=\infty$) if and only if

$$\rho_p\left(A_0|_{V(\nabla v)}, A_1|_{V(\nabla v)}\right) < 2^{1/p}. \tag{3.7}$$

Proof. Let ϕ be the normalized solution of the refinement equation (1.1). Suppose ϕ belongs to $L_p(\mathbb{R})$ (ϕ is continuous in the case $p=\infty$). Let v be given by

$$v(\alpha) := (\phi * B_m)(\alpha), \qquad \alpha \in \mathbb{Z},$$

where B_m is the cardinal B-spline of order m. It is easily seen that $\phi * B_m$ is continuous and refinable with b as its mask. Hence,

$$v(\alpha) = \sum_{\beta\in\mathbb{Z}} b(\beta)v(2\alpha-\beta) = \sum_{\beta\in\mathbb{Z}} b(2\alpha-\beta)v(\beta) \qquad \forall\alpha\in\mathbb{Z}.$$

In other words, $T_b v = v$. Since the shifts of B_m form a partition of unity, we have

$$\sum_{\alpha\in\mathbb{Z}} v(\alpha) = \int_{\mathbb{R}} \phi(x)\sum_{\alpha\in\mathbb{Z}} B_m(\alpha-x)\,dx = \int_{\mathbb{R}}\phi(x)\,dx = \hat{\phi}(0) = 1.$$

Hence, v is an eigenvector of T_b corresponding to eigenvalue 1. Clearly, $\|\nabla_{2^{-n}}\phi\|_p$ converges to 0 as n goes to ∞. By Lemma 3 we have

$$\lim_{n\to\infty} 2^{-n/p}\|a_n*(\nabla v)\|_p = 0. \tag{3.8}$$

Let $\rho_p := \rho_p(A_0|_{V(\nabla v)}, A_1|_{V(\nabla v)})$. Then $\rho_p \le \|\mathcal{A}^n|_{V(\nabla v)}\|_p^{1/n}$ for all n. By (3.1) and Lemma 2, this yields

$$\rho_p^n \le \left\|\mathcal{A}^n|_{V(\nabla v)}\right\|_p \le C\left\|\mathcal{A}^n(\nabla v)\right\|_p = C\,\|a_n*(\nabla v)\|_p,$$

where C is a constant independent of n. Taking (3.8) into account, we deduce that

$$\lim_{n\to\infty}\left(2^{-1/p}\rho_p\right)^n = \lim_{n\to\infty} 2^{-n/p}\rho_p^n = 0.$$

Therefore, we must have $2^{-1/p}\rho_p < 1$, i.e., $\rho_p < 2^{1/p}$. Thus, we have established the necessity part of the theorem.

It remains to establish the sufficiency part of the theorem. By our assumption, $\tilde{b}(z) = \tilde{a}(z)(1+z)^m/2^m$ and T_b has 1 as its simple eigenvalue. Let v be an eigenvector of T_b corresponding to eigenvalue 1. In light of the discussions in Section 2, we have $\sum_{\alpha\in\mathbb{Z}} v(\alpha) \ne 0$. Thus, without loss of any generality, we may assume that $\sum_{\alpha\in\mathbb{Z}} v(\alpha) = 1$. Let u be the sequence on $\mathbb{Z}$ given by

$$u(\beta) := 2^{-m}\sum_{j=0}^{m}\binom{m}{j} v(\beta-j), \qquad \beta\in\mathbb{Z}.$$

Then the symbol of u is $\tilde{u}(z) = 2^{-m}(1+z)^m\tilde{v}(z)$. Hence, we have $A_0 u = T_a u = v$, by Lemma 4.

Let us consider the case $1 \le p < \infty$ first. The initial function ϕ_0 is chosen as

$$\phi_0(x) := \sum_{\beta\in\mathbb{Z}} u(\beta)B_1(x-\beta), \qquad x\in\mathbb{R},$$

where $B_1 = \chi_{[0,1)}$. Since $\sum_{\beta\in\mathbb{Z}} u(\beta) = 1$, we have $\hat{\phi}_0(0) = 1$. It follows that

$$\left(\widehat{Q_a\phi_0}\right)(0) = \sum\nolimits_{\alpha\in\mathbb{Z}} a(\alpha)\hat{\phi}_0(0)/2 = 1 .$$

For $n = 1, 2, \ldots$, set $\phi_n := Q_a^n\phi_0$. Then $\hat{\phi}_n(0) = 1$ for $n = 1, 2, \ldots$. We wish to show that $(\phi_n)_{n=1,2,\ldots}$ converges in the L_p-norm. For this purpose, we observe that

$$\phi_n(x) = \sum_{\alpha\in\mathbb{Z}} a_n(\alpha)\phi_0\left(2^n x - \alpha\right) = \sum_{\alpha\in\mathbb{Z}}\sum_{\beta\in\mathbb{Z}} a_n(\alpha)u(\beta)B_1\left(2^n x - \alpha - \beta\right) .$$

Consequently,

$$\phi_n(x) = \sum\nolimits_{\beta\in\mathbb{Z}} (a_n{*}u)(\beta)B_1\left(2^n x - \beta\right) .$$

Since $B_1(x) = B_1(2x) + B_1(2x-1)$, we have

$$\phi_n(x) = \sum_{\alpha\in\mathbb{Z}} (a_n{*}u)(\alpha)\left(B_1\left(2^{n+1}x - 2\alpha\right) + B_1\left(2^{n+1}x - 2\alpha - 1\right)\right) .$$

Moreover,

$$\phi_{n+1}(x) = \sum_{\alpha\in\mathbb{Z}} (a_{n+1}{*}u)(2\alpha)B_1\left(2^{n+1}x - 2\alpha\right) + \sum_{\alpha\in\mathbb{Z}} (a_{n+1}{*}u)(2\alpha+1)B_1\left(2^{n+1}x - 2\alpha - 1\right) .$$

Subtracting the first equation from the second, we obtain

$$\phi_{n+1}(x) - \phi_n(x) = \sum_{\alpha\in\mathbb{Z}} w_{0,n}(\alpha)B_1\left(2^{n+1}x - 2\alpha\right) + \sum_{\alpha\in\mathbb{Z}} w_{1,n}(\alpha)B_1\left(2^{n+1}x - 2\alpha - 1\right) ,$$

where

$$w_{0,n}(\alpha) := (a_{n+1}{*}u)(2\alpha) - (a_n{*}u)(\alpha)$$

and

$$w_{1,n}(\alpha) := (a_{n+1}{*}u)(2\alpha+1) - (a_n{*}u)(\alpha) .$$

It follows that

$$\|\phi_{n+1} - \phi_n\|_p \le 2^{1-(n+1)/p}\left(\left\|w_{0,n}\right\|_p + \left\|w_{1,n}\right\|_p\right) . \tag{3.9}$$

