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An improved model function method for choosing regularization parameters in linear inverse problems

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Abstract
This paper proposes a new model function iterative method, that improves our earlier work (Kunisch K and Zou J 1998 *Inverse Problems* **14** 1247–64), on finding some reasonable regularization parameters in the widely used output least squares formulations of linear inverse problems, based on the Morozov and damped Morozov principles. The new algorithm updates the model parameters in a computationally more stable manner. In addition, the method can be rigorously shown to have global convergence, in particular, its convergence is carried on strictly monotone decreasingly. This property seems especially useful and important in real applications as it enables us to start with some larger regularization parameters, and thus with more stable least squares problems. Numerical experiments for one- and two-dimensional elliptic inverse problems and an inverse integral problem are presented to illustrate the efficiency of the proposed algorithm.

1. Introduction

Inverse problems can find wide applications in engineering and scientific computation [7]. Most inverse problems are ill-posed. The hardest issue in numerical solutions of inverse problems is the instability of the solutions with respect to the noise in the observation data, namely, small perturbations in the observation data may lead to large effects on the considered solutions. To ensure a feasible and stable numerical resolution, some type of regularization should be introduced, which entails the necessity of choosing an appropriate regularization parameter. In fact, the effectiveness of a regularization method depends strongly on the choice of a good regularization parameter.

Also, in testing the Tikhonov regularization method for solving an inverse problem, one often needs to try a large number of regularization parameters in order to find a reasonably good parameter. This is often very time-consuming. One may save lots of time and computational

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cost if there is an iterative method which can give us a reasonable regularization parameter within a practically acceptable number of iterations.

There exists a significant amount of research in the literature on the development of appropriate strategies for selecting regularization parameters, see [1, 2, 4, 8, 13] and references therein, while much less work has been carried out on the numerical realization of such strategies. In fact, it appears that very few of the strategies are utilized for practical applications.

Motivated by a similar technique developed by Hebden *et al* (cf [3]) for quasi-Newton methods, Ito and Kunisch proposed a model function approach (with four parameters) to solve some nonlinear parameter identification problems [9]. In [11], Kunisch and Zou proposed a two-parameter algorithm to choose some reasonable regualrization parameters in the Tikhonov regularization formulation for linear inverse problems. The basic tool is to use the well known Morozov discrepancy principle and the damped Morozov discrepancy principle [5, 6, 9–12].

This paper intends to make some further contribution to the subject in proposing some practical parameter choice strategies. We first derive a new model function which is quite different from the model used in [11] and is more consistent with our original approximation of the Morozov function. Some nice properties of the new model are analysed. Then we combine the model function with an approximation technique to propose a new two-parameter algorithm to solve the Morozov equation. The new algorithm updates the model parameters in a computationally more stable manner. In addition, the method can be rigorously shown to have global convergence, in particular, its convergence is carried on strictly monotone decreasingly. This property seems especially useful and important in real applications as it enables us to start with some larger regularization parameters, and thus with more stable least squares problems. Numerical experiments for three linear elliptic and integral inverse problems are presented to illustrate the efficiency of the proposed algorithm. In addition, the method and techniques adopted in this paper may be generalized to solve some other regularization parameter choice principles, though only the Morozov discrepancy principles are focused upon here. This is one of our future research topics.

2. Output least squares formulation with Tikhonov regularization

In this section, we shall first formulate our problem and review some basic notation and useful results from [11]. We shall consider inverse problems of the form

$$Tf = z$$

where T is a linear bounded operator mapping the parameter space X into the observation space Y, and z is the observation data. In applications, z is often corrupted by some error and the noisey data of z with noise level δ are denoted by z^δ. Such ill-posed problems are often solved through the following well-posed minimization problem of the Tikhonov functional:

$$\min_{f \in X} J_\beta(f) = \frac{1}{2}\|Tf - z^\delta\|_Y^2 + \frac{\beta}{2}\|f\|_X^2 \tag{2.1}$$

where $\beta > 0$ is the regularization parameter, and $\|\cdot\|_Y$ and $\|\cdot\|_X$ denote the norms in the Hilbert spaces Y and X, respectively. The problem (2.1) has a unique minimizer for any fixed β, denoted as $f(\beta)$, and it can be characterized as the solution to the system

$$T^*Tf + \beta f = T^*z^\delta$$

or in variational form

$$(Tf, Tg)_Y + \beta(f, g)_X = (z^\delta, Tg)_Y \qquad \text{for all } g \in X. \tag{2.2}$$

