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THE REGULARITY OF L^p-SCALING FUNCTIONS*

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Abstract. The existence of L^p-scaling function and the L^p-Lipschitz exponent have been studied in [Ji] and [LW] and a criterion is given in terms of a series of product of matrices. In this paper we make some further study of the criterion. In particular we show that for p an even integer or for all $p \geq 1$ in some special cases, the criterion can be simplified to a computationally efficient form.

1. Introduction. The solution f of a 2-scale dilation equation

$$f(x) = \sum_{n=0}^{N} c_n f(2x-n), \quad x \in \mathbb{R} \tag{1.1}$$

is called a *scaling function*. This class of functions has been studied in detail in recent literature in connection with wavelet theory [D] and constructive approximations [DGL]. The question of existence of continuous, L^1 and L^2 solutions was treated in Daubechies [D], Daubechies and Lagarias [DL1], Collela and Heil [CH], Eirola [E], Heil [H], and Micchelli and Prautzsch [MP]. The regularity of such solutions was studied, in addition to the above papers, in Cohen and Daubechies [CD], Daubechies and Lagarias [DL2,3], Herve [He], Lau, Ma and Wang [LWM] and Villemos [V1,2]. Also the existence of L^p-solutions has been characterized by Lau and Wang in [LW] and Jia[Ji].

In this paper, we will adopt the previous notations as in [CH], [DL1] and [LW]. Let $T_0 = [c_{2i-j-1}]_{1\leq i,j\leq N}$ and $T_1 = [c_{2i-j}]_{1\leq i,j\leq N}$ be the associated matrices of the coefficients $\{c_n\}$, i.e.,

$$T_0 = \begin{pmatrix} c_0 & 0 & 0 & \dots & 0 \\ c_2 & c_1 & c_0 & \dots & 0 \\ c_4 & c_3 & c_2 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & c_{N-1} \end{pmatrix}, \quad T_1 = \begin{pmatrix} c_1 & c_0 & 0 & \dots & 0 \\ c_3 & c_2 & c_1 & \dots & 0 \\ c_5 & c_4 & c_3 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & c_N \end{pmatrix}.$$

It is known that if $\sum_{n=0}^{N} c_n = 2$, then 2 is always an eigenvalue of $(T_0 + T_1)$. Furthermore, if $\sum c_{2n} = \sum c_{2n+1} = 1$, then 1 is an eigenvalue of both T_0 and T_1. Let $\mathbf{v}$ be a 2-eignevector of $(T_0 + T_1)$ (which means a right eigenvector associated with the eigenvalue 2) and let

$$\tilde{\mathbf{v}} := (T_0 - I)\mathbf{v} = (I - T_1)\mathbf{v}.$$

In [LW] the following theorem was proved:

THEOREM A. *Suppose $1 \leq p < \infty$ and $\sum_{n=0}^{N} c_n = 2$. Then equation (1.1) has a nonzero compactly supported L^p-solution (notation: L^p_c-solution) if and only if there exists a 2-eigenvector $\mathbf{v}$ of $(T_0 + T_1)$ satisfying*

$$\lim_{n\to\infty} \frac{1}{2^n} \sum_{|J|=n} \|T_J \tilde{\mathbf{v}}\|^p = 0.$$

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In [Ji], Jai studied the same existence question by means of a "hat' function and obtained a similar criterion independently. Furthermore he showed that the existence of an L^p-solution implies that $\sum c_{2n} = \sum c_{2n+1} = 1$.

In this paper we will consider the regularity of the solution. We use the L^p-*Lipschitz exponent* to describe the regularity. It is defined by

$$\mathrm{Lip}_p(f) = \liminf_{h \to 0^+} \frac{\ln \|\Delta_h f\|_p}{\ln h},$$

where $\Delta_h f(x) = f(x+h) - f(x)$. It is well known that for $1 \le p < \infty$, if

$$\limsup_{h \to 0^+} \frac{1}{h} \|\Delta_h f\|_p < \infty$$

(which implies $\mathrm{Lip}_p(f) = 1$), then f' exists a.e. and is in L^p and f is the indefinite integral of f'. Recall that the q-Sobolev exponent of a function f is defined as

$$\sup\{\alpha : \int (1 + |\xi|^{q\alpha}) |\hat{f}(\xi)|^q d\xi < \infty\}.$$

For $p = q = 2$, the 2-Sobolev exponent equals to the L^2-Lipschitz exponent, and they are different when $p, q \ne 2$. In general the L^p-Lipschitz exponent describes the regularity of f more accurately than the Sobolev exponent. The q-Sobolev exponent has been studied in [He]. The L^p-Lipschitz exponent (in a slightly different terminology) has been used to investigate the multifractal structure of scaling functions in [DL3] and [J1,2].

Let $H(\tilde{\mathbf{v}})$ be the complex subspace spanned by $\{\, T_J \tilde{\mathbf{v}} : J \text{ is a multi-index}\,\}$. (We use complex scalar because it will be more convenient to deal with the complex eigenvalues and eigenvectors.).

THEOREM B. *Suppose that* $\sum c_{2n} = \sum c_{2n+1} = 1$ *and either (i)* 1 *is a simple eigenvalue of* T_0 *and* T_1 *or (ii)* $H(\tilde{\mathbf{v}}) = \{\, \mathbf{u} \in \mathbb{C}^N : \sum_{i=1}^N u_i = 0 \,\}$. *Then for* f *an* L^p_c*-solution of* (1.1), $1 \le p < \infty$,

$$\mathrm{Lip}_p(f) = \liminf_{n \to \infty} \frac{\ln(2^{-n} \sum_{|J|=n} \|T_J \tilde{\mathbf{v}}\|^p)}{p \ln(2^{-n})}. \tag{1.2}$$

We remark that Jia [Ji, Theorem 6.2] proved that the above formula by replacing $\|T_J \tilde{\mathbf{v}}\|$ with $\|T_J / H\|$ where H denoted the hyperplane in (ii). Our special perference on $\|T_J \tilde{\mathbf{v}}\|$ is that it allows us to calculate the sum in many cases (see Section 4 and 5). Even though there are some overlaps with Jia's result, we like to give a full proof of Theorem B because of completeness and the consistence of the development in the in the sequel.

To reduce the formula in Theorem B, we only consider the 4-coefficient dilation equation for simplicity. We show that if in addition $c_0 + c_3 = 1$, then

$$\mathrm{Lip}_p(f) = \frac{\ln((|c_0|^p + |1 - c_0|^p)/2)}{-p \ln 2},$$

(Proposition 4.3) and if $c_0 + c_3 = 1/2$, then

$$\mathrm{Lip}_p(f) = \min\{1,\ \frac{\ln((|c_0|^p + |\frac{1}{2} - c_0|^p)/2)}{-p \ln 2}\}.$$

(Proposition 4.5). Note that the second case contains Daubechies scaling function D_4. The formula was actually obtained in [DL3] using a different method and assuming further $1/2 < c_0 < 3/4$. For the general case we show that if p is an even integer, then

$$\sum_{|J|=n} \|T_J\tilde{\mathbf{v}}\|^p = \mathbf{a}W_p^n\mathbf{b} \tag{1.3}$$

for an auxillary $(p+1)\times(p+1)$ matrix W_p depends only on the coefficients of the dilation equation and for some vectors **a** and **b** (Proposition 5.1). In particular for $p = 2$, the matrix W_2 is equivalent to the transition matrix used in [CD], [LW] and [V] for the existence of L^2-scaling function. By using (1.3) it is easy to show that the necessary and sufficient condition in Theorem A reduces to $\rho(W_p) < 2$ and (1.2) becomes

$$\mathrm{Lip}_p(f) = \frac{\ln(\rho(W_p)/2)}{-p\ln 2}$$

where $\rho(W_p)$ is the spectral radius of W_p (Theorem 5.3).

The paper is organized as follows. In Section 2 we include some preliminary results concerning the eigen-properties of the matrices T_0, T_1 and $T_0 + T_1$ that we need. We give a complete proof of Theorem B in Section 3. In Section 4, we will apply Theorem B to obtain explicit expressions for the two special cases described above. Finally in Section 5 we construct the matrix W_p in (1.3) and use the spectral radius of W_p to determine $\mathrm{Lip}_p(f)$ when p is an even integer. We also make some remarks concerning extensions of the construction and discuss some unsolved questions.

2. Preliminaries. Throughout this paper, unless otherwise specified, we assume that $1 \le p < \infty$, $c_n \in \mathbb{R}$, c_0, $c_N \ne 0$ and $\sum c_{2n} = \sum c_{2n+1} = 1$. For any $k \ge 1$, let $J = (j_1, \dots, j_k)$, $j_i = 0$ or 1, be the multi-index and $|J|$ the length of J. Let $I_J = I_{(j_1,\dots,j_k)}$ be the dyadic interval $[0.j_1\cdots j_k,\, 0.j_1\cdots j_k + 2^{-k})$. The matrix T_J represents the product $T_{j_1}\cdots T_{j_k}$. If **v** is a 2-eigenvector of (T_0+T_1), it is clear that

$$\frac{1}{2^k}\sum_{|J|=k} T_J\mathbf{v} = \frac{1}{2^k}(T_0+T_1)^k\mathbf{v} = \mathbf{v}. \tag{2.1}$$

For any $g \in L^p(\mathbb{R})$ with support in $[0,\, N]$, let $\mathbf{g} : [0,1] \to \mathbb{R}^N$

$$\mathbf{g}(x) = [g(x),\, g(x+1),\, \dots,\, g(x+(N-1))]^t, \quad x \in [0,\, 1)$$

be the vector-valued function representing g and let

$$\mathbf{Tg}(x) = \begin{cases} T_0\mathbf{g}(2x) & \text{if } x \in [0,\, \frac{1}{2}), \\ T_1\mathbf{g}(2x-1) & \text{if } x \in [\frac{1}{2},\, 1). \end{cases}$$

It is easy to show that f is a solution of (1.1) if and only if $\mathbf{f} = \mathbf{Tf}$ [DL1]. With no confusion, we use $\|\cdot\|$ to denote the L^p-norm of g as well as the vector-valued function **g**. Also for a vector $\mathbf{u} \in \mathbb{R}^n$, $\|\mathbf{u}\|$ will denote the ℓ^p_N-norm in $\mathbb{R}^N$.

Let g_I be the average $|I|^{-1}\int_I g(x)\,dx$ of g on an interval I.