Let us estimate $\|w_{0,n}\|_p$ and $\|w_{1,n}\|_p$. By (3.4), for $\alpha \in \mathbb{Z}$,

$$(a_{n+1}{*}u)(2\alpha) = \sum_{\beta\in\mathbb{Z}}\sum_{\gamma\in\mathbb{Z}} a_n(\gamma)a(2\alpha - \beta - 2\gamma)u(\beta) = a_n{*}(A_0u)(\alpha) .$$

Let $w_0 := A_0u - u$. Then

$$\left\|w_{0,n}\right\|_p = \|a_n{*}w_0\|_p = \left\|\mathcal{A}^n w_0\right\|_p, \qquad 1 \le p < \infty . \tag{3.10}$$

Similarly, we have

$$\left\|w_{1,n}\right\|_p = \left\|\mathcal{A}^n w_1\right\|_p, \qquad 1 \le p < \infty , \tag{3.11}$$

where $w_1 := A_1u - u$. By Lemma 4 we have $A_0u = v$. Hence,

$$w_0 = A_0u - u = v - u = 2^{-m}\sum_{j=0}^{m}\binom{m}{j}[v - v(\cdot - j)] .$$

This shows $w_0 \in W$, where W denotes the linear span of $\{\nabla v(\cdot - j) : j \in \mathbb{Z}\}$. Furthermore,

$$w_1 = A_1 u - u = (A_1 - A_0) u + (A_0 u - u) = A_0(u(\cdot + 1) - u) + (A_0 u - u) .$$

Clearly, $u(\cdot + 1) - u$ lies in W. Therefore, by (3.5) there exists a positive constant C_1 independent of n such that

$$\|\mathcal{A}^n w_0\|_p \le C_1 \left\| \mathcal{A}^n|_{V(\nabla v)} \right\|_p \quad \text{and} \quad \|\mathcal{A}^n A_0(u(\cdot + 1) - u)\|_p \le C_1 \left\| \mathcal{A}^n|_{V(\nabla v)} \right\|_p .$$

Suppose (3.7) is valid. By what has been proved, there exist a constant $C_2 > 0$ and a constant t between 0 and 1 such that

$$\|\mathcal{A}^n w_0\|_p \le C_2 \left(2^{1/p} t\right)^n \quad \text{and} \quad \|\mathcal{A}^n w_1\|_p \le C_2 \left(2^{1/p} t\right)^n \tag{3.12}$$

hold true for all $n = 1, 2, \ldots$. Combining (3.9)–(3.12) together, we see that there exists a constant $C > 0$ such that

$$\|\phi_{n+1} - \phi_n\|_p \le C t^n, \qquad n = 1, 2, \ldots .$$

Since $0 < t < 1$, this shows that there exists $\phi \in L_p(\mathbb{R})$ such that $\|\phi_n - \phi\|_p \to 0$ as $n \to \infty$. From $\hat{\phi}_n(0) = 1$ we deduce $\hat{\phi}(0) = 1$, since all the functions ϕ_n are supported in a common closed interval.

For the case $p = \infty$, we choose the initial function ϕ_0 to be $\sum_{\beta \in \mathbb{Z}} u(\beta) B_2(\cdot - \beta)$. The existence of continuous solutions can be proved in an analogous way. □

Let us apply Theorem 3 to the example mentioned in the beginning of this article.

Example 2. Let a be the sequence on $\mathbb{Z}$ given by its symbol

$$\tilde{a}(z) = 1 - z + z^2 + z^3 - z^4 + z^5 .$$

Let ϕ be the normalized solution of the refinement equation associated with the mask a. Then ϕ lies in $L_p(\mathbb{R})$ for $1 \le p < \infty$, but ϕ does not lie in $C(\mathbb{R})$.

Proof. The mask a is supported on $[0, 5]$. A simple computation gives

$$\operatorname{spec}\left(T_a|_{\ell([0,5])}\right) = \{2, -2, 1, 1, 1, -1\} .$$

If the sequence b is given by $\tilde{b}(z) = \tilde{a}(z)(1 + z)^2/2^2$, then 1 is a simple eigenvalue of $T_b|_{\ell([0,7])}$, by Lemma 1. Indeed, we have

$$\operatorname{spec}\left(T_b|_{[0,7]}\right) = \{1, 1/2, 1/2, -1/2, 1/4, 1/4, 1/4, -1/4\} .$$

We may identify $\ell([0, 7])$ with $\mathbb{C}^8$. By computation we find that the vector

$$v := [0, 1, 3, 5, 5, 3, 1, 0]^T/18$$

satisfies $T_b v = v$ and $\sum_{\alpha \in \mathbb{Z}} v(\alpha) = 1$. It follows that

$$\nabla v = [0, 1, 2, 2, 0, -2, -2, -1]^T/18 .$$

A simple computation yields

$$w_0 := A_0(\nabla v) = [0, 1, 1, 1, -1, -1, -1, 0]^T/18$$

and

$$w_1 := A_1(\nabla v) = [1, 1, 1, -1, -1, -1, 0, 0]^T/18 .$$

Moreover,

$$A_0 \begin{bmatrix} \nabla v \\ w_0 \\ w_1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \nabla v \\ w_0 \\ w_1 \end{bmatrix}$$

and

$$A_1 \begin{bmatrix} \nabla v \\ w_0 \\ w_1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \nabla v \\ w_0 \\ w_1 \end{bmatrix}.$$

The above two 3×3 matrices are triangular, by [10, Lemma 4.8] and [16, Lemma 4.2] we obtain

$$\rho_p\left(A_0|_{V(\nabla v)}, A_1|_{V(\nabla v)}\right) = 1, \qquad 1 \le p \le \infty .$$

Since $1 < 2^{1/p}$ for $1 \le p < \infty$, by Lemma 3 we conclude that ϕ lies in $L_p(\mathbb{R})$. But ϕ is not continuous. □

The following example shows that the invariant subspace $V(\nabla v)$ in Theorem 3 may be a proper subspace of U given by (1.4), even if the shifts of ϕ are stable.

Example 3. Let a be the sequence on $\mathbb{Z}$ given by its symbol

$$\tilde{a}(z) = \frac{15}{16} + \frac{3}{4}z + \frac{1}{8}z^2 + \frac{1}{4}z^3 - \frac{1}{16}z^4 .$$

Let ϕ be the normalized solution of the refinement equation associated with the mask a. Then ϕ is continuous. Choose $m = 1$ in Theorem 3. Then $\dim V(\nabla v) = 2$. Hence $V(\nabla v)$ is a proper subspace of U given by (1.4). Moreover, the shifts of ϕ are stable.