We will frequently use the minimal cost functional of (2.1):

$$F(\beta) = \frac{1}{2}\|Tf(\beta) - z^\delta\|_Y^2 + \frac{\beta}{2}\|f(\beta)\|_X^2. \tag{2.3}$$

It is known (cf [11]) that both $f(\beta)$ and $F(\beta)$ are infinitely differentiable with respect to β, and $w = f'(\beta) \in X$ solves

$$(Tw, Tg)_Y + \beta(w, g)_X = -(f(\beta), g)_X \qquad \text{for all } g \in X. \tag{2.4}$$

Moreover, we have

$$F'(\beta) = \tfrac{1}{2}\|f(\beta)\|_X^2, \qquad F''(\beta) = (f(\beta), f'(\beta))_X. \tag{2.5}$$

It is easy to see that $f(\beta) = 0$ if and only if $z^\delta \in \ker T^*$. Therefore, we shall always assume $z^\delta \notin \ker T^*$ in the subsequent discussions. This implies by (2.4) and (2.5)

$$F'(\beta) > 0, \qquad F''(\beta) < 0. \tag{2.6}$$

The Morozov and damped Morozov principles

The very popular Morozov principle has received a considerable amount of attention in linear inverse problems (cf [1, 7, 12, 13]) and turns out to be very effective for many inverse problems [4, 5, 9–11]. This principle suggests choosing the regularization parameter β in such a way that the error due to the regularization is equal to the error due to the observation data. That is, β is chosen according to

$$\|Tf(\beta) - z^\delta\|_Y^2 + \beta^\gamma\|f(\beta)\|_X^2 = \delta^2, \tag{2.7}$$

where $\gamma \in [1, \infty]$, and δ is the noise level defined by $\delta = \|z - z^\delta\|_Y$. In terms of $F(\beta)$, the Morozov equation (2.7) can be written as

$$F(\beta) + (\beta^\gamma - \beta)F'(\beta) = \tfrac{1}{2}\delta^2. \tag{2.8}$$

Then the entire difficulty of choosing the regularization parameter β lies in how to solve this highly nonlinear equation (2.8) of β effectively. One may apply Newton's method or the quasi-Newton's method to solve (2.8) with quadratical or superlinear convergence, see [11]. But these methods converge only locally, and they must start with some very good initial regularization parameters. This is certainly not practical as these good initial parameters may already be good enough to serve as the required regularization parameters in most applications. We shall propose a more practical and global convergent iterative method in the subsequent sections.

3. The model function for $F(\beta)$

This section is devoted to the derivation of a new model function for $F(\beta)$ and to some of its nice properties. This model function is the basis for our new iterative method for solving the Morozov equation (2.8). It is quite different from the one proposed in [11], which is based on the assumption $F(0) = 0$, and is computationally more stable.

The two-parameter algorithm in [11] is based on the following important identity:

$$2F(\beta) + 2\beta F'(\beta) + \|Tf(\beta)\|_Y^2 = 2C_0 \tag{3.1}$$

where C_0 is an integration constant. By further assuming

$$(Tf(\beta), Tf(\beta))_Y \approx \tilde{T}(f(\beta), f(\beta))_X, \tag{3.2}$$

where $\tilde{T}$ is a constant, one derives the model function $m(\beta)$ from (3.1):

$$m(\beta) = C_0 + \frac{\tilde{C}}{\tilde{T} + \beta}.$$

Then by assuming $F(0) = 0$, or $m(0) = 0$, one can remove the constant C_0 and arrive at the following two-parameter model function (cf [11]):

$$m(\beta) = c\left(1 - \frac{t}{t + \beta}\right). \tag{3.3}$$

With this model function, Kunisch and Zou [11] proposed the following two-parameter algorithm to solve the Morozov equation (2.8):

Two-parameter algorithm I

Given $\beta_0 > 0$ and $\varepsilon > 0$, set $k = 0$.