Proposition 2.1. *Let f be an L^p_c-solution of* (1.1) *and* $\mathbf{v} = [f_{[0,1)}, \dots, f_{[N-1,N)}]^t$ *be the vector defined by the average of f on the N subintervals. Then*

(i) **v** *is a 2-eigenvector of* (T_0+T_1).

(ii) *Let* $\mathbf{f}_0(x) = \mathbf{v}$, $x \in [0,\, 1)$, *and let* $\mathbf{f}_{n+1} = \mathbf{T}\mathbf{f}_n$, $n = 0,\, 1, \dots,$ *then*

$$\mathbf{f}_n(x) = \sum_{|J|=n} (T_J\mathbf{v})\chi_{I_J}(x), \qquad x \in [0,\, 1),$$

and $\|\mathbf{f}_{n+1} - \mathbf{f}_n\|^p = 2^{-n} \sum_{|J|=n} \|T_J\tilde{\mathbf{v}}\|^p$.

(iii) $\|\mathbf{f} - \mathbf{f}_n\|^p \le \frac{c}{2^n} \sum_{|J|=n} \|T_J\tilde{\mathbf{v}}\|^p$ *for some* $c > 0$ *and* $\mathbf{f}_n \to \mathbf{f}$ *in* $L^p([0,\, 1], \mathbb{R}^N)$.

Proof. The proof of these statements can be found in [LW, Proposition 2.3, Lemma 2.4, 2.5, and Theorem 2.6]. In particular to prove the last identity in (ii), we observe that

$$\begin{aligned}\|\mathbf{f}_{n+1} - \mathbf{f}_n\|^p =& \frac{1}{2^{n+1}} \sum_{|J|=n} \left(\|T_J(T_0 - I)\mathbf{v}\|^p + \|T_J(T_1 - I)\mathbf{v}\|^p\right)\\ =& \frac{1}{2^n} \sum_{|J|=n} \|T_J\tilde{\mathbf{v}}\|^p. \qquad \square\end{aligned}$$

Let

$$\alpha = \liminf_{n\to\infty} \frac{\ln(2^{-n} \sum_{|J|=n} \|T_J\tilde{\mathbf{v}}\|^p)}{\ln(2^{-n})}.$$

Then α is the rate of convergence of $2^{-n} \sum_{|J|=n} \|T_J\tilde{\mathbf{v}}\|^p$ to 0 in the sense that the sum is of order $o(2^{-\beta n})$ for any $\beta < \alpha$. Let $H(\tilde{\mathbf{v}})$ be the subspace (with complex scalar) spanned by $\{\, T_J\tilde{\mathbf{v}} : J$ is a multi-index $\}$.

LEMMA 2.2. *Under the same conditions and notations as in Proposition 2.1, for any* $\mathbf{u} \in H(\tilde{\mathbf{v}})$,

$$\liminf_{n\to\infty} \frac{\ln(2^{-n} \sum_{|J|=n} \|T_J\mathbf{u}\|^p)}{\ln(2^{-n})} \ge \alpha.$$

Furthermore equality holds if $H(\mathbf{u}) = H(\tilde{\mathbf{v}})$.

Proof. Since $H(\tilde{\mathbf{v}})$ is finite dimensional, it suffices to consider $\mathbf{u} = T_{J'}\tilde{\mathbf{v}}$ for some J'. Let $|J'| = k$, then

$$\frac{1}{2^n} \sum_{|J|=n} \|T_J\mathbf{u}\|^p = \frac{1}{2^n} \sum_{|J|=n} \|T_J T_{J'}\tilde{\mathbf{v}}\|^p \le 2^k \frac{1}{2^{n+k}} \sum_{|J|=n+k} \|T_J\tilde{\mathbf{v}}\|^p.$$

It follows that

$$\frac{\ln(2^{-n} \sum_{|J|=n} \|T_J\mathbf{u}\|^p)}{\ln(2^{-n})} \ge \frac{\ln(2^k)}{\ln(2^{-n})} + \frac{\ln(2^{-(n+k)} \sum_{|J|=n+k} \|T_J\tilde{\mathbf{v}}\|^p)}{\ln(2^{-(n+k)})},$$

which implies the stated inequality. For the last statement we need only change the roles of $\mathbf{u}$ and $\tilde{\mathbf{v}}$ and make use of the inequality we just proved. $\square$

Let $M = [c_{2i-j}]_{1\le i,j\le N-1}$, i.e.,

$$M = \begin{pmatrix} c_1 & c_0 & 0 & \dots & 0\\ c_3 & c_2 & c_1 & \dots & 0\\ c_5 & c_4 & c_3 & \dots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & 0 & \dots & c_{N-1} \end{pmatrix}$$

be the common submatrix of T_0 and T_1. If $\sum c_{2n} = \sum c_{2n+1} = 1$, then 1 is an eigenvalue of M and $[1,1,\dots,1]$ is the corresponding left 1-eigenvector. Let $H = \{\mathbf{u} \in \mathbb{C}^N : \sum_{i=1}^N u_i = 0\}$.

LEMMA 2.3. *There exist* $\mathbf{v}_0$, $\mathbf{v}_1 \notin H$ *(i.e.,* $\sum(\mathbf{v}_0)_i = \sum(\mathbf{v}_1)_i \neq 0$*) such that* $(T_0 - I)^m\mathbf{v}_0 = (T_1 - I)^m\mathbf{v}_1 = 0$ *for some* $m > 0$*, and*

$$T_0\mathbf{v}_1 = T_1\mathbf{v}_0. \tag{2.2}$$

Remark. When $m = 1$, $\mathbf{v}_0$ and $\mathbf{v}_1$ are 1-eigenvectors of T_0 and T_1 respectively.

Proof. Let $E_\lambda = \{\mathbf{u} \in \mathbb{C}^{N-1} : (M - \lambda I)^m\mathbf{u} = 0 \text{ for some } m > 0\}$. Observe that for $\lambda \neq 1$ and for $\mathbf{u} \in E_\lambda$,

$$0 = [1,1,\dots,1](M - \lambda I)^m\mathbf{u} = (1-\lambda)^m \sum_{i=1}^{N-1} u_i$$

for some $m > 0$, so that $\sum u_i = 0$. In view of $\mathbb{C}^{N-1} = E_1 \oplus \sum_{\lambda\neq 1} E_\lambda$, there exists $\mathbf{a} \in E_1$ such that $\sum a_i \neq 0$. If 1 is a simple eigenvalue of M, $\dim E_1 = 1$ and hence the above $\mathbf{a}$ is a 1-eigenvector of M. Let

$$\mathbf{v}_0 := [0,\, a_1, \dots, a_{N-1}]^t, \qquad \mathbf{v}_1 := [a_1, \dots, a_{N-1},\, 0]^t. \tag{2.3}$$

Then $\mathbf{v}_0$ and $\mathbf{v}_1$ are 1-eigenvectors of T_0 and T_1 respectively, and $\mathbf{v}_0$, $\mathbf{v}_1 \notin H$. Moreover, by the definitions of T_0 and T_1, we have

$$(T_0\mathbf{v}_1)_i = \sum c_{2i-j-1}a_j = \sum c_{2i-j}a_{j+1} = (T_1\mathbf{v}_0)_i. \tag{2.4}$$

so that $T_0\mathbf{v}_1 = T_1\mathbf{v}_0$. If 1 is not a simple eigenvalue of M, we let m be the smallest positive integer so that $(M - I)^m\mathbf{a} = 0$. Define $\mathbf{a}^{(1)} = \mathbf{a}, \cdots, \mathbf{a}^{(m)} = (M - I)^{m-1}\mathbf{a}$, and let

$$\mathbf{v}_0^{(i)} = [0, \mathbf{a}^{(i)}]^t \quad \text{and} \quad \mathbf{v}_1^{(i)} = [\mathbf{a}^{(i)}, 0]^t, \quad 1 \le i \le m. \tag{2.5}$$

Then $\mathbf{v}_j^{(1)} \notin H$ and $\mathbf{v}_j^{(m)}$ are eigenvectors of T_j, $j = 0, 1$ and

$$T_j\mathbf{v}_j^{(i)} = \mathbf{v}_j^{(i)} + \mathbf{v}_j^{(i+1)}, \quad 1 \le i \le m-1, \;\; j = 0, 1. \tag{2.6}$$

If we let $\mathbf{v}_0 = \mathbf{v}_0^{(1)}$ and $\mathbf{v}_1 = \mathbf{v}_1^{(1)}$, then a similar calculation like (2.4) implies that $T_0\mathbf{v}_1 = T_1\mathbf{v}_0$ again. □

COROLLARY 2.4. *Let* $\mathbf{v}_0$, $\mathbf{v}_1$ *be chosen as in the proof of Lemma 2.3, Then*

(i) $T_0\mathbf{v}_1^{(i)} = T_1\mathbf{v}_0^{(i)}$ *for* $1 \le i \le m$.

(ii) $T_1T_0^{k-1}\mathbf{v}_0 = T_0T_1^{k-1}\mathbf{v}_1$ *for* $k \ge 1$.

(iii) $(T_0^n\mathbf{v}_0)_1 = (T_1^n\mathbf{v}_1)_N = 0$ *and* $(T_0^n\mathbf{v}_0)_i = (T_1^n\mathbf{v}_1)_{i-1}$ *for* $2 \le i \le N$.

Proof. (i) and (ii) follows directly from the same calculation as in the proof of the above lemma. The first identity in (iii) is a consequence of $(\mathbf{v}_0)_1 = (\mathbf{v}_1)_N = 0$ as in (2.3). For the second identity, if $\mathbf{v}_0$ and $\mathbf{v}_1$ are 1-eigenvectors of T_0 and T_1 respectively, (2.2) implies that

$$(T_0^n\mathbf{v}_0)_i = (\mathbf{v}_0)_i = (\mathbf{v}_1)_{i-1} = (T_1^n\mathbf{v}_1)_{i-1}.$$

For the general case we need only apply

$$T_j^n\mathbf{v}_j^{(1)} = \begin{cases} \sum_{i=0}^{n} \binom{n}{i}\mathbf{v}_j^{(i+1)} & \text{if } n < m \\ \sum_{i=0}^{m-1} \binom{n}{i}\mathbf{v}_j^{(i+1)} & \text{if } n \ge m \end{cases}$$

which can be checked directly by using (2.6). □

LEMMA 2.5. *Let* $\mathbf{v}$ *be a 2-eigenvector of* (T_0+T_1) *and* $\tilde{\mathbf{v}} = (T_0 - I)\mathbf{v}$. *Then* $H(\tilde{\mathbf{v}})$ *is a subspace of* H. *Moreover, if (i)* 1 *is a simple eigenvalue of* T_0 *and* T_1*; or (ii)* $H(\tilde{\mathbf{v}}) = H$, *then for* $\mathbf{v}_0$, $\mathbf{v}_1 \notin H$ *as defined in Lemma 2.3, there exists a constant* c *such that*

$$\mathbf{v} = c\mathbf{v}_0 + \mathbf{h}_0 = c\mathbf{v}_1 + \mathbf{h}_1$$

for some $\mathbf{h}_0$, $\mathbf{h}_1 \in H(\tilde{\mathbf{v}})$.