Proof. The mask a is supported on $[0, 4]$. A simple computation gives

$$\operatorname{spec}\left(T_a|_{\ell([0,4])}\right) = \{1, 15/16, 1/8, -1/16, 0\} .$$

If the sequence b is given by $\tilde{b}(z) = \tilde{a}(z)(1+z)/2$, then 1 is a simple eigenvalue of $T_b|_{\ell([0,5])}$, by Lemma 1. Indeed, we have

$$\operatorname{spec}\left(T_b|_{[0,5]}\right) = \{1, 1/2, 15/32, 1/16, -1/32, 0\} .$$

We may identify $\ell([0,5])$ with $\mathbb{C}^6$. By computation we find that the vector

$$v := [0, 3/4, 1/4, 0, 0, 0]^T$$

satisfies $T_b v = v$ and $\sum_{\alpha\in\mathbb{Z}} v(\alpha) = 1$. It follows that

$$\nabla v = [0, 3/4, -1/2, -1/4, 0, 0]^T .$$

A simple computation yields

$$A_0(\nabla v) = \frac{1}{8}\nabla v$$

and

$$A_1(\nabla v) = [45/64, -33/64, -13/64, 1/64, 0, 0]^T .$$

Moreover, $\operatorname{span}\{\nabla v, A_1(\nabla v)\}$ is invariant under A_0 and A_1. This is the subspace $V(\nabla v)$. If we choose a basis as $\{\nabla v, w := 8A_1(\nabla v)\}$, then

$$A_0 \begin{bmatrix} \nabla v \\ w \end{bmatrix} = \begin{bmatrix} 1/8 & 0 \\ -1/16 & 15/16 \end{bmatrix} \begin{bmatrix} \nabla v \\ w \end{bmatrix}$$

and

$$A_1 \begin{bmatrix} \nabla v \\ w \end{bmatrix} = \begin{bmatrix} 0 & 1/8 \\ 1/16 & 1/16 \end{bmatrix} \begin{bmatrix} \nabla v \\ w \end{bmatrix} .$$

Take the norm of the above two 2×2 matrices as the maximum of the sums of each column. Then

$$\rho_\infty \left(A_0|_{V(\nabla v)} , A_1|_{V(\nabla v)} \right) \leq \max \left\{ \| A_0|_{V(\nabla v)} \| , \| A_1|_{V(\nabla v)} \| \right\} = 15/16 .$$

Since $\rho_\infty(A_0|_{V(\nabla v)}, A_1|_{V(\nabla v)}) < 1$, by Lemma 3 we conclude that ϕ is continuous. Note that $\dim V(\nabla v) = 2$, while $\dim U = 3$ in this example. Then $V(\nabla v)$ is a proper subspace of U.

Observe that the symbol of the mask can be factorized as

$$\tilde{a}(z) = (1+z)(5-z)\left(3+z^2\right)/16 .$$

By the criteria on stability and linear independence in [18], the shifts of ϕ are stable, but linearly dependent. □

4. Characterization of L_p Regularity

In this section we give a characterization for the regularity of a refinable function in terms of the corresponding refinement mask.

Let $a \in \ell_0(\mathbb{Z})$ be a refinement mask such that $\sum_{\alpha\in\mathbb{Z}} a(\alpha) = 2$. Recall that A_ε ($\varepsilon = 0, 1$) are the linear operators defined in (1.3).

Theorem 4.

Suppose the normalized solution ϕ of the refinement equation (1.1) *with mask a lies in $L_p(\mathbb{R})$. Let m be a positive integer such that the transition operator T_b induced by $\tilde{b}(z) := \tilde{a}(z)(1+z)^m/2^m$ has 1 as its simple eigenvalue. Let v be the element in $\ell_0(\mathbb{Z})$ such that $T_b v = v$ and $\sum_{\alpha\in\mathbb{Z}} v(\alpha) = 1$. Let k be the smallest positive integer such that*

$$k > 1/p - \log_2 \rho_p \left(A_0|_{V(\nabla^k v)} , A_1|_{V(\nabla^k v)} \right) ,$$

where $V(\nabla^k v)$ denotes the minimal common invariant subspace of A_0 and A_1 generated by $\nabla^k v$. Then

$$\nu_p(\phi) = 1/p - \log_2 \rho_p \left(A_0|_{V(\nabla^k v)} , A_1|_{V(\nabla^k v)} \right) . \tag{4.1}$$

Proof. Let us prove the theorem for $1 \leq p < \infty$. The case $p = \infty$ can be treated similarly.

Write $\rho_{p,k}$ for $\rho_p\big(A_0|_{V(\nabla^k v)}, A_1|_{V(\nabla^k v)}\big)$. Since $\nabla^{k+1} v = \nabla^k v - \nabla^k v(\cdot - 1)$, we have $\rho_{p,k+1} \leq \rho_{p,k}$ for $k = 1, 2, \ldots$. We shall first establish the following fact:

$$1/p - \log_2 \rho_{p,k} > k - 1 \Longrightarrow \nu_p(\phi) \geq 1/p - \log_2 \rho_{p,k} \qquad \forall k \in \mathbb{N} . \tag{4.2}$$

Fix a positive integer k. Suppose $1/p - \log_2 \rho_{p,k} > k - 1$. Let w be the element in $\ell_0(\mathbb{Z})$ given by

$$w(\beta) = 2^{1-k-m} \sum_{j=0}^{k+m-1} \binom{k+m-1}{j} v(\beta - j), \qquad \beta \in \mathbb{Z} .$$

Then the symbol of w is $\tilde{w}(z) := \tilde{v}(z)(1+z)^{k+m-1}/2^{k+m-1}$. By Lemma 4 we have $A_0(\nabla^{k-1} w) = T_a(\nabla^{k-1} w) = 2^{1-k}\nabla^{k-1} v$. The initial function ϕ_0 is chosen as

$$\phi_0(x) := \sum\nolimits_{\alpha\in\mathbb{Z}} w(\alpha) B_k(x - \alpha), \qquad x \in \mathbb{R} ,$$

where B_k is the cardinal B-spline of order k. For $n = 1, 2, \ldots$, let $\phi_n := Q_a^n \phi_0$. It was proved in Section 3 that

$$\phi_n(x) = \sum\nolimits_{\alpha\in\mathbb{Z}} (a_n * w)(\alpha) B_k\left(2^n x - \alpha\right) , \tag{4.3}$$

where a_n is given by (3.4). From the proof of Theorem 3 we see that $(\phi_n)_{n=1,2,\ldots}$ converges to ϕ in the L_p-norm. Differentiating both sides of (4.3) $k-1$ times, we obtain

$$\phi_n^{(k-1)}(x) = 2^{n(k-1)} \sum\nolimits_{\alpha\in\mathbb{Z}} \left(a_n * \left(\nabla^{k-1} w\right)\right)(\alpha) B_1\left(2^n x - \alpha\right) , \tag{4.4}$$

where we have used the fact $B_k^{(k-1)} = \nabla^{k-1} B_1$.