(1) Solve (2.2) for $f(\beta_k)$ and thus compute $F(\beta_k)$ and $F'(\beta_k) = \frac{1}{2}\|f(\beta_k)\|_X^2$.
Then update t_k and c_k from

$$m(\beta_k) = c_k\left(1 - \frac{t_k}{t_k + \beta_k}\right) = F(\beta_k), \tag{3.4}$$

$$m'(\beta_k) = \frac{c_k t_k}{(t_k + \beta_k)^2} = F'(\beta_k). \tag{3.5}$$

(2) Set the kth model function

$$m(\beta) = c_k\left(1 - \frac{t_k}{t_k + \beta}\right),$$

and solve for β_{k+1} the approximate Morozov's equation

$$m(\beta) + (\beta^\gamma - \beta)m'(\beta) = \tfrac{1}{2}\delta^2. \tag{3.6}$$

(3) STOP if $|\beta_{k+1} - \beta_k| \leqslant \varepsilon$; otherwise set $k := k + 1$, GOTO (1).

The following formulae can be easily found from (3.4) and (3.5) for computing t_k and c_k:

$$t_k = \frac{\beta_k^2 F'(\beta_k)}{F(\beta_k) - \beta_k F'(\beta_k)}, \qquad c_k = \frac{F^2(\beta_k)}{F(\beta_k) - \beta_k F'(\beta_k)}. \tag{3.7}$$

We remark that the denominators in both t_k and c_k can be written as

$$F(\beta_k) - \beta_k F'(\beta_k) = \tfrac{1}{2}\|Tf(\beta_k) - z^\delta\|_Y^2, \tag{3.8}$$

which may approach zero when β_k becomes close to its convergence limit, and so may cause some computational instability in applications.

A new model function

Now, we are going to present a new model function, which is intended to be computationally more stable. Unlike in [11], we will not assume $F(0) = 0$ here but are still able to derive a model function with only two parameters. In fact we will find the exact value of the integration constant C_0 in (3.1). To do so, by (2.3) we have

$$F(\beta) = \frac{1}{2}\|z^\delta\|_Y^2 - (Tf(\beta), z^\delta)_Y + \frac{1}{2}\|Tf(\beta)\|_Y^2 + \frac{\beta}{2}\|f(\beta)\|_X^2. \tag{3.9}$$

Taking $g = f(\beta)$ in (2.2) we know

$$(Tf(\beta), z^\delta)_Y = \|Tf(\beta)\|_Y^2 + \beta\|f(\beta)\|_X^2.$$

Substituting this into (3.9), we obtain the following identity:

$$F(\beta) = \frac{1}{2}\|z^\delta\|_Y^2 - \frac{1}{2}\|Tf(\beta)\|_Y^2 - \frac{\beta}{2}\|f(\beta)\|_X^2, \tag{3.10}$$

which, by (2.5), can be rewritten as

$$2F(\beta) + 2\beta F'(\beta) + \|Tf(\beta)\|_Y^2 = \|z^\delta\|_Y^2. \tag{3.11}$$

This is exactly the identity (3.1), but C_0 is now explicitly given by $\frac{1}{2}\|z^\delta\|_Y^2$. Using the approximation (3.2), where $\tilde{T}$ is a constant, equation (3.11) reduces to

$$\beta m'(\beta) + m(\beta) + \tilde{T}m'(\beta) = \tfrac{1}{2}\|z^\delta\|_Y^2, \tag{3.12}$$

where $m(\beta)$ is the model function for $F(\beta)$.

Solving the ordinary differential equation (3.12), we obtain a new model function

$$m(\beta) = \frac{1}{2}\|z^\delta\|_Y^2 + \frac{\tilde{C}}{\tilde{T} + \beta}. \tag{3.13}$$

This model function still keeps only two parameters, $\tilde{C}$ and $\tilde{T}$, without assuming $F(0) = 0$.

Replacing the model function $m(\beta)$ in the two-parameter algorithm I by our new model function (3.13), we can formulate a new two-parameter algorithm as follows.

The two-parameter algorithm II

Given $\beta_0 > 0$ and $\varepsilon > 0$, set $k = 0$.

(1) Solve (2.2) for $f(\beta_k)$ and thus compute $F(\beta_k)$ and $F'(\beta_k) = \frac{1}{2}\|f(\beta_k)\|_X^2$.
Then update T_k and C_k from

$$m_k(\beta_k) = \frac{1}{2}\|z^\delta\|_Y^2 + \frac{C_k}{T_k + \beta_k} = F(\beta_k), \tag{3.14}$$

$$m_k'(\beta_k) = -\frac{C_k}{(T_k + \beta_k)^2} = F'(\beta_k). \tag{3.15}$$

(2) Set the kth model function

$$m_k(\beta) = \frac{1}{2}\|z^\delta\|_Y^2 + \frac{C_k}{T_k + \beta}, \tag{3.16}$$

and solve for β_{k+1} the approximate Morozov's equation

$$m_k(\beta) + (\beta^\gamma - \beta)m_k'(\beta) = \tfrac{1}{2}\delta^2. \tag{3.17}$$

(3) STOP if $|\beta_{k+1} - \beta_k| \leqslant \varepsilon$; otherwise set $k := k + 1$, GOTO (1).