Proof. Note that $[1, 1, \ldots, 1]^t$ is a left 1-eigenvector of T_0, so that $(T_0 - I)\mathbf{u} \in H$ for every $\mathbf{u} \in \mathbb{C}^n$. In particular, $\tilde{\mathbf{v}} = (T_0 - I)\mathbf{v}$ must be in H. Also it is easy to show that H is invariant under T_0 and T_1, hence $H(\tilde{\mathbf{v}})$ is a subspace of H. Let $\mathbf{v}_0$ and $\mathbf{v}_1$ be as in Lemma 2.3, then $a = \sum(\mathbf{v}_0)_i = \sum(\mathbf{v}_1)_i \neq 0$. Let $c = \sum_{i=1}^N v_i/a$, where v_i's are the coordinates of $\mathbf{v}$ and let

$$\mathbf{h}_0 = \mathbf{v} - c\mathbf{v}_0 \quad \text{and} \quad \mathbf{h}_1 = \mathbf{v} - c\mathbf{v}_1.$$

By the choice of c, we have $\mathbf{h}_0$, $\mathbf{h}_1 \in H$ which implies case (ii) because $H = H(\tilde{\mathbf{v}})$. In case (i), we observe that if 1 is a simple eigenvalue of T_0, then $T_0 - I$ restricted on H is bijective; it is hence also bijective on the $(T_0 - I)$-invariant subspace $H(\tilde{\mathbf{v}})$. Consequently,

$$\tilde{\mathbf{v}} = (T_0 - I)\mathbf{v} = (T_0 - I)(c\mathbf{v}_0 + \mathbf{h}_0) = (T_0 - I)\mathbf{h}_0$$

so that $\mathbf{h}_0$ must be in $H(\tilde{\mathbf{v}})$. The same proof holds for $\mathbf{h}_1$. □

3. Proof of Theorem B. Let f be an L^p_c-solution of (1.1) and let $\mathbf{v} = [f_{[0,1)}, \ldots, f_{[N-1,N)}]^t$ be the vector defined by the average of f over the N-subintervals (see Proposition 2.1), then $\mathbf{v}$ is a 2-eigenvector of $(T_0 + T_1)$. Let

$$\mathbf{f}_n(x) = \sum_{|J|=n} (T_J\mathbf{v})\chi_{I_J}(x)$$

and let f_n be the corresponding real valued function of $\mathbf{f}_n$ defined on $[0, N]$.

LEMMA 3.1. *For* $n \geq 1$ *and* $\ell \geq 0$,

$$\int_0^{1-2^{-n}} \|\mathbf{f}_{n+\ell}(x + 2^{-n}) - \mathbf{f}_{n+\ell}(x)\|^p \, dx = \frac{1}{2^{n+\ell}} \sum_{|J'|=\ell} \left(\sum_{i=1}^{n} \sum_{|J|=n-i} \|T_J(T_1T_0^{i-1} - T_0T_1^{i-1})T_{J'}\mathbf{v}\|^p \right).$$

Proof. We divide the interval $[0, 1-2^{-n})$ into $2^n - 1$ equal subintervals. For each subinterval, we further divide it into 2^ℓ equal parts. In this way we have $2^\ell(2^n - 1)$ equal subintervals with length $2^{-(n+\ell)}$. For each such dyadic interval, we can write down its binary representation with length $2^{n+\ell}$, say $I_{(j_1,\ldots,j_n,j'_1,\ldots,j'_\ell)}$. Since it is contained in $[0, 1-2^{-n})$, at least one of the $j_1, \ldots, j_n$ must equal 0. Suppose $x \in I_{(j_1,\ldots,j_n,j'_1,\ldots,j'_\ell)}$ with j_{n-i+1} as the last zero in $\{j_1, \ldots, j_n\}$, i.e., $x \in I_{(j_1,\ldots,j_{n-i},0,1,\ldots,1,j'_1,\ldots,j'_\ell)}$, then

$x+2^{-n} \in I_{(j_1,\dots,j_{n-i},1,0,\dots,0,j'_1,\dots,j'_\ell)}$. It follows that

$$\begin{aligned}
&\mathbf{f}_{n+\ell}(x+2^{-n}) - \mathbf{f}_{n+\ell}(x) \\
&\quad = T_{j_1}\cdots T_{j_{n-i}} T_1 T_0^{i-1}(T_{j'_1}\cdots T_{j'_\ell}\mathbf{v}) - T_{j_1}\cdots T_{j_{n-i}} T_0 T_1^{i-1}(T_{j'_1}\cdots T_{j'_\ell}\mathbf{v}) \\
&\quad = T_{j_1}\cdots T_{j_{n-i}}(T_1T_0^{i-1} - T_0T_1^{i-1})T_{j'_1}\cdots T_{j'_\ell}\mathbf{v}.
\end{aligned}$$

Since $\mathbf{f}_{n+\ell}(x+2^{-n}) - \mathbf{f}_{n+\ell}(x)$ is a constant function on each dyadic interval of size $2^{-(n+\ell)}$, an integration over the interval $[0,\, 1-2^{-n})$ yields the lemma immediately. □

We first give a lower bound estimate of $\|\Delta_{2^{-n}} f\|$.

Proposition 3.2. *For $n \ge 1$,*

$$\|\Delta_{2^{-n}} f\|^p \ge \frac{2^{p-1}}{2^{n-1}} \sum_{|J|=n-1} \|T_J \tilde{\mathbf{v}}\|^p.$$

Proof. Fix $n \ge 1$ and for any $\ell \ge 0$,

$$\begin{aligned}
\|\Delta_{2^{-n}} f_{n+\ell}\|^p &= \int_{-2^{-n}}^{N} |f_{n+\ell}(x+2^{-n}) - f_{n+\ell}(x)|^p\, dx \\
&\ge \int_0^{1-2^{-n}} \|\mathbf{f}_{n+\ell}(x+2^{-n}) - \mathbf{f}_{n+\ell}(x)\|^p\, dx \\
&= \frac{1}{2^{n+\ell}} \sum_{|J'|=\ell} \left(\sum_{i=1}^{n} \sum_{|J|=n-i} \|T_J(T_1T_0^{i-1} - T_0T_1^{i-1})T_{J'}\mathbf{v}\|^p \right) \\
&\hspace{20em} \text{(by Lemma 3.1)} \\
&\ge \frac{1}{2^{n+\ell}} \sum_{|J'|=\ell} \sum_{|J|=n-1} \|T_J(T_1 - T_0)T_{J'}\mathbf{v}\|^p \\
&\ge \frac{1}{2^n} \sum_{|J|=n-1} \|T_J(T_1-T_0)\big(\frac{1}{2^\ell}\sum_{|J'|=\ell} T_{J'}\mathbf{v}\big)\|^p \\
&= \frac{1}{2^n} \sum_{|J|=n-1} \|T_J(T_1-T_0)\mathbf{v}\|^p \qquad \text{(by (2.1))} \\
&= 2^{p-1}\frac{1}{2^{n-1}} \sum_{|J|=n-1} \|T_J\tilde{\mathbf{v}}\|^p \qquad \text{(use } (T_1-T_0)\mathbf{v} = -2\tilde{\mathbf{v}}).
\end{aligned}$$

The assertion now follows by letting $\ell \to \infty$. □

For the upper bound of $\|\Delta_h f\|$, we need an estimation of the integral of $|\Delta_h f_n(x)|$ near the integers $k = 0, \dots, N$.

Lemma 3.3. *Under the same assumptions as in Lemma 2.5, for $n > 0$ and for $0 < h < 2^{-n}$,*

$$\int_{E_n} |\Delta_h f_n(x)|^p\, dx \le 2^p\, h(\|T_0^n \mathbf{h}_0\|^p + \|T_1^n \mathbf{h}_1\|^p)$$

where $E_n = \bigcup_{k=0}^{N} [k-2^{-n},\, k)$.

Proof. Since f_n is a constant function on the dyadic intervals of size 2^{-n}, we have

$$\begin{aligned}\int_{E_n} |\Delta_h f_n(x)|^p\,dx &= \sum_{k=0}^{N} \int_{k-2^{-n}}^{k} |f_n(x+h) - f_n(x)|^p\,dx \\ &= \sum_{k=0}^{N} \int_{k-h}^{k} |f_n(x+h) - f_n(x)|^p\,dx \\ &= h\Big(|(T_0^n \mathbf{v})_1|^p + \sum_{i=2}^{N} |(T_0^n \mathbf{v})_i - (T_1^n \mathbf{v})_{i-1}|^p + |-(T_1^n \mathbf{v})_N|^p\Big).\end{aligned}$$

Recall that $\mathbf{v} = c\mathbf{v}_0 + \mathbf{h}_0 = c\mathbf{v}_1 + \mathbf{h}_1$ as in Lemma 2.5. Therefore, by Corollary 2.4(iii),

$$(T_0^n \mathbf{v})_1 = c(T_0^n \mathbf{v}_0)_1 + (T_0^n \mathbf{h}_0)_1 = (T_0^n \mathbf{h}_0)_1$$

and similarly $(T_1^n \mathbf{v})_N = (T_1^n \mathbf{h}_1)_N$. Also for $2 \le i \le N$, by Corollary 2.4(iii) again,

$$\begin{aligned}(T_0^n \mathbf{v})_i - (T_1^n \mathbf{v})_{i-1} =& c(T_0^n \mathbf{v}_0)_i + (T_0^n \mathbf{h}_0)_i - c(T_1^n \mathbf{v}_1)_{i-1} - (T_1^n \mathbf{h}_1)_{i-1} \\ =& (T_0^n \mathbf{h}_0)_i - (T_1^n \mathbf{h}_1)_{i-1}.\end{aligned}$$