As was done in the proof of Theorem 3, the following estimate can be derived from (4.4):

$$\left\|\phi_{n+1}^{(k-1)} - \phi_n^{(k-1)}\right\|_p \le C_1 2^{n(k-1)-n/p} \left(\left\|\mathcal{A}^n w_0\right\|_p + \left\|\mathcal{A}^n w_1\right\|_p\right) , \tag{4.5}$$

where C_1 is a positive constant independent of n,

$$w_0 := 2^{k-1} A_0\left(\nabla^{k-1} w\right) - \nabla^{k-1} w \quad \text{and} \quad w_1 := 2^{k-1} A_1\left(\nabla^{k-1} w\right) - \nabla^{k-1} w .$$

Let W be the linear span of $\{\nabla^k v(\cdot - j) : j \in \mathbb{Z}\}$. For $w \in W$, we have the following estimate similar to (3.5):

$$\left\|\mathcal{A}^n w\right\|_p \le C_2 \left\|\mathcal{A}^n|_{V(\nabla^k v)}\right\|_p \qquad \forall n \in \mathbb{N} . \tag{4.6}$$

Since $2^{k-1} A_0(\nabla^{k-1} w) = \nabla^{k-1} v$, we have

$$w_0 = \nabla^{k-1}(v - w) = 2^{1-k-m} \sum_{j=0}^{k+m-1} \binom{k+m-1}{j} \left[\nabla^{k-1} v - \nabla^{k-1} v(\cdot - j)\right] .$$

This shows that w_0 lies in W. Furthermore,

$$\begin{aligned} w_1 &= 2^{k-1}\left(A_1 - A_0\right) \nabla^{k-1} w + 2^{k-1} A_0\left(\nabla^{k-1} w\right) - \nabla^{k-1} w \\ &= 2^{k-1} A_0\left(\nabla^{k-1} w(\cdot + 1) - \nabla^{k-1} w\right) + \left(\nabla^{k-1} v - \nabla^{k-1} w\right) . \end{aligned}$$

Clearly, $\nabla^{k-1} w(\cdot + 1) - \nabla^{k-1} w$ lies in W. Note that $\rho_{p,k} = \rho_p(A_0|_{V(\nabla^k v)}, A_1|_{V(\nabla^k v)})$. For given $\varepsilon > 0$, in light of (4.6), there exists a positive constant C_3 such that

$$\left\|\mathcal{A}^n w_0\right\|_p + \left\|\mathcal{A}^n w_1\right\|_p \le C_3\left(\rho_{p,k} + \varepsilon\right)^n \qquad \forall n = 1, 2, \ldots . \tag{4.7}$$

Combining (4.5) and (4.7) together, we see that there exists a positive constant C independent of n such that

$$\left\|\phi_{n+1}^{(k-1)} - \phi_n^{(k-1)}\right\|_p \le C 2^{n(k-1-1/p)} \left(\rho_{p,k} + \varepsilon\right)^n = C 2^{-\mu n} , \tag{4.8}$$

where $\mu := 1/p - (k-1) - \log_2(\rho_{p,k} + \varepsilon)$. Since $1/p - \log_2 \rho_{p,k} > k - 1$, we have $\mu > 0$ if $\varepsilon > 0$ is sufficiently small. Thus, (4.8) tells us that $(\phi_n^{(k-1)})_{n=1,2,\ldots}$ is a Cauchy sequence in $L_p(\mathbb{R})$. There exists $g \in L_p(\mathbb{R})$ such that $\|\phi_n^{(k-1)} - g\|_p \to 0$ as $n \to \infty$. But $(\phi_n)_{n=1,2,\ldots}$ converges to ϕ in $L_p(\mathbb{R})$, so we must have $g = \phi^{(k-1)}$.

We claim that $\phi^{(k-1)}$ belongs to Lip*$(\mu, L_p(\mathbb{R}))$. To justify our claim, we deduce from (4.4) that

$$\nabla_{2^{-n}}\phi_n^{(k-1)}(x) = 2^{n(k-1)} \sum_{\alpha\in\mathbb{Z}} \left(a_n * \left(\nabla^k w\right)\right)(\alpha) B_1\left(2^n x - \alpha\right) .$$

In what follows, C_1, C_2, C_3, and C denote positive constants independent of n. Since $\nabla^k w$ lies in W, by (4.6) the following estimate is valid:

$$\left\| \nabla_{2^{-n}}\phi_n^{(k-1)} \right\|_p \le C_1 2^{-\mu n} . \tag{4.9}$$

It follows from (4.8) and (4.9) that

$$\begin{aligned} \left\| \nabla_{2^{-n}}\phi^{(k-1)} \right\|_p &= \left\| \phi^{(k-1)} - \phi^{(k-1)}(\cdot - 1/2^n) \right\|_p \\ &\le \left\| \nabla_{2^{-n}}\phi_n^{(k-1)} \right\|_p + 2\sum_{j=n}^{\infty} \left\| \phi_{j+1}^{(k-1)} - \phi_j^{(k-1)} \right\|_p \\ &\le C_1 2^{-\mu n} + 2C_2 \sum_{j=n}^{\infty} 2^{-\mu j} \le C_3 2^{-\mu n} . \end{aligned}$$

Suppose

$$h = \frac{1}{2^n} + \frac{d_1}{2^{n+1}} + \frac{d_2}{2^{n+2}} + \cdots ,$$

where $d_1, d_2, \cdots \in \{0, 1\}$. By what has been proved, we have

$$\left\| \phi^{(k-1)} - \phi^{(k-1)}(\cdot - h) \right\|_p \le C_3 \sum_{j=n}^{\infty} 2^{-j\mu} \le C h^{\mu} .$$

This shows that $\phi^{(k-1)}$ belongs to Lip*(μ, L_p). Hence,

$$\nu_p(\phi) \ge k - 1 + \mu = 1/p - \log_2 \left(\rho_{p,k} + \varepsilon\right) .$$

Letting $\varepsilon \to 0$ in the above inequality, we obtain $\nu_p(\phi) \ge 1/p - \log_2 \rho_{p,k}$, as desired.