This new algorithm has some advantages over the two-parameter algorithm I. From the equalities (3.14) and (3.15) we can easily derive the formulae for updating the two parameters C_k and T_k:

$$T_k = \frac{\|Tf(\beta_k)\|_Y^2}{\|f(\beta_k)\|_X^2}, \qquad C_k = -\frac{(\|Tf(\beta_k)\|_Y^2 + \beta_k\|f(\beta_k)\|_X^2)^2}{2\|f(\beta_k)\|_X^2}. \tag{3.18}$$

It is interesting to note that the formula for updating T_k in (3.18) is exactly the approximation form (3.2) that we have made for deriving the model function. However, the formula for t_k in (3.7) derived in [11] does not have this property. So the new model function appears more consistent with the original approximation principle. Clearly the inconsistency of the formulation in (3.7) comes from the assumption $F(0) = 0$ made in [11]. In addition, the new formulae in (3.18) are more computationally stable than those in (3.7), see (3.8).

Also, one can easily see from (3.18) and (3.16) that for any $\beta > 0$,

$$m_k'(\beta) = -\frac{C_k}{(T_k+\beta)^2} > 0, \qquad m_k''(\beta) = \frac{2C_k}{(T_k+\beta)^3} < 0. \tag{3.19}$$

So the new model function $m(\beta)$ in (3.16) still preserves the monotonicity and concavity of $F(\beta)$.

Monotonicities of the parameters T_k and C_k

In the rest of this section, we shall present some observations about the two parameters T_k and C_k in (3.18), which are very important for us to greatly improve our new two-parameter algorithm II: both T_k and C_k are monotonic with respect to β_k. To see this, we define

$$g(\beta) = \frac{\|Tf(\beta)\|_Y^2}{\|f(\beta)\|_X^2}, \qquad h(\beta) = \frac{(\|Tf(\beta)\|_Y^2 + \beta\|f(\beta)\|_X^2)^2}{\|f(\beta)\|_X^2} \tag{3.20}$$

associated with T_k and C_k respectively, see (3.18), and their quotient

$$q(\beta) = \frac{(\|Tf(\beta)\|_Y^2 + \beta\|f(\beta)\|_X^2)^2}{\|Tf(\beta)\|_Y^2}. \tag{3.21}$$

Lemma 3.1. *For any $\beta > 0$, we have $g'(\beta) \geqslant 0$.*

Proof. It is easy to verify that

$$g'(\beta) = \frac{2(Tf'(\beta), Tf(\beta))_Y\|f(\beta)\|_X^2 - 2(f'(\beta), f(\beta))_X\|Tf(\beta)\|_Y^2}{\|f(\beta)\|_X^4},$$

so it suffices to show the numerator is non-negative, or equivalently

$$\varphi(\beta) \equiv (Tf'(\beta), Tf(\beta))_Y\|f(\beta)\|_X^2 - (f'(\beta), f(\beta))_X\|Tf(\beta)\|_Y^2 \geqslant 0. \tag{3.22}$$

From (2.2) and (2.4) we have the following identities:

$$(Tf'(\beta), Tf(\beta))_Y = -\|f(\beta)\|_X^2 - \beta(f'(\beta), f(\beta))_X, \tag{3.23}$$

$$(f'(\beta), f(\beta))_X = -\|Tf'(\beta)\|_Y^2 - \beta\|f'(\beta)\|_X^2. \tag{3.24}$$

Substituting them into (3.22), we derive

$$\varphi(\beta) = (\|Tf(\beta)\|_Y^2 + \beta\|f(\beta)\|_X^2)(\|Tf'(\beta)\|_Y^2 + \beta\|f'(\beta)\|_X^2) - \|f(\beta)\|_X^4. \tag{3.25}$$

But using the Cauchy–Schwarz inequality, we have by (3.23)

$$\|f(\beta)\|_X^2 \leqslant \|Tf'(\beta)\|_Y\|Tf(\beta)\|_Y + \beta\|f'(\beta)\|_X\|f(\beta)\|_X.$$