We can continue the above estimation:

$$\begin{aligned}\int_{E_n} |\Delta_h f_n(x)|^p\,dx &= h\Big(|(T_0^n \mathbf{h}_0)_1|^p + \sum_{i=2}^{N} |(T_0^n \mathbf{h}_0)_i - (T_1^n \mathbf{h}_1)_{i-1}|^p + |(T_1^n \mathbf{h}_1)_N|^p\Big) \\ &\le 2^p\, h\big([\|T_0^n \mathbf{h}_0\|^p + \|T_1^n \mathbf{h}_1\|^p\big)\end{aligned}$$

and complete the proof. ∎

PROPOSITION 3.4. *Under the same assumptions as in Lemma 2.5, we have for* $0 < h < 2^{-n}$,

$$\|\Delta_h f_n\|^p \le \frac{2^{p+1}}{2^n}\Big(\sum_{|J|=n} \|T_J \mathbf{h}_0\|^p + \sum_{|J|=n} \|T_J \mathbf{h}_1\|^p\Big).$$

Proof. Let $E_n = \bigcup_{k=0}^{N} [k - 2^{-n},\, k)$ and $\tilde{E}_n = [-2^{-n},\, N) \setminus E_n = \bigcup_{k=0}^{N-1} [k,\, k+1-2^{-n})$. Since f_n is supported by $[0,\, N]$, we have

$$\begin{aligned}\|\Delta_h f_n\|^p &= \int_{-2^{-n}}^{N} |\Delta_h f_n(x)|^p\,dx \\ &= \int_{E_n} |\Delta_h f_n(x)|^p\,dx + \int_{\tilde{E}_n} |\Delta_h f_n(x)|^p\,dx \\ &:= I_1 + I_2.\end{aligned}$$

Lemma 3.3 implies that

$$I_1 \le 2^p\, h(\|T_0^n \mathbf{h}_0\|^p + \|T_1^n \mathbf{h}_1\|^p).$$

On the other hand, if we write I_2 in the vector form, we have

$$\begin{aligned} I_2 &= \int_0^{1-2^{-n}} \|\mathbf{f}_n(x+h) - \mathbf{f}_n(x)\|^p \, dx \\ &= h \sum_{k=1}^{n} \sum_{|J|=n-k} \|T_J(T_1 T_0^{k-1} - T_0 T_1^{k-1})\mathbf{v}\|^p. \end{aligned}$$

From Corollary 2.4(ii) we conclude that

$$\begin{aligned} (T_1 T_0^{k-1} - T_0 T_1^{k-1})\mathbf{v} &= T_1 T_0^{k-1}(c\mathbf{v}_0 + \mathbf{h}_0) - T_0 T_1^{k-1}(c\mathbf{v}_1 + \mathbf{h}_1) \\ &= T_1 T_0^{k-1}\mathbf{h}_0 - T_0 T_1^{k-1}\mathbf{h}_1, \end{aligned}$$

and therefore

$$\begin{aligned} I_2 &\le 2^p h \Big(\sum_{k=1}^{n} \sum_{|J|=n-k} \|T_J T_1 T_0^{k-1}\mathbf{h}_0\|^p + \sum_{k=1}^{n} \sum_{|J|=n-k} \|T_J T_0 T_1^{k-1}\mathbf{h}_1\|^p \Big) \\ &\le 2^p h \Big(\sum_{|J|=n} \|T_J \mathbf{h}_0\|^p + \sum_{|J|=n} \|T_J \mathbf{h}_1\|^p \Big). \end{aligned}$$

The lemma then follows from the two estimates of I_1 and I_2. □

We can now state and prove our main theorem of this section (i.e. Theorem B in Section 1).

THEOREM 3.5. *Suppose that either (i)* 1 *is a simple eigenvalue of* T_0 *and* T_1 *or (ii)* $H(\tilde{\mathbf{v}}) = \{ \mathbf{u} \in \mathbb{C}^N : \sum_{i=1}^N u_i = 0 \}$. *If* f *is a* L_c^p*-solution of* (1.1), *then*

$$\mathrm{Lip}_p(f) = \liminf_{n\to\infty} \frac{\ln(2^{-n}\sum_{|J|=n} \|T_J\tilde{\mathbf{v}}\|^p)}{p\ln(2^{-n})}.$$

Proof. As a direct consequence of Proposition 3.2, we have

$$\mathrm{Lip}_p(f) \le \liminf_{n\to\infty} \frac{\ln\|\Delta_{2^{-n}} f\|}{\ln(2^{-n})} \le \liminf_{n\to\infty} \frac{\ln(2^{-n}\sum_{|J|=n} \|T_J\tilde{\mathbf{v}}\|^p)}{p\ln(2^{-n})}.$$

To prove the reverse inequality we first observe that $\|\Delta_h f\| \le 2\|f - f_n\| + \|\Delta_h f_n\|$. Proposition 2.1 (iii) and Proposition 3.4 imply that

$$\|\Delta_h f\|^p \le C\Big(2^{-n}\sum_{|J|=n}\|T_J\tilde{\mathbf{v}}\|^p + 2^{-n}\sum_{|J|=n}\|T_J\mathbf{h}_0\|^p + 2^{-n}\sum_{|J|=n}\|T_J\mathbf{h}_1\|^p\Big)$$

for some constant C independent of n. Since $\tilde{\mathbf{v}},\ \mathbf{h}_0,\ \mathbf{h}_1$ are all in $H(\tilde{\mathbf{v}})$, we can apply Lemma 2.2 to have the reverse inequality. □

4. $\mathrm{Lip}_p(f)$ for some special cases. For the 2-coefficient dilation equation $f(x) = f(2x) + f(2x-1)$, the scaling function is $\chi_{[0,1)}$ and it is easy to calculate that $\mathrm{Lip}_p(f) = 1/p$ from the definition.

PROPOSITION 4.1. *If* f *is an* L_c^p*-solution of* $f(x) = c_0 f(2x) + c_1 f(2x-1) + c_2 f(2x-2)$ *with* $c_0 + c_2 = 1$, $c_1 = 1$, *and* c_0, $c_2 \ne 0$, *then*

$$\mathrm{Lip}_p(f) = \frac{\ln((|c_0|^p + |1-c_0|^p)/2)}{-p\ln 2}.$$

Proof. In this case,

$$T_0 = \begin{pmatrix} c_0 & 0 \\ 1-c_0 & 1 \end{pmatrix} \quad \text{and} \quad T_1 = \begin{pmatrix} 1 & c_0 \\ 0 & 1-c_0 \end{pmatrix},$$

and $\mathbf{v} = [c_0,\, c_0 - 1]^t$ is a 2-eigenvector of $(T_0 + T_1)$. Then

$$\tilde{\mathbf{v}} = (T_0 - I)\mathbf{v} = \begin{pmatrix} c_0(c_0-1) \\ -c_0(c_0-1) \end{pmatrix} \neq 0.$$

Note that $\tilde{\mathbf{v}}$ is an c_0-eigenvector of T_0 and $(1-c_0)$-eigenvector of T_1. A straight-forward calculation yields

$$\begin{aligned} \frac{1}{2^n}\sum_{|J|=n}\|T_J\tilde{\mathbf{v}}\|^p &= \frac{1}{2^n}\Big(\sum_{k=0}^{n}\binom{n}{k}\left(|c_0|^p\right)^k\left(|1-c_0|^p\right)^{n-k}\Big)\|\tilde{\mathbf{v}}\|^p \\ &= \Big(\frac{|c_0|^p + |1-c_0|^p}{2}\Big)^n \|\tilde{\mathbf{v}}\|^p. \end{aligned}$$

This implies that

$$\mathrm{Lip}_p(f) = \frac{\ln((|c_0|^p + |1-c_0|^p)/2)}{-p\ln 2}. \qquad \square$$

We now turn to the 4-coefficient dilation equation

$$f(x) = c_0 f(2x) + c_1 f(2x-1) + c_2 f(2x-2) + c_3 f(2x-3) \tag{4.1}$$

with $c_0 + c_2 = c_1 + c_3 = 1$ and $c_0,\, c_3 \neq 0$. We first observe that

$$T_0 = \begin{pmatrix} c_0 & 0 & 0 \\ 1-c_0 & 1-c_3 & c_0 \\ 0 & c_3 & 1-c_0 \end{pmatrix}, \quad T_1 = \begin{pmatrix} 1-c_3 & c_0 & 0 \\ c_3 & 1-c_0 & 1-c_3 \\ 0 & 0 & c_3 \end{pmatrix}. \tag{4.2}$$

The eigenvalues of $(T_0 + T_1)$ are 2, 1, and $(1 - c_0 - c_3)$, and the 2-eigenvector $\mathbf{v}$ is

$$\mathbf{v} = \begin{pmatrix} c_0(1+c_0-c_3) \\ (1+c_0-c_3)(1-c_0+c_3) \\ c_3(1-c_0+c_3) \end{pmatrix}. \tag{4.3}$$

Therefore

$$\tilde{\mathbf{v}} = (T_0 - I)\mathbf{v} = \begin{pmatrix} c_0(c_0-1)(1+c_0-c_3) \\ -c_0(c_0-1)(1+c_0-c_3) + c_3(c_3-1)(1-c_0+c_3) \\ -c_3(c_3-1)(1-c_0+c_3) \end{pmatrix}. \tag{4.4}$$

Note that in Proposition 4.1, the computation can be made easier if $\tilde{\mathbf{v}}$ is an eigenvector of both T_0 and T_1. Here we have

LEMMA 4.2. *Let T_0 and T_1 be as in (4.2) and let $\mathbf{v}$ be the 2-eigenvector of (T_0+T_1) as in (4.3) and let $\tilde{\mathbf{v}} = (T_0 - I)\mathbf{v}$. Then $\tilde{\mathbf{v}}$ is an eigenvector of both T_0 and T_1 (not necessary to the same eigenvalue) if and only if $c_0 + c_3 = 1$.*

Proof. Suppose $c_0 + c_3 = 1$, then $c_0 = c_1$, $c_2 = c_3$, and (4.2) reduces to

$$T_0 = \begin{pmatrix} c_0 & 0 & 0 \\ 1-c_0 & c_0 & c_0 \\ 0 & 1-c_0 & 1-c_0 \end{pmatrix}, \quad T_1 = \begin{pmatrix} c_0 & c_0 & 0 \\ 1-c_0 & 1-c_0 & c_0 \\ 0 & 0 & 1-c_0 \end{pmatrix},$$

and $\tilde{\mathbf{v}} = [-2c_0^2c_3,\ 2c_0^2c_3 - 2c_0c_3^2,\ 2c_0c_3^2]^t \neq 0$. By a direct calculation, $\tilde{\mathbf{v}}$ is a c_0-eigenvector of T_0 and $(1 - c_0)$-eigenvector of T_1.