From the preceding discussion we see that $1/p - \log_2 \rho_{p,k} \ge k$ implies $\nu_p(\phi) \ge k$. But $\nu_p(\phi) < \infty$. Hence, there exists a positive integer k such that

$$k > 1/p - \log_2 \rho_{p,k} .$$

Let k be the smallest positive integer satisfying the above inequality. We claim that $\nu_p(\phi) \ge 1/p - \log_2 \rho_{p,k}$. By our assumption, $\phi \in L_p(\mathbb{R})$; hence $1/p - \log_2 \rho_{p,1} > 0 = 1 - 1$, by Theorem 3. Thus, if $k = 1$, it follows from (4.2) that $\nu_p(\phi) \ge 1/p - \log_2 \rho_{p,1}$. Suppose $k \ge 2$. Then by the very definition of k we have $1/p - \log_2 \rho_{p,k-1} \ge k - 1$. If $\rho_{p,k} < \rho_{p,k-1}$, then

$$1/p - \log_2 \rho_{p,k} > 1/p - \log_2 \rho_{p,k-1} \ge k - 1 .$$

Hence, by (4.2) we obtain $\nu_p(\phi) \ge 1/p - \log_2 \rho_{p,k}$. Otherwise, we have $\rho_{p,k} = \rho_{p,k-1}$. Since $1/p - \log_2 \rho_{p,k-1} > (k-1) - 1$, in light of (4.2) we have

$$\nu_p(\phi) \ge 1/p - \log_2 \rho_{p,k-1} = 1/p - \log_2 \rho_{p,k} .$$

In order to complete the proof of (4.1), it remains to prove $\nu_p(\phi) \le 1/p - \log_2 \rho_{p,k}$. Suppose to the contrary that $\nu_p(\phi) > 1/p - \log_2 \rho_{p,k}$. Then there exists μ such that

$$\min\left\{k, \nu_p(\phi)\right\} > \mu > 1/p - \log_2 \rho_{p,k} . \tag{4.10}$$

It follows from $\nu_p(\phi) > \mu$ that $\phi \in$ Lip*$(\mu, L_p(\mathbb{R}))$. Consequently,

$$\left\| \nabla_{2^{-n}}^k \phi \right\|_p \le C_1 2^{-n\mu} .$$

From the proof of Theorem 3 we have $v(\alpha) = (\phi * B_m)(\alpha)$ for all $\alpha \in \mathbb{Z}$. Hence, by Lemma 3 we obtain

$$\left\| a_n * \left(\nabla^k v\right) \right\|_p \le 2^{n/p} \left\| \nabla^k_{2^{-n}} \phi \right\|_p \le C_2 2^{n(1/p-\mu)} .$$

By Lemma 2 we have $\|\mathcal{A}^n(\nabla^k v)\|_p = \|a_n * (\nabla^k v)\|_p$. Therefore,

$$\rho_p\left(A_0|_{V(\nabla^k v)}, A_1|_{V(\nabla^k v)}\right) = \lim_{n\to\infty} \left\| \mathcal{A}^n \left(\nabla^k v\right) \right\|_p^{1/n} \le 2^{1/p-\mu} .$$

In other words, $\log_2 \rho_{p,k} \le 1/p - \mu$, which contradicts (4.10). The proof of Theorem 4 is complete. □

The L_p regularity of refinable functions was considered by many authors. Usually, it is assumed that the shifts of the refinable function ϕ are stable. See [14] and [18] for discussions on stability of compactly supported functions. Under the stability condition on ϕ, Villemoes [26] employed the factorization technique to provide a characterization for the L_p regularity of the refinable function ϕ in terms of the spectral radius of a certain subdivision operator associated to the corresponding mask a. In [13] Jia used the p-norm joint spectral radius to investigate the L_p regularity of ϕ. Without stability condition imposed on ϕ, Lau and Ma in [21] provided a characterization of the L_p regularity of ϕ in terms of the mask. But they still required that 1 be a *simple* eigenvalue of the transition operator T_a. Example 2 demonstrates that this condition might fail to hold in some cases.

The case $p = 2$ is of particular interest. It was observed by Deslauriers and Dubuc [5] that the optimal Lipschitz exponent of a refinable function ϕ can be computed by calculating the spectral radius of a certain finite matrix associated with the mask a, provided the symbol of the mask is non-negative. Their idea was employed by Eirola [7] and Villemoes [26] to give a formula for $\nu_2(\phi)$ when the shifts of ϕ are stable. In [8], Goodman, Micchelli, and Ward established a formula for the spectral radius of the subdivision operator S_a in $\ell_2(\mathbb{Z})$. Lau, Ma, and Wang [22] gave sharp estimates for the L_2 regularity of refinable functions. Recently, motivated by the work of Lau and Ma in [21], Zhou [31] showed that the 2-norm joint spectral radius of a finite collection of square matrices is equal to the spectral radius of a certain finite matrix derived from the given matrices.

Here we give a brief discussion on the L_2 regularity of a refinable function without the stability condition. Our discussion is based on the work [9]. For two elements u, v in $\ell_0(\mathbb{Z})$, $u \odot v$ is the sequence on $\mathbb{Z}$ given by

$$u \odot v(\alpha) := \sum_{\beta\in\mathbb{Z}} u(\alpha+\beta)\overline{v(\beta)}, \qquad \alpha \in \mathbb{Z} .$$

Let ϕ be the normalized solution of the refinement equation (1.1) with mask a. Define $c := a \odot a/2$. Then T_c is the transition operator associated with c. Let v be the element in $\ell_0(\mathbb{Z})$ as given in Theorem 4. For a positive integer k, let W_k be the minimal invariant subspace of T_c generated by $\Delta^k w$, where $w := v \odot v$. Then we have

$$\nu_2(\phi) = -\frac{1}{2}\log_2 \rho\left(T_c|_{W_k}\right) ,$$

provided $k > -\log_2 \rho(T_c|_{W_k})/2$.

Let us show the applicability of our characterization on the regularity without assuming stability by a simple example.

