THE INVERSE TRIGONOMETRIC FUNCTIONS

PO-LAM YUNG

We knew that sin is a strictly increasing function from $[-\frac{\pi}{2}, \frac{\pi}{2}]$, and cos is a strictly decreasing function on $[0, \pi]$. The image of both is $[-1, 1]$. This allows us to *define* the inverse functions to these, by

$$\arcsin\colon [-1,1] \to [-\frac{\pi}{2}, \frac{\pi}{2}],$$

and

$$\arccos\colon [-1,1] \to [0,\pi],$$

so that

$$\sin(\arcsin(x)) = x$$

and

$$\cos(\arccos(x)) = x$$

for all $x \in [-1, 1]$.

By the inverse function theorem, arcsin and arccos are differentiable functions on the open interval $(-1, 1)$, and their derivatives are given by

$$\frac{d}{dx}\arcsin(x) = \frac{1}{\cos(\arcsin(x))} = \frac{1}{\sqrt{1-x^2}},$$

$$\frac{d}{dx}\arccos(x) = -\frac{1}{\sin(\arccos(x))} = -\frac{1}{\sqrt{1-x^2}}$$

for all $x \in (-1, 1)$. (The last equalities follow from $\sin^2 y + \cos^2 y = 1$ with $y = \arcsin(x)$ and $y = \arccos(x)$ respectively.)

The tangent function is a strictly increasing function on $(-\frac{\pi}{2}, \frac{\pi}{2})$, and its image is $\mathbb{R}$. Hence we can *define* the inverse function to tan, by

$$\arctan\colon \mathbb{R} \to \left(-\frac{\pi}{2}, \frac{\pi}{2}\right),$$

by

$$\tan(\arctan(x)) = x$$

for all $x \in \mathbb{R}$.

By the inverse function theorem, arctan is a differentiable function on $\mathbb{R}$, and its derivative is given by

$$\frac{d}{dx}\arctan(x) = \frac{1}{\sec^2(\arctan(x))} = \frac{1}{1+x^2}$$

for all $x \in (-1, 1)$. (The last equality follows from $\sec^2 y = 1 + \tan^2 y$ with $y = \arctan(x)$.)

The above gives us some useful formula in computing the anti-derivatives of

$$\frac{1}{\sqrt{1-x^2}} \quad \text{and} \quad \frac{1}{1+x^2}$$

later on. In addition, we now know how to expand $\arctan x$ in power series:

Proposition 1. *For any $|x| < 1$, we have*

$$\arctan(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} x^{2k+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots$$

Proof. The radius of convergence of the power series on the right hand side is equal to 1. Hence it defines a differentiable function on $(-1, 1)$. Let's call this function $f(x)$. Then for $|x| < 1$, we have

$$f'(x) = \sum_{k=0}^{\infty} \frac{d}{dx} \frac{(-1)^k}{2k+1} x^{2k+1} = \sum_{k=0}^{\infty} (-1)^k x^{2k} = \frac{1}{1+x^2},$$

the last equality following from the formula for a geometric series. This shows that

$$\frac{d}{dx}(f(x) - \arctan(x)) = 0$$

for all $|x| < 1$. In particular, by a corollary to the mean-value theorem, $f(x) - \arctan(x)$ is a constant, and hence

$$f(x) - \arctan(x) = f(0) - \arctan(0) = 0$$

for all $|x| < 1$, as desired. □