Conversely, suppose $\tilde{\mathbf{v}}$ is an eigenvector of both T_0 and T_1. Let $\mathbf{u}_0 = [0,\ 1,\ -1]^t$ and $\mathbf{u}_1 = [1,\ -1,\ 0]^t$, then $\tilde{\mathbf{v}} = a\mathbf{u}_0 + b\mathbf{u}_1$ where a and b is determined by (4.4). By using $\mathbf{u}_0$ and $\mathbf{u}_1$ as a basis of the subspace $H = \{\mathbf{u} \in \mathbb{C}^3 : \sum u_i = 0\}$, we can rewrite T_0, T_1 (restricted on H) and $\tilde{\mathbf{v}}$ as follows:

$$(4.5) \qquad T_0 = \begin{pmatrix} 1 - c_0 - c_3 & c_3 \\ 0 & c_0 \end{pmatrix}, \quad T_1 = \begin{pmatrix} c_3 & 0 \\ c_0 & 1 - c_0 - c_3 \end{pmatrix}, \quad \text{and} \quad \tilde{\mathbf{v}} = \begin{pmatrix} a \\ b \end{pmatrix}.$$

Note that T_0 has c_0 and $1 - c_0 - c_3$ as eigenvalues while T_1 has c_3 and $1 - c_0 - c_3$ as eigenvalues. We claim that $\tilde{\mathbf{v}}$ is an c_0-eigenvector of T_0. For otherwise, $\tilde{\mathbf{v}}$ is an $(1 - c_0 - c_3)$-eigenvector of T_0, then b must be zero and $\tilde{\mathbf{v}} = [a,\ 0]^t$. But this contradicts to the assumption that $\tilde{\mathbf{v}}$ is an eigenvector of T_1. Similarly, $\tilde{\mathbf{v}}$ must be a c_3-eigenvector of T_1. Hence,

$$(T_0 + T_1)\tilde{\mathbf{v}} = (c_0 + c_3)\tilde{\mathbf{v}}.$$

There are only three choices of the eigenvalues of $T_0 + T_1$: $2, 1$ or $1 - c_0 - c_3$. By a direct check we conclude that $c_0 + c_3 = 1$ is the only allowable case. □

In view of Lemma 4.2 we can use the same technique as in Proposition 4.1 to prove the next proposition

Proposition 4.3. *If f is an L_c^p-solution of* (4.1) *with the additional assumption that $c_0 + c_3 = 1$, then*

$$(4.6) \qquad \operatorname{Lip}_p(f) = \frac{\ln((|c_0|^p + |1 - c_0|^p))/2}{-p \ln 2}.$$

In Figure 1, we draw the graphs of some scaling functions satisfying the assumption in the above proposition and their L^p-Lipschitz exponents. Note that if $\operatorname{Lip}_p(f) = 1$ for all $1 \le p < \infty$, then f is differentiable almost everywhere and the derivative is in L^p for all $1 \le p < \infty$. This is the case for $c_0 = 0.5$ and is obvious from the graph of the corresponding scaling function. For the graph of $c_0 = 1.125$, we see that $\operatorname{Lip}_p(f)$ is undefined for $p > 6$. Indeed $f \notin L^p(\mathbb{R})$, for $p > 6$, making use of the criterion in Theorem A.

We conclude this section by giving a formula of $\operatorname{Lip}_p(f)$ with the coefficients satisfying $c_0 + c_3 = \frac{1}{2}$ instead of $c_0 + c_3 = 1$. It includes Daubechies scaling function D_4 which corresponds to $c_0 = (1 + \sqrt{3})/4$, $c_3 = (1 - \sqrt{3})/4$. This formula has been obtained in [DL3] using a different method and assuming in addition that $\frac{1}{2} < c_0 < \frac{3}{4}$. Here, we need an estimation on the product of two non-commutative matrices.

Lemma 4.4. *Let $\beta_0, \beta_1 \in \mathbb{R}$. Let*

$$P_0 = \begin{pmatrix} 1 & 0 \\ 0 & \beta_0 \end{pmatrix} \quad \text{and} \quad P_1 = \begin{pmatrix} 1 & 0 \\ \beta_0 & \beta_1 \end{pmatrix}.$$

For any multi-index $J = (j_1, j_2, \ldots, j_n)$, we let $P_J = P_{j_1} \cdots P_{j_n}$. Then

$$P_J = \begin{pmatrix} 1 & 0 \\ \lambda_J & \mu_J \end{pmatrix}$$

where $\lambda_J = \beta_0(j_1 + j_2\beta_{j_1} + \cdots + j_n(\beta_{j_{n-1}} \cdots \beta_{j_1}))$ and $\mu_J = \beta_{j_n}\beta_{j_{n-1}} \cdots \beta_{j_1}$. Let

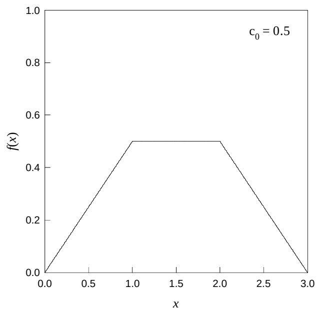

Figure 1a

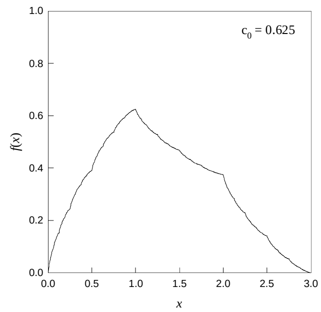

Figure 1b

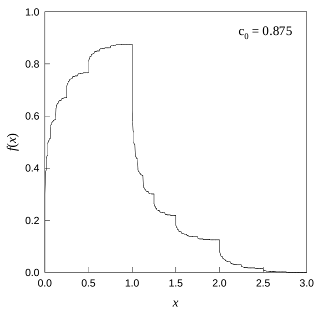

Figure 1c

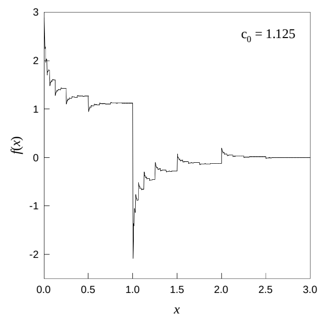

Figure 1d

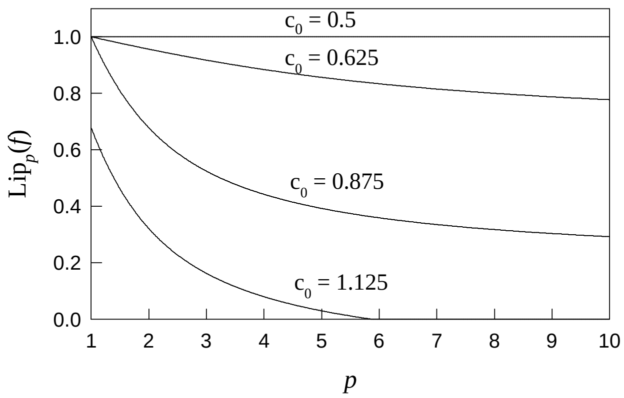

Figure 1e

$\gamma = (|\beta_0|^p + |\beta_1|^p)/2$. *Then*

$$2^{-n} \sum_{|J|=n} |\mu_J|^p = \gamma^n \qquad \textit{and} \qquad 2^{-n} \sum_{|J|=n} |\lambda_J|^p \le C\, n^p \max\{1, \gamma^n\}$$

for some constant $C > 0$ *independent of* n.

Proof. The explicit form of the product P_J can easily be shown by induction. For the second part of the lemma, the first identity follows from

$$2^{-n} \sum_{|J|=n} |\mu_J|^p = 2^{-n} \sum_{j_1,\dots,j_n=0,1} |\beta_{j_n} \cdots \beta_{j_1}|^p = \gamma^n.$$

For the second identity we observe that

$$\begin{aligned}
\Big(\sum_{|J|=n} |\lambda_J|^p \Big)^{\frac{1}{p}} &= |\beta_0| \Big(\sum_{j_1,\dots,j_n=0,1} \Big| j_1 + \sum_{i=2}^{n} j_i(\beta_{j_{i-1}} \cdots \beta_{j_1}) \Big|^p \Big)^{\frac{1}{p}} \\
&\le |\beta_0| \Big(2^{(n-1)/p} + \sum_{i=2}^{n} \Big(\sum_{j_1,\dots,j_n=0,1} |j_i(\beta_{j_{i-1}} \cdots \beta_{j_1})|^p \Big)^{\frac{1}{p}} \Big) \\
&\hspace{6cm} \text{(by Minkowski inequality)} \\
&= |\beta_0| \Big(2^{(n-1)/p} + \sum_{i=2}^{n} \big(|\beta_0|^p + |\beta_1|^p \big)^{(i-1)/p} \Big) \\
&\le |\beta_0|\, 2^{(n-1)/p} \sum_{i=1}^{n} (\gamma^{1/p})^{i-1} \\
&\le |\beta_0|\, n\, 2^{(n-1)/p} \max\{1, (\gamma^{1/p})^n\}.
\end{aligned}$$

The last identity now follows. □

PROPOSITION 4.5. *If* f *is the* L^p_c*-solution of* (4.1) *with the additional assumption that* $c_0 + c_3 = \frac{1}{2}$, *then*

$$\text{Lip}_p(f) = \min\left\{ 1,\ \frac{\ln((|c_0|^p + |\frac{1}{2} - c_0|^p)/2)}{-p \ln 2} \right\}. \tag{4.7}$$

Proof. In this case, (4.2) reduces to

$$T_0 = \begin{pmatrix} c_0 & 0 & 0 \\ 1 - c_0 & \frac{1}{2} + c_0 & c_0 \\ 0 & \frac{1}{2} - c_0 & 1 - c_0 \end{pmatrix} \quad \text{and} \quad T_1 = \begin{pmatrix} \frac{1}{2} + c_0 & c_0 & 0 \\ \frac{1}{2} - c_0 & 1 - c_0 & \frac{1}{2} + c_0 \\ 0 & 0 & \frac{1}{2} - c_0 \end{pmatrix}.$$