Example 4. Let a be the sequence on $\mathbb{Z}$ given by its symbol

$$\tilde{a}(z) = \frac{3}{4} + \frac{1}{2}z + \frac{1}{2}z^2 + \frac{1}{2}z^3 - \frac{1}{4}z^4 .$$

Let ϕ be the normalized solution of the refinement equation associated with the mask a. Then ϕ is continuous. The shifts of ϕ are not stable. The critical exponent of ϕ is given by

$$\nu_p(\phi) = 2 + \frac{1}{p} - \frac{1}{p}\log_2\left(3^p + 1\right), \qquad 1 \le p \le \infty .$$

Proof. The method is the same as that in Example 3. The mask a is supported on [0, 4] and

$$\operatorname{spec}\left(T_a|_{\ell([0,4])}\right) = \{1, 3/4, 1/2, -1/4, 0\}.$$

If the sequence b is given by $\tilde{b}(z) = \tilde{a}(z)(1+z)/2$, then by Lemma 1, 1 is a simple eigenvalue of $T_b|_{\ell([0,5])}$ and

$$\operatorname{spec}\left(T_b|_{[0,5]}\right) = \{1, 1/2, 3/8, 1/4, -1/8, 0\}.$$

We may identify $\ell([0, 5])$ with $\mathbb{C}^6$. By computation we find that the vector

$$v := [0, 1/2, 1/2, 0, 0, 0]^T$$

satisfies $T_b v = v$ and $\sum_{\alpha\in\mathbb{Z}} v(\alpha) = 1$. It follows that

$$\nabla v = [0, 1/2, 0, -1/2, 0, 0]^T.$$

A simple computation yields

$$A_0(\nabla v) = \frac{1}{2}\nabla v$$

and

$$A_1(\nabla v) = [3/8, -1/8, -3/8, 1/8, 0, 0]^T.$$

Moreover, span$\{\nabla v, A_1(\nabla v)\}$ is invariant under A_0 and A_1. This is the subspace $V(\nabla v)$. If we choose a basis as $\{\nabla v, w := A_1(\nabla v) - \nabla v/2\}$, then

$$A_0\begin{bmatrix}\nabla v\\ w\end{bmatrix} = \begin{bmatrix}1/2 & 0\\ 0 & 3/4\end{bmatrix}\begin{bmatrix}\nabla v\\ w\end{bmatrix}$$

and

$$A_1\begin{bmatrix}\nabla v\\ w\end{bmatrix} = \begin{bmatrix}1/2 & 1\\ 0 & -1/4\end{bmatrix}\begin{bmatrix}\nabla v\\ w\end{bmatrix}.$$

The above two 2×2 matrices are triangular, by [10, Lemma 4.8] and [16, Lemma 4.2] we obtain

$$\rho_p\left(A_0|_{V(\nabla v)}, A_1|_{V(\nabla v)}\right) = \max\left\{2^{1/p-1}, \left(3^p+1\right)^{1/p}/4\right\} = \left(3^p+1\right)^{1/p}/4, \qquad 1\le p\le\infty.$$

In particular, $\rho_\infty(A_0|_{V(\nabla v)}, A_1|_{V(\nabla v)}) = 3/4 < 1$. By Lemma 3 we conclude that ϕ is continuous.

Note that $\tilde{a}(i) = \tilde{a}(-i) = 0$. Hence $\tilde{a}$ has symmetric zeros on the unit circle. The criterion on stability given in [18] tells us that the shifts of ϕ are not stable.

Take $k = 1$ in Theorem 4. We know that for $p > 1$,

$$1/p - \log_2\rho_p\left(A_0|_{V(\nabla v)}, A_1|_{V(\nabla v)}\right) < 1.$$

Then Theorem 4 shows that

$$\nu_p(\phi) = 2 + \frac{1}{p} - \frac{1}{p}\log_2\left(3^p+1\right), \qquad 1 < p \le \infty.$$

Notice that $0 < \nu_p(\phi) < 1$ for $1 < p \le \infty$. The case $p = 1$ follows from the argument in [17]. This proves all the conclusions. □

Note again that $\dim V(\nabla v) = 2$, while $\dim U = 3$ in this example. Then $V(\nabla v)$ is a proper subspace of U.

5. Vector Refinement Equations

The results of the previous sections can be extended to vector refinement equations. A vector refinement equation is a functional equation of the form

$$\phi(x) = \sum_{\alpha\in\mathbb{Z}} a(\alpha)\phi(2x-\alpha), \qquad x \in \mathbb{R}, \tag{5.1}$$

where $\phi = (\phi_1, \ldots, \phi_r)^T$ is an $r \times 1$ vector of functions on $\mathbb{R}$, and each $a(\alpha)$ is an $r \times r$ (complex) matrix.

The existence of compactly supported distributional solutions of the vector refinement equation (5.1) was discussed by Heil and Colella [11], and by Cohen, Daubechies, and Plonka [2]. Here we assume that the matrix $M := \sum_{\alpha\in\mathbb{Z}} a(\alpha)/2$ has 1 as its simple eigenvalue and does not have eigenvalues of the form 2^n for any positive integer n. Let y be a right eigenvector of M corresponding to eigenvalue 1. Then there exists a unique compactly supported distributional solution ϕ of (5.1) subject to the condition $\hat{\phi}(0) = y$. This result was established by Zhou [30] for the case $r = 2$, and by Jiang and Shen [20] for the general case.

Let $(\ell_0(\mathbb{Z}))^r$ denote the linear space of all finitely supported sequences of $r \times 1$ vectors. Similarly, we define $(\ell_0(\mathbb{Z}))^{r\times r}$ to be the linear space of all finitely supported sequences of $r \times r$ matrices. For $v \in (\ell_0(\mathbb{Z}))^r$, define $\nabla v := v - v(\cdot - 1)$.

Let a be an element of $(\ell_0(\mathbb{Z}))^{r\times r}$. For $\varepsilon \in \{0, 1\}$, let A_ε be the linear operator on $(\ell_0(\mathbb{Z}))^r$ given by

$$A_\varepsilon v(\alpha) = \sum_{\beta\in\mathbb{Z}} a(\varepsilon + 2\alpha - \beta)v(\beta), \qquad \alpha \in \mathbb{Z}, \quad v \in (\ell_0(\mathbb{Z}))^r . \tag{5.2}$$

By $V(v)$ we denote the minimal common invariant subspace of A_0 and A_1 generated by v. We write T_a for A_0 and call it the transition operator associated with a.

For $a \in (\ell_0(\mathbb{Z}))^{r\times r}$, its symbol is defined by

$$\tilde{a}(z) := \sum_{j\in\mathbb{Z}} a(j)z^j, \qquad z \in \mathbb{C}\setminus\{0\} .$$

Let b be the mask given by $\tilde{b}(z) = \tilde{a}(z)(1+z)^m/2^m$, where m is a positive integer. If m is sufficiently large, then the transition operator T_b has 1 as its simple eigenvalue.

Concerning L_p solutions and L_p regularity of solutions of vector refinement equations, we state the following two results. Their proofs are similar to those of Theorem 3 and Theorem 4.

Theorem 5.