Taking squares on both sides and then applying the Cauchy–Schwarz inequality leads to

$$\begin{aligned}\|f(\beta)\|_X^4 &\leqslant 2\beta\|Tf'(\beta)\|_Y\|Tf(\beta)\|_Y\|f'(\beta)\|_X\|f(\beta)\|_X + \|Tf'(\beta)\|_Y^2\|Tf(\beta)\|_Y^2 \\ &\quad + \beta^2\|f'(\beta)\|_X^2\|f(\beta)\|_X^2 \leqslant \beta(\|Tf'(\beta)\|_Y^2\|f(\beta)\|_X^2 + \|Tf(\beta)\|_Y^2\|f'(\beta)\|_X^2) \\ &\quad + \|Tf'(\beta)\|_Y^2\|Tf(\beta)\|_Y^2 + \beta^2\|f'(\beta)\|_X^2\|f(\beta)\|_X^2 \\ &= (\|Tf(\beta)\|_Y^2 + \beta\|f(\beta)\|_X^2)(\|Tf'(\beta)\|_Y^2 + \beta\|f'(\beta)\|_X^2),\end{aligned}$$

which, together with (3.25), implies $\varphi(\beta) \geqslant 0$. □

Lemma 3.2. *For any $\beta > 0$, we have $h'(\beta) \geqslant 0$.*

Proof. We rewrite $h(\beta)$ as $h(\beta) = \xi(\beta)\eta(\beta)$, where

$$\xi(\beta) = \frac{\|Tf(\beta)\|_Y^2 + \beta\|f(\beta)\|_X^2}{\|f(\beta)\|_X^2} = g(\beta) + \beta \tag{3.26}$$

and

$$\eta(\beta) = \|Tf(\beta)\|_Y^2 + \beta\|f(\beta)\|_X^2. \tag{3.27}$$

By (3.23), we have

$$\eta'(\beta) = 2(Tf'(\beta), Tf(\beta))_Y + 2\beta(f'(\beta), f(\beta))_X + \|f(\beta)\|_X^2 = -\|f(\beta)\|_X^2. \tag{3.28}$$

Therefore

$$\begin{aligned} h'(\beta) &= \xi'(\beta)\eta(\beta) + \xi(\beta)\eta'(\beta) \\ &= (g'(\beta)+1)(\|Tf(\beta)\|_Y^2 + \beta\|f(\beta)\|_X^2) - \frac{\|Tf(\beta)\|_Y^2 + \beta\|f(\beta)\|_X^2}{\|f(\beta)\|_X^2}\|f(\beta)\|_X^2 \\ &= g'(\beta)(\|Tf(\beta)\|_Y^2 + \beta\|f(\beta)\|_X^2), \end{aligned} \tag{3.29}$$

which implies $h'(\beta) \geqslant 0$ by lemma 3.1. □

Lemma 3.3. *For any $\beta > 0$, we have $q'(\beta) \leqslant 0$.*

Proof. We rewrite $q(\beta)$ as

$$q(\beta) = \eta(\beta)\left(1 + \frac{\beta}{g(\beta)}\right)$$

where $\eta(\beta)$ is as in (3.27). Then a straightforward computation together with (3.28) gives

$$\begin{aligned} q'(\beta) &= \eta'(\beta) + \frac{\eta(\beta) + \beta\eta'(\beta)}{g(\beta)} - \frac{\beta\eta(\beta)g'(\beta)}{g^2(\beta)} \\ &= -\frac{\beta g'(\beta)(\|Tf(\beta)\|_Y^2 + \beta\|f(\beta)\|_X^2)}{g^2(\beta)} \end{aligned}$$

which completes the proof by lemma 3.1. □

4. An improved two-parameter algorithm

In this section, we first show the solvability of the approximate Morozov equation (3.17) and then propose an improved two-parameter algorithm.