Note that $\mathbf{h} = [1,\, -2,\, 1]^t$ is a c_0-eigenvector of T_0 and also a $(\frac{1}{2} - c_0)$-eigenvector of T_1. It is clear that

$$2^{-n} \sum_{|J|=n} \|T_J \mathbf{h}\|^p = 2^{-n} \left(|c_0|^p + |\frac{1}{2} - c_0|^p \right)^n \|\mathbf{h}\|^p.$$

Since $\mathbf{h} \in H(\tilde{\mathbf{v}})$, by Lemma 2.2, we have

$$\text{Lip}_p(f) \le \frac{\ln((|c_0|^p + |\frac{1}{2} - c_0|^p)/2)}{-p \ln 2}.$$

Next observe that $\mathbf{u} = [0,\, 1,\, -1]^t$ is a $\frac{1}{2}$-eigenvector of T_0 and $T_1\mathbf{u} = \frac{1}{2}\mathbf{u} + c_0\mathbf{h}$. Therefore, by using $\mathbf{u}$ and $\mathbf{h}$ as a basis of the subspace $H(\tilde{\mathbf{v}})$, we can rewrite T_0, T_1, restricted on H $(= H(\tilde{\mathbf{v}})$ in this case), and $\tilde{\mathbf{v}}$, as

$$T_0 = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & c_0 \end{pmatrix}, \quad T_1 = \begin{pmatrix} \frac{1}{2} & 0 \\ c_0 & \frac{1}{2} - c_0 \end{pmatrix}, \quad \text{and} \quad \tilde{\mathbf{v}} = \begin{pmatrix} a \\ b \end{pmatrix},$$

where $a = -\frac{1}{4}$ and $b = \frac{2}{3}c_0(c_0 - 1)(\frac{1}{2} + 2c_0)$. Let $\beta_0 = 2c_0$ and $\beta_1 = 1 - \beta_0$. For $J = (j_1, j_2, \ldots, j_n)$, we have $T_J = \frac{1}{2^n}P_J$, so that

$$\|T_J\tilde{\mathbf{v}}\|^p = |a2^{-n}|^p + |2^{-n}(a\lambda_J + b\mu_J)|^p,$$

where λ_J and μ_J are defined as in Lemma 4.4. This implies $\|T_J\tilde{\mathbf{v}}\|^p \geq |a2^{-n}|^p$ and $\mathrm{Lip}_p(f) \leq 1$ by Theorem 3.5. Hence

$$\mathrm{Lip}_p(f) \leq \min\left\{1,\ \frac{\ln((|c_0|^p + |\frac{1}{2} - c_0|^p)/2)}{-p\ln 2}\right\}. \tag{4.8}$$

On the other hand,

$$\|T_J\tilde{\mathbf{v}}\|^p \leq |a2^{-n}|^p + 2^p|a2^{-n}\lambda_J|^p + 2^p|b2^{-n}\mu_J|^p.$$

By Lemma 4.4, we have

$$\begin{aligned} 2^{-n}\sum_{|J|=n}\|T_J\tilde{\mathbf{v}}\|^p &\leq C\,n^p\,2^{-pn}\max\{1,\ ((|\beta_0|^p + |\beta_1|^p)/2)^n\} \\ &= C\,n^p\max\left\{2^{-pn},\ \left((|\beta_0/2|^p + |\beta_1/2|^p)/2\right)^n\right\}. \end{aligned}$$

Consequently we have the reverse inequality of (4.8) and completes the proof. ∎

In figures 2a–e we again sketch some L^p_c-scaling functions from Proposition 4.5 ($c_0 + c_3 = \frac{1}{2}$) and their L^p-Lipschitz exponents $\mathrm{Lip}_p(f)$ of p. The case for $c_0 = 0.25$ corresponding to $\chi_{[0,1]} * \chi_{[0,1]}$, it is differentiable and hence $\mathrm{Lip}_p(f) = 1$ for all p. The case corresponding to $c_0 = 0.683...$ is the Daubechies scaling function D_4. From the picture of $\mathrm{Lip}_p(f)$, D_4 has L^p-derivative for $1 \leq p < 2$. It is known that for $p = 2$, D_4 is differentiable almost everywhere but the derivative is not in L^2. Also it is known that the Hölder exponent of D_4 is $2 - \ln(1 + \sqrt{3})/\ln 2$, which is the same number as the formula in the proposition when $p \to \infty$.

5. $\mathrm{Lip}_p(f)$ when p is a positive even integer. The computation of $\mathrm{Lip}_p(f)$ in Section 4 depends on the existence of an eigenvector of both T_0 and T_1 (which may be associated with different eigenvalues). This technique cannot be used for the general case. In this section we show that if p is a positive even integer, then $\mathrm{Lip}_p(f)$ is related to the spectral radius of a matrix W_p whose entries are induced from the coefficients of the dilation equation. For simplicity, we only give the construction of W_p for the 4-coefficient dilation equation. It is not hard to extend this to the case with more coefficients.

In view of Theorem 3.5, we will first develop a simple expression for the sum $2^{-n}\sum_{|J|=n}\|T_J\tilde{\mathbf{v}}\|^p$ for p a positive even integer. Let $[0,\, 1,\, -1]^t$ and $[1,\, -1,\, 0]^t$ be a basis of $H = \{\mathbf{u} \in \mathbb{C}^3 : \sum u_i = 0\}$. Then T_0 and T_1 can be written as in (4.5). Let $\mathbf{e}_0 = [1,\, 0]$, $\mathbf{e}_1 = [0,\, 1]$. For a fixed $\mathbf{u} = [\alpha,\, \beta]^t \in H(\tilde{\mathbf{v}})$ (to be determined later), we

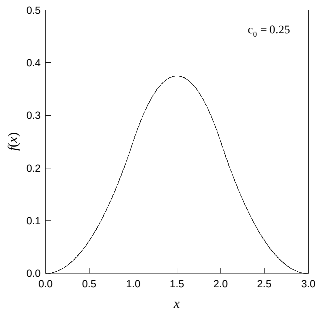

Figure 2a

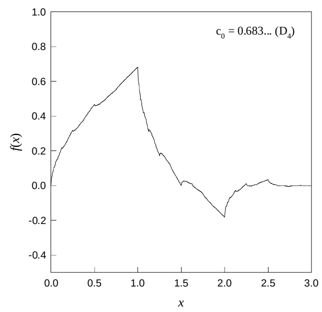

Figure 2b

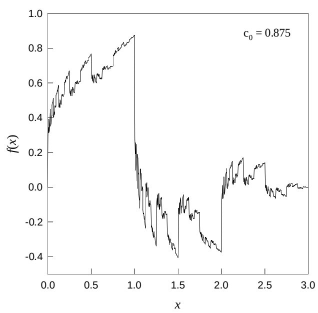

Figure 2c

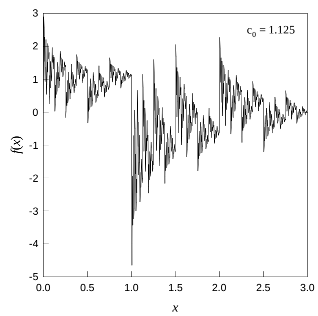

Figure 2d

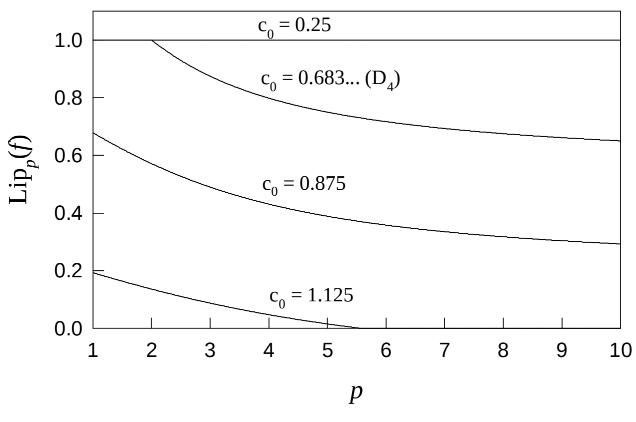

Figure 2e

define the vector $\mathbf{a}_n$ with the i-th entry by

$$(\mathbf{a}_n)_i = \sum_{|J|=n} (\mathbf{e}_0 T_J \mathbf{u})^{p-i} (\mathbf{e}_1 T_J \mathbf{u})^i, \qquad i = 0, \ldots, p.$$

If p is an even integer, then

$$\begin{aligned} \sum_{|J|=n} \|T_J \mathbf{u}\|^p &= \sum_{|J|=n} (|\mathbf{e}_0 T_J \mathbf{u}|^p + |\mathbf{e}_1 T_J \mathbf{u}|^p) \\ &= (\mathbf{a}_n)_0 + (\mathbf{a}_n)_p = |(\mathbf{a}_n)_0| + |(\mathbf{a}_n)_p|. \end{aligned} \tag{5.1}$$