Let m be a positive integer such that the transition operator T_b induced by $\tilde{b}(z) := \tilde{a}(z)(1+z)^m/2^m$ has 1 as its simple eigenvalue. Let y be a nonzero $r \times 1$ vector such that $My = y$. Then (5.1) *has a compactly supported L_p solution ϕ (continuous solution in the case $p = \infty$) with $\hat{\phi}(0) = y$ if and only if the element v in $(\ell_0(\mathbb{Z}))^r$ determined by $T_b v = v$ and $\sum_{\alpha\in\mathbb{Z}} v(\alpha) = y$ satisfies the following condition:*

$$\rho_p\left(A_0|_{V(\nabla v)}, A_1|_{V(\nabla v)}\right) < 2^{1/p} .$$

Theorem 6.

Suppose the conditions of Theorem 5 are satisfied. Let k be the smallest positive integer such that $k > 1/p - \log_2 \rho_p(A_0|_{V(\nabla^k v)}, A_1|_{V(\nabla^k v)})$. Then

$$\nu_p(\phi) = 1/p - \log_2 \rho_p\left(A_0|_{V(\nabla^k v)}, A_1|_{V(\nabla^k v)}\right) .$$

Under the stability condition on ϕ, Jia, Riemenschneider, and Zhou [17] characterized the L_p regularity of ϕ in terms of the p-norm joint spectral radius of A_0 and A_1 restricted to a certain common invariant subspace. Stability and linear independence of refinable vectors of functions were discussed by Hogan [12] and Wang [27]. However, it is still difficult to check stability or linear independence of a refinable vector of functions in terms of the corresponding mask. In Theorem 6, the L_p regularity of ϕ was characterized without any consideration of stability. Therefore, it has significant advantages over the previous results.

We demonstrate applicability of Theorems 5 and 6 by the following example.

Example 5. Let $a \in (\ell_0(\mathbb{Z}))^{2\times 2}$ be supported in $[0, 2]$ and given by

$$a(0) = \begin{bmatrix} 1/2 & s/2 \\ t & 1/4+2st \end{bmatrix}, \quad a(1) = \begin{bmatrix} 1 & 0 \\ 0 & 1/2 \end{bmatrix}, \quad a(2) = \begin{bmatrix} 1/2 & -s/2 \\ -t & 1/4+2st \end{bmatrix}, \tag{5.3}$$

where s, t are real parameters satisfying $t \neq 0$ and $1/2+2st \notin \{2^{n-1} : n \in \mathbb{N}\}$. Then $M = [a(0) + a(1) + a(2)]/2$ has 1 as its simple eigenvalue and an associated eigenvector $y = (1, 0)^T$. Moreover, M has no eigenvalues of the form 2^n, $n \in \mathbb{N}$. Let ϕ be the compactly supported distributional solution of the vector refinement equation (5.1) subject to the condition $\hat{\phi}(0) = y$. Then ϕ is in L_p (continuous in the case $p = \infty$) if and only if $-3/4 < st < 1/4$. In this case, the critical exponent of ϕ is given by

$$\nu_p(\phi) = \begin{cases} 2+\frac{1}{p} & \text{if } |st+1/4| \le 2^{-3-1/p} \text{ and } s \neq 0\,, \\ -\log_2\left|\frac{1}{2}+2st\right| & \text{if } 2^{-3-1/p} < |st+1/4| < 1/2 \text{ and } s \neq 0\,, \\ 1+1/p & \text{if } s = 0\,. \end{cases}$$

This example was discussed in [16] and [17]. Under the restriction $|st + 1/4| < 1/2$, i.e., $-3/4 < st < 1/4$, it was proved in [16, Example 6.3] that the subdivision scheme associated with a converges uniformly. Consequently, the solution ϕ is continuous. But the question was not answered whether the soltuion ϕ is in L_p if $|st + 1/4| \ge 1/2$. Under the restriction $-3/4 < st < 1/4$, the smoothness of ϕ was analyzed in [17, Example 4.2] by considering the stable and nonstable cases separately. In contrast, Theorem 5 and Theorem 6 enable us to give a unified approach for the existence and smoothness of the solution without the restriction on the parameters.

Proof. Let $m = 1$ and $\tilde{b}(z) = \tilde{a}(z)(1+z)/2$. Then 1 is a simple eigenvalue of T_b. The element $v \in (\ell_0(\mathbb{Z}))^2$ determined by $T_b v = v$ and $\sum_{\alpha\in\mathbb{Z}} v(\alpha) = y = (1, 0)^T$ is supported in $[1, 2]$ and given by

$$v(1) = \begin{bmatrix} 1/2 \\ 2t/3 \end{bmatrix}, \qquad v(2) = \begin{bmatrix} 1/2 \\ -2t/3 \end{bmatrix}.$$

Since a is supported in $[0, 2]$, $(\ell([0, 3]))^2$ is invariant under A_0 and A_1. By acting A_0 and A_1 iteratively on $\nabla v \in (\ell([0, 3]))^2$, we find that the elements ∇v, $A_0\nabla v$, $A_1\nabla v$, $A_0^2\nabla v$, $A_1^2\nabla v$ and $A_0A_1\nabla v$ generate an invariant subspace of A_0 and A_1. This is the subspace $V(\nabla v)$. The elements form a basis of $V(\nabla v)$ when $s \neq 0$, while the first five elements form a basis when $s = 0$.

Let us choose a basis of $V(\nabla v)$ (when $s \neq 0$) as

$$v_1 := \begin{bmatrix} 1 \\ 4t \end{bmatrix}\delta + \begin{bmatrix} -1 \\ 4t \end{bmatrix}\delta_1, \qquad v_2 := \begin{bmatrix} 0 \\ 1 \end{bmatrix}(\delta - \delta_1), \qquad v_3 := \begin{bmatrix} 1 \\ 0 \end{bmatrix}(\delta - \delta_1),$$

$$v_4 := -\begin{bmatrix} 1 \\ 4t \end{bmatrix}\delta + \begin{bmatrix} 2 \\ 0 \end{bmatrix}\delta_1 + \begin{bmatrix} -1 \\ 4t \end{bmatrix}\delta_2, \qquad v_5 := \begin{bmatrix} 0 \\ 1 \end{bmatrix}(\delta_1 - \delta_2), \qquad v_6 := \nabla v\,.$$