Note that $\mathbf{a}_0 = [\alpha^p,\, \alpha^{p-1}\beta,\, \ldots,\, \alpha\beta^{p-1},\, \beta^p]^t$. If we let $d = 1 - c_0 - c_3$, we have, in view of (4.5),

$$\begin{aligned} \mathbf{e}_0 T_0 &= d\mathbf{e}_0 + c_3 \mathbf{e}_1, & \mathbf{e}_1 T_0 &= c_0 \mathbf{e}_1, \\ \mathbf{e}_0 T_1 &= c_3 \mathbf{e}_0, & \mathbf{e}_1 T_1 &= c_0 \mathbf{e}_0 + d\mathbf{e}_1, \end{aligned}$$

and hence

$$\begin{aligned} (\mathbf{a}_{n+1})_i &= \sum_{|J|=n+1} (\mathbf{e}_0 T_J \mathbf{u})^{p-i} (\mathbf{e}_1 T_J \mathbf{u})^i \\ &= \sum_{|J|=n} (\mathbf{e}_0 T_0 T_J \mathbf{u})^{p-i} (\mathbf{e}_1 T_0 T_J \mathbf{u})^i + \sum_{|J|=n} (\mathbf{e}_0 T_1 T_J \mathbf{u})^{p-i} (\mathbf{e}_1 T_1 T_J \mathbf{u})^i \\ &= \sum_{|J|=n} \big((d\mathbf{e}_0 + c_3 \mathbf{e}_1) T_J \mathbf{u}\big)^{p-i} (c_0 \mathbf{e}_1 T_J \mathbf{u})^i \\ &\quad + \sum_{|J|=n} \big((c_3 \mathbf{e}_0) T_J \mathbf{u}\big)^{p-i} (c_0 \mathbf{e}_0 + d\mathbf{e}_1 T_J \mathbf{u})^i \\ &= \sum_{|J|=n} \left(\sum_{\ell=0}^{p-i} \binom{p-i}{\ell} d^{p-i-\ell} (\mathbf{e}_0 T_J \mathbf{u})^{p-i-\ell} c_3^\ell (\mathbf{e}_1 T_J \mathbf{u})^\ell \right) \big(c_0^i (\mathbf{e}_1 T_J \mathbf{u})^i\big) \\ &\quad + \sum_{|J|=n} \big(c_3^{p-i} (\mathbf{e}_0 T_J \mathbf{u})^{p-i}\big) \left(\sum_{\ell=0}^{i} \binom{i}{\ell} c_0^{i-\ell} (\mathbf{e}_0 T_J \mathbf{u})^{i-\ell} d^\ell (\mathbf{e}_1 T_J \mathbf{u})^\ell \right) \\ &= \sum_{\ell=0}^{p-i} \binom{p-i}{\ell} c_0^i c_3^\ell d^{p-i-\ell} (\mathbf{a}_n)_{i+\ell} + \sum_{\ell=0}^{i} \binom{i}{\ell} c_0^{i-\ell} c_3^{p-i} d^\ell (\mathbf{a}_n)_\ell. \end{aligned}$$

Summarizing the above, we have

PROPOSITION 5.1. *For any integer $p \geq 1$, let W_p be the $(p+1) \times (p+1)$ matrix defined by*

$$(W_p)_{ij} = \begin{cases} \binom{p-i}{j-i} c_0^i c_3^{j-i} d^{p-j} & \text{for } 0 \leq i < j \leq p \\ c_0^i d^{p-i} + c_3^{p-i} d^i & \text{for } i = j \\ \binom{i}{j} c_0^{i-j} c_3^{p-i} d^j & \text{for } 0 \leq j < i \leq p \end{cases}$$

where $d = 1 - c_0 - c_3$. Then

$$\mathbf{a}_{n+1} = W_p \mathbf{a}_n = W_p^{n+1} \mathbf{a}_0$$

where $\mathbf{a}_0 = [\alpha^p, \alpha^{p-1}\beta, \ldots, \alpha\beta^{p-1}, \beta^p]^t$. *In particular if p is an even integer, then*

$$\sum_{|J|=n} \|T_J\mathbf{u}\|^p = [1,0,0,\cdots,0,1]W_p^n\mathbf{a}_0.$$

The matrix W_p can be written as $W_p = W_p^{(L)} + W_p^{(U)}$, where $W_p^{(L)}$ and $W_p^{(U)}$ are the lower and upper triangular part of W_p, in a very symmetric manner. For example,

$$W_2^{(L)} = \begin{pmatrix} \binom{0}{0}c_3^2 & 0 & 0 \\ \binom{1}{0}c_0c_3 & \binom{1}{1}c_3d & 0 \\ \binom{2}{0}c_0^2 & \binom{2}{1}c_0d & \binom{2}{2}d^2 \end{pmatrix}, \quad W_2^{(U)} = \begin{pmatrix} \binom{2}{0}d^2 & \binom{2}{1}c_3d & \binom{2}{2}c_3^2 \\ 0 & \binom{1}{0}c_0d & \binom{1}{1}c_0c_3 \\ 0 & 0 & \binom{0}{0}c_0^2 \end{pmatrix};$$

$$W_4^{(L)} = \begin{pmatrix} \binom{0}{0}c_3^4 & 0 & 0 & 0 & 0 \\ \binom{1}{0}c_0c_3^3 & \binom{1}{1}c_3^3d & 0 & 0 & 0 \\ \binom{2}{0}c_0^2c_3^2 & \binom{2}{1}c_0c_3^2d & \binom{2}{2}c_3^2d^2 & 0 & 0 \\ \binom{3}{0}c_0^3c_3 & \binom{3}{1}c_0^2c_3d & \binom{3}{2}c_0c_3d^2 & \binom{3}{3}c_3d^3 & 0 \\ \binom{4}{0}c_0^4 & \binom{4}{1}c_0^3d & \binom{4}{2}c_0^2d^2 & \binom{4}{3}c_0d^3 & \binom{4}{4}d^4 \end{pmatrix},$$

$$W_4^{(U)} = \begin{pmatrix} \binom{4}{0}d^4 & \binom{4}{1}c_3d^3 & \binom{4}{2}c_3^2d^2 & \binom{4}{3}c_3^3d & \binom{4}{4}c_3^4 \\ 0 & \binom{3}{0}c_0d^3 & \binom{3}{1}c_0c_3d^2 & \binom{3}{2}c_0c_3^2d & \binom{3}{3}c_0c_3^3 \\ 0 & 0 & \binom{2}{0}c_0^2d^2 & \binom{2}{1}c_0^2c_3d & \binom{2}{2}c_0^2c_3^2 \\ 0 & 0 & 0 & \binom{1}{0}c_0^3d & \binom{1}{1}c_0^3c_3 \\ 0 & 0 & 0 & 0 & \binom{0}{0}c_0^4 \end{pmatrix}.$$

Recall from basic linear algebra that if $\rho(A)$ is the spectral radius of an $N \times N$ matrix A, then $\rho(A) = \max\{|\lambda| : \lambda \text{ is an eigenvalue of } A\}$ and

$$\lim_{n\to\infty} \|A^n\|^{1/n} = \rho(A). \tag{5.2}$$

Let λ be any eigenvalue of A, and $E_\lambda = \{\mathbf{u} \in \mathbb{C}^N : (A - \lambda I)^m\mathbf{u} = 0 \text{ for some } m > 0\}$, then $\mathbb{C}^N = E_\lambda \oplus Z$ for some A-invariant subspace Z of $\mathbb{C}^N$. We say that $\mathbf{u}$ has a component in E_λ if $\mathbf{u} = \mathbf{u}_\lambda + \mathbf{z}$ with $\mathbf{u}_\lambda \neq 0$. It is clear that if $\mathbf{u} \in \mathbb{C}^N$ has a component in E_λ, then there is a constant $C > 0$ such that $\|A^n\mathbf{u}\| \geq C|\lambda|^n$ for all $n > 0$.

LEMMA 5.2. *Let* λ *be the eigenvalue of* W_p *such that* $|\lambda| = \rho(W_p)$ *and let* E_λ *be defined as above. Suppose* $\dim H(\tilde{\mathbf{v}}) = 2$. *Then there exists* $\mathbf{u} = \alpha\mathbf{b}_0 + \beta\mathbf{b}_1 \in H(\tilde{\mathbf{v}})$, *where* $\mathbf{b}_0 = [0, 1, -1]^t$, $\mathbf{b}_1 = [1, -1, 0]^t$, *such that* $H(\mathbf{u}) = H(\tilde{\mathbf{v}})$ *and the corresponding* $\mathbf{a}_0 = [\alpha^p, \alpha^{p-1}\beta, \ldots, \alpha\beta^{p-1}, \beta^p]^t$ *has a component of* E_λ. *For such* $\mathbf{u}$, *there is a constant* $C > 0$ *such that*

$$C\,\rho(W_p)^n \leq \sum_{|J|=n} \|T_J\mathbf{u}\|^p \qquad \text{for all } n > 0.$$

Proof. We choose $p+1$ vectors $\mathbf{u}_i = \alpha_i\mathbf{b}_0 + \beta_i\mathbf{b}_1$, $i = 0, \ldots, p$, such that $H(\mathbf{u}_i) = H(\tilde{\mathbf{v}})$ and

$$\alpha_i\beta_j - \alpha_j\beta_i \neq 0, \qquad \text{for } i \neq j.$$

Then the corresponding vectors

$$\boldsymbol{\gamma}_i = [\alpha_i^p,\, \alpha_i^{p-1}\beta_i,\, \dots,\, \alpha_i\beta_i^{p-1},\, \beta_i^p]^t, \quad 0 \le i \le p$$

form a basis of $\mathbb{C}^{p+1}$ because the matrix with the vectors $\boldsymbol{\gamma}_i$'s as rows is a Vandermonde matrix

$$A = \begin{pmatrix} \alpha_0^p & \alpha_0^{p-1}\beta_0 & \cdots & \alpha_0\beta_0^{p-1} & \beta_0^p \\ \alpha_1^p & \alpha_1^{p-1}\beta_1 & \cdots & \alpha_1\beta_1^{p-1} & \beta_1^p \\ \vdots & \vdots & & \vdots & \vdots \\ \alpha_p^p & \alpha_p^{p-1}\beta_p & \cdots & \alpha_p\beta_p^{p-1} & \beta_p^p \end{pmatrix}$$

and $\det A = \prod_{0 \le j,k \le p} (\alpha_j\beta_k - \alpha_k\beta_j) \ne 0$. Hence, one of the $\boldsymbol{\gamma}_i$'s has a component of E_λ. Let $\mathbf{u}$ be the corresponding $\mathbf{u}_i$ and the first part of the lemma follows. To prove the second part, we observe that for any J with $|J| = n$ and $0 \le j \le p$,

$$\begin{aligned} |\mathbf{e}_0 T_J \mathbf{u}|^{p-j} |\mathbf{e}_1 T_J \mathbf{u}|^j &\le \max\{|\mathbf{e}_0 T_J \mathbf{u}|^p,\, |\mathbf{e}_1 T_J \mathbf{u}|^p\} \\ &\le |\mathbf{e}_0 T_J \mathbf{u}|^p + |\mathbf{e}_1 T_J \mathbf{u}|^p. \end{aligned}$$

Hence

$$\begin{aligned} |(\mathbf{a}_n)_j| &= \sum_{|J|=n} |\mathbf{e}_0 T_J \mathbf{u}|^{p-j} |\mathbf{e}_1 T_J \mathbf{u}|^j \\ &\le \sum_{|J|=n} |\mathbf{e}_0 T_J \mathbf{u}|^p + \sum_{|J|=n} |\mathbf{e}_1 T_J \mathbf{u}|^p = |(\mathbf{a}_n)_0| + |(\mathbf{a}_n)_p| \\ &= \sum_{|J|=n} \|T_J \mathbf{u}\|^p \qquad \text{(by (5.1))} \end{aligned}$$

It follows that $\|\mathbf{a}_n\|_1 = \sum_{j=0}^p |(\mathbf{a}_n)_j| \le p \sum_{|J|=n} \|T_J \mathbf{u}\|^p$. Since $\mathbf{a}_0$ has a component of E_λ, there exists a constant $C > 0$ such that

$$C\,\rho(W_p)^n \le \|W_p^n \mathbf{a}_0\|_1 = \|\mathbf{a}_n\|_1 \le p \sum_{|J|=n} \|T_J \mathbf{u}\|^p. \qquad \square$$

For the 4-coefficient dilation equation in (4.1), it is easy to check that $\dim H(\tilde{\mathbf{v}}) = 0$ if and only if $(c_0, c_3) \in \{(0,0),\, (1,0),\, (0,1),\, (1,1)\}$, and the solutions are characteristic functions ([LW, Lemma 3.3]). Hence $\mathrm{Lip}_p(f) = 1/p$. Also $\dim H(\tilde{\mathbf{v}}) = 1$ if and only if $c_0 + c_3 = 1$ (Lemma 4.2), and in Proposition 4.3 we have given a formula of $\mathrm{Lip}_p(f)$ for this case. It remains to consider the case $H(\tilde{\mathbf{v}}) = 2$, which will complete all the cases for all 4-coefficient scaling functions.

THEOREM 5.3. *Consider the 4-coefficient dilation equation in (4.1) with the assumption that* $\dim H(\tilde{\mathbf{v}}) = 2$. *For p a positive even integer, the equation has a (unique) L_c^p-solution f if and only if $\rho(W_p)/2 < 1$, and in this case*

$$\mathrm{Lip}_p(f) = \frac{\ln(\rho(W_p)/2)}{-p \ln 2}.$$

Proof. Note that for any $\mathbf{u} \in H(\tilde{\mathbf{v}})$ and for any $\epsilon > 0$, we have for large n

$$\begin{aligned}\sum_{|J|=n} \|T_J\mathbf{u}\|^p &= |(\mathbf{a}_n)_0| + |(\mathbf{a}_n)_p| && \text{(by (5.1))}\\ &\le \|\mathbf{a}_n\|_1 = \|W_p^n \mathbf{a}_0\|_1 && \text{(by Proposition 5.1)}\\ &\le \|W_p^n\| \, \|\mathbf{a}_0\|_1 \\ &\le \|\mathbf{a}_0\|_1 (\rho(W_p)+\epsilon)^n. && \text{(by (5.2))}\end{aligned}$$

If we choose $\mathbf{u} \in H(\tilde{\mathbf{v}})$ as in Lemma 5.2, combining with Theorem A in Section 1, we have the first conclusion. The second assertion follows from Lemma 2.2, the estimation of $\sum_{|J|=n} \|T_J\mathbf{u}\|^p$ from above and Lemma 5.2. □

Figure 3 shows the domain of (c_0, c_3) for the existence of L_c^p-solutions for even integers using the above criterion $\rho(W_p)/2 < 1$. The curves are $\rho(W_p)/2 = 1$ corresponds to $p = 2$, 4, 6, 10, 20, and 40. Note that when $p \to \infty$ the limit is the triangular region which is the approximate region plotted in [H] for the existence of continuous 4-coefficient scaling functions using the joint spectral radius. However, we are not able to prove this assertion yet, i.e., $\lim_{p\to\infty} \mathrm{Lip}_p(f)$ is the Hölder exponent. Also we do not have a criterion for the existence of an L_c^∞-solution.

Figure 4 is the graph of $\mathrm{Lip}_4(f)$ plotted against the (c_0, c_3)-plane. It shows the overall picture of $\mathrm{Lip}_4(f)$ for the 4-coefficient case. It looks similar to the graph of $\mathrm{Lip}_2(f)$ plotted in [LMW].

We remark that if $c_0 > 0$, $c_3 > 0$, and $1 - c_0 - c_3 > 0$, then T_0 and T_1 in (4.5) are non-negative matrices. Also the vector $\mathbf{u}$ in Lemma 5.2 can be chosen to be a positive vector. Hence (5.1) still holds if p is a positive odd integer. Consequently, we have

PROPOSITION 5.4. *Consider the* 4*-coefficient dilation equation* (4.1) *with* $c_0 > 0$, $c_3 > 0$, *and* $1 - c_0 - c_3 > 0$. *Suppose* $\dim H(\tilde{\mathbf{v}}) = 2$, *then for* p *a positive integer,*

$$\mathrm{Lip}_p(f) = \frac{\ln(\rho(W_p)/2)}{-p\ln 2}. \tag{5.3}$$

Without such positivity assumption on the coefficients, the expression in (5.3) does not necessarily give $\mathrm{Lip}_p(f)$ for p odd integers. For example Figure 5a is the graph of $-\ln(\rho(W_p)/2)/(p\ln 2)$, $p = 1, \cdots, 10$, of the scaling function corresponding to $c_0 = 0.5$ and $c_3 = -0.4$. The points bounce up and down but $\mathrm{Lip}_p(f)$ should be convex in that region. Figure 5b corresponds to Daubechies scaling function D_4 ($c_3 < 0$). On this graph, the points are obtained by $-\ln(\rho(W_p)/2)/(p\ln 2)$ while the curve is $\mathrm{Lip}_p(f)$ given by (4.7). It shows that for even integer p, they coincide. For odd integer p, the values obtained by (5.3) are different to $\mathrm{Lip}_p(f)$ but surprisely close (see Table 1). Also when $p \to \infty$, in our numerical and graphical experiments, the values obtained from $-\ln(\rho(W_p)/2)/(p\ln 2)$, p odd integers, seems to converge to $\mathrm{Lip}_p(f)$ rather rapidly.

Finally we remark that for the dilation equation with $N+1$ ($N > 3$) coefficients, if $\dim H(\tilde{\mathbf{v}}) = 1$ then $\tilde{\mathbf{v}}$ is an eigenvector of both T_0 and T_1, say $T_0\tilde{\mathbf{v}} = a\tilde{\mathbf{v}}$ and $T_1\tilde{\mathbf{v}} = b\tilde{\mathbf{v}}$. Then the same technique as in the proof of Proposition 4.1 yields

$$\mathrm{Lip}_p(f) = \frac{\ln((|a|^p + |b|^p)/2)}{-p\ln 2}$$

for $1 \le p < \infty$. For the case $\dim H(\tilde{\mathbf{v}}) \ge 2$, we can use a similar method to that in

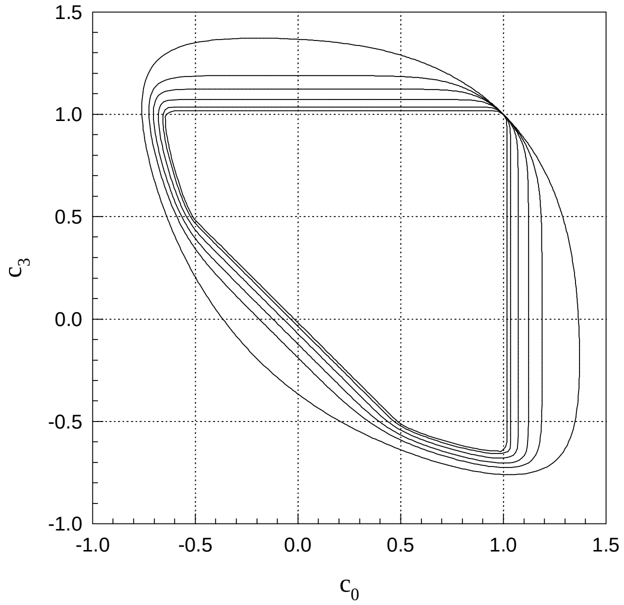

Figure 3

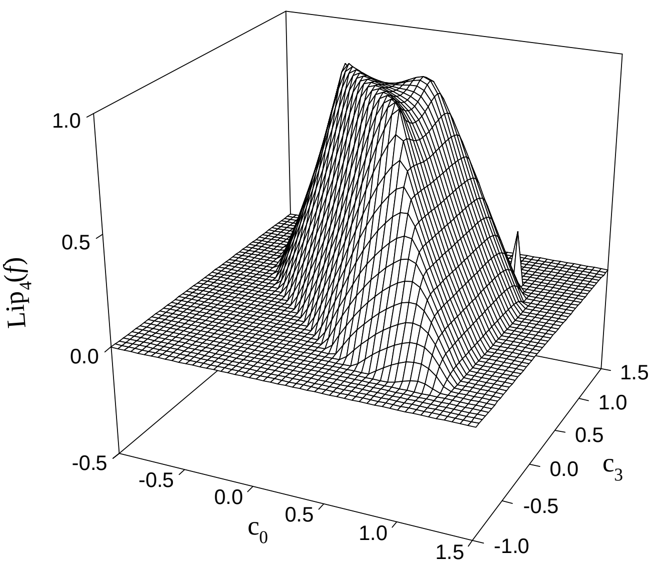

Figure 4

Proposition 5.1 to obtain a $(p+1)^{N-2} \times (p+1)^{N-2}$ square matrix W_p, and Theorem 5.3 and Proposition 5.4 will still hold for $\dim H(\tilde{\mathbf{v}}) \geq 2$.

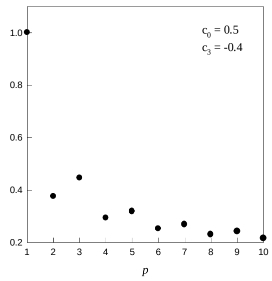

Figure 5a

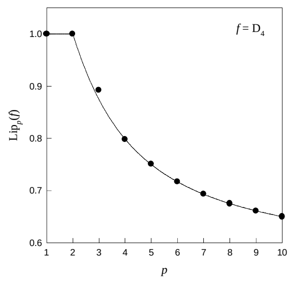

Figure 5b

p	$\mathrm{Lip}_p(f)$	value obtained by (5.3)
3	0.874185416	0.892690635
5	0.749617426	0.750414497
7	0.692852392	0.692893269
9	0.661125656	0.661127939

Table 1

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