Let $V = [v_1, v_2, v_3, v_4, v_5, v_6]$. Then we have

$$A_0 V = V \begin{bmatrix} 1/2+2st & s/2 & 0 & 0 & -s/2 & -2st/3 \\ 0 & 1/4 & t & 0 & 0 & 2t \\ 0 & 0 & 1/2 & 0 & 0 & 1/2 \\ 0 & 0 & 0 & 1/2+2st & -s/2 & 1/2-2st/3 \\ 0 & 0 & 0 & 0 & 1/4 & 2t \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{5.4}$$

and

$$A_1 V = V \begin{bmatrix} 1/2+2st & -s/2 & 0 & 0 & s/2 & 2st/3 \\ 0 & 1/4 & -t & 0 & 1/4 & 5t/3 \\ 0 & 0 & 1/2 & 0 & 0 & 1/2 \\ 0 & 0 & 0 & 0 & 0 & 1/4+st/3 \\ 0 & 0 & 0 & 0 & 0 & t/3 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}. \tag{5.5}$$

When $s \neq 0$, the matrix representations of $A_0|_{V(\nabla v)}$ and $A_1|_{V(\nabla v)}$ under the basis $\{v_j\}_{j=1}^6$ are the matrices on the right side of (5.4) and (5.5). When $s = 0$, the matrix representations under the basis $\{v_j\}_{j=2}^6$ are the submatrices by deleting the first rows and columns of these matrices. Thus we have

$$\rho_p\left(A_0|_{V(\nabla v)}, A_1|_{V(\nabla v)}\right) = \begin{cases} \max\left\{2^{1/p}|1/2+2st|, 2^{1/p-1}\right\}, & \text{if } s \neq 0, \\ \max\left\{|1/2+2st|, 2^{1/p-1}\right\} = 2^{1/p-1}, & \text{if } s = 0. \end{cases}$$

Therefore, by Theorem 5, the vector refinement equation (5.1) has a compactly supported L_p solution ϕ (continuous solution in the case $p = \infty$) with $\hat{\phi}(0) = y$ if and only if $|1/2+2st| < 1$ when $s \neq 0$, i.e., $-3/4 < st < 1/4$.

To analyze the smoothness of ϕ, we take $k = 3$ and generate the subspace $V(\nabla^3 v)$. By acting A_0 and A_1 iteratively on $\nabla^3 v$, we see that the linear span of the vectors $\{\nabla^3 v, A_0\nabla^3 v, A_1\nabla^3 v, A_0^2\nabla^3 v, A_0A_1\nabla^3 v, A_1^2\nabla^3 v\}$ is invariant under both A_0 and A_1. This is $V(\nabla^3 v)$. Moreover, these 6 vectors form a basis of $V(\nabla^3 v)$, while $A_0^2\nabla^3 v = 0$ and the remaining 5 vectors form a basis when $s = 0$.

Choose the following vectors

$$w_1 := \begin{bmatrix} 1 \\ 4t \end{bmatrix}\delta + \begin{bmatrix} -1 \\ 4t \end{bmatrix}\delta_1, \qquad w_2 := \begin{bmatrix} 1 \\ 4t \end{bmatrix}\delta + \begin{bmatrix} -2 \\ 0 \end{bmatrix}\delta_1 + \begin{bmatrix} 1 \\ -4t \end{bmatrix}\delta_2, \qquad w_3 := e_2\nabla^2\delta,$$
$$w_4 := e_1\nabla^3\delta, \qquad w_5 := e_2\nabla^3\delta, \qquad w_6 := \nabla^3 v.$$

Then $\{w_j\}_{j=1}^6$ form a basis of $V(\nabla^3 v)$ when $s \neq 0$, while $\{w_j\}_{j=2}^6$ form a basis when $s = 0$. Moreover, with $W = [w_1, w_2, w_3, w_4, w_5, w_6]$, we have

$$A_0 W = W \begin{bmatrix} 1/2+2st & 0 & s & 0 & 2s & -8st/3 \\ 0 & 1/2+2st & -s/2 & 1/2 & -3s/2 & 4st \\ 0 & 0 & 1/4 & -t & 1/4 & -4t/3-32st^2/3 \\ 0 & 0 & 0 & 0 & 0 & -4st/3 \\ 0 & 0 & 0 & 0 & 0 & 4t/3+16st^2/3 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{5.6}$$

and

$$A_1 W = W \begin{bmatrix} 1/2+2st & 0 & -s & 0 & -2s & 8st/3 \\ 0 & 0 & 0 & -1/2 & s/2 & -8st/3 \\ 0 & 0 & 0 & -t & -1/4 & 4t/3+8st^2/3 \\ 0 & 0 & 0 & 0 & 0 & 1/4+st/3 \\ 0 & 0 & 0 & 0 & 0 & -2t/3-4st^2/3 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}. \tag{5.7}$$

When $s \neq 0$, the matrix representations of $A_0|_{V(\nabla^3 v)}$ and $A_1|_{V(\nabla^3 v)}$ under the basis $\{w_j\}_{j=1}^6$ are the matrices on the right side of (5.6) and (5.7). When $s = 0$, the matrix representations under the basis $\{w_j\}_{j=2}^6$ are the submatrices obtained by deleting the first row and the first column of each matrix. Thus we have

$$\rho_p\left(A_0|_{V(\nabla^3 v)}, A_1|_{V(\nabla^3 v)}\right) = \begin{cases} \max\left\{2^{1/p}|1/2+2st|, 1/4\right\}, & \text{if } s \neq 0, \\ \max\{|1/2+2st|, 1/4\} = 1/2, & \text{if } s = 0. \end{cases}$$

It follows that for $s \neq 0$, $p > 1$,

$$1/p - \log_2 \rho_p\left(A_0|_{V(\nabla^3 v)}, A_1|_{V(\nabla^3 v)}\right) \leq 1/p - \log_2(1/4) = 2 + 1/p < 3.$$

When $s = 0$, $1/p - \log_2 \rho_p\big(A_0|_{V(\nabla^3 v)}, A_1|_{V(\nabla^3 v)}\big) = 1 + 1/p < 3$. By Theorem 6, we have

$$\nu_p(\phi) = \begin{cases} 2 + \frac{1}{p} & \text{if } |st + 1/4| \leq 2^{-3-1/p}, p > 1 \text{ and } s \neq 0, \\ -\log_2\left|\frac{1}{2} + 2st\right| & \text{if } 2^{-3-1/p} < |st + 1/4| < 1/2, p > 1 \text{ and } s \neq 0, \\ 1 + 1/p & \text{if } s = 0. \end{cases}$$

The case $p = 1$ follows from the argument in [17]. This completes the smoothness analysis. □

When $s = 0$, the solution ϕ is not stable. This shows that Theorems 5 and 6 can be used to handle the existence and smoothness of unstable refinable vectors of functions.

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Received August 16, 1999

Revision received July 13, 2000

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