We denote $\mu = \inf_{f\in X} \|Tf - z^\delta\|_Y$. If the error-free data z is obtainable, i.e. there exists some $f^* \in X$ such that $Tf^* = z$, then $\mu \leqslant \delta$. Without loss of generality, we assume

$$0 \leqslant \mu < \delta, \qquad \mu < \|z^\delta\|_Y. \tag{4.1}$$

One can show (cf [10]) that

$$\lim_{\beta\to 0+} F(\beta) = \tfrac{1}{2}\mu^2, \qquad \lim_{\beta\to 0+} \beta\|f(\beta)\|_X^2 = 0, \tag{4.2}$$

based on which we obtain for $q(\beta)$, see (3.21), that

$$\lim_{\beta\to 0+} q(\beta) = \|z^\delta\|_Y^2 - \mu^2. \tag{4.3}$$

For $\gamma \in [1, \infty]$, we introduce two functions of β as follows:

$$G(\beta) = F(\beta) + (\beta^\gamma - \beta)F'(\beta), \tag{4.4}$$

$$G_k(\beta) = m_k(\beta) + (\beta^\gamma - \beta)m_k'(\beta), \tag{4.5}$$

then we can show the following result about the solvability of the approximate Morozov equation (3.17).

Theorem 4.1. *Assuming that $G_k(\beta_k) > \delta^2/2$ and β_k is small enough, then there exists a unique solution β_{k+1} to (3.17) and $G(\beta_{k+1}) < \frac{1}{2}\delta^2$.*

Proof. By (3.4), (3.5), (4.4) and (4.5), we have

$$G_k(\beta_k) = G(\beta_k). \tag{4.6}$$

From (3.18), (3.16) and the definition of $q(\beta)$, we know

$$G_k(0) = \tfrac{1}{2}\|z^\delta\|_Y^2 - \tfrac{1}{2}q(\beta_k), \tag{4.7}$$

which together with lemma 3.3 and equation (4.3) indicates $G_k(0) < \frac{1}{2}\delta^2$ when β_k is very close to zero. Therefore, β_{k+1} is well defined whenever β_k is small enough. Moreover, it follows from (3.19) that

$$G_k'(\beta) = \gamma\beta^{\gamma-1}m_k'(\beta) + (\beta^\gamma - \beta)m_k''(\beta) > 0, \qquad \beta \in (0, 1]. \tag{4.8}$$

This monotonicity of $G_k(\beta)$ implies $\beta_{k+1} < \beta_k$. To show $G(\beta_{k+1}) < \frac{1}{2}\delta^2$, noting (4.6) and $G_k(\beta_{k+1}) = \frac{1}{2}\delta^2$, it is sufficient to prove $G_{k+1}(\beta_{k+1}) < G_k(\beta_{k+1})$. By direct computation we have

$$G_{k+1}(\beta_{k+1}) = \frac{1}{2}\|z^\delta\|_Y^2 - \frac{h(\beta_{k+1})}{2(\beta_{k+1} + g(\beta_{k+1}))^2}\{g(\beta_{k+1}) + 2\beta_{k+1} - \beta_{k+1}^\gamma\}, \tag{4.9}$$

$$G_k(\beta_{k+1}) = \frac{1}{2}\|z^\delta\|_Y^2 - \frac{h(\beta_k)}{2(\beta_{k+1} + g(\beta_k))^2}\{g(\beta_k) + 2\beta_{k+1} - \beta_{k+1}^\gamma\}. \tag{4.10}$$

Consider the following function with $a = \beta_{k+1}$:

$$\varphi(\beta) = \frac{h(\beta)(g(\beta) + 2a - a^\gamma)}{(g(\beta) + a)^2}, \qquad \beta \in (\beta_{k+1}, \beta_k). \tag{4.11}$$

The desired result then follows if we can show that $\varphi(\beta)$ is decreasing.

In fact, by the definition of $g(\beta)$, $h(\beta)$ and (3.29), we have

$$h(\beta) = \|f(\beta)\|_X^2(g(\beta) + \beta)^2, \qquad h'(\beta) = \|f(\beta)\|_X^2 g'(\beta)(g(\beta) + \beta). \tag{4.12}$$

Then by direct computation and rearrangement, we obtain

$$\varphi'(\beta) = \frac{\|f(\beta)\|_X^2 g'(\beta)(g(\beta) + \beta)}{(g(\beta) + a)^3}\{(a^\gamma - \beta)g(\beta) + \beta(2a^\gamma - 3a) + a(2a - a^\gamma)\}. \tag{4.13}$$

Among the three terms in the brackets, the first one is negative and the rest are linear functions of β. The linear function is decreasing due to $2a^\gamma - 3a < 0$ and its value at $\beta = a$ is $a(2a^\gamma - 3a) + a(2a - a^\gamma) = a(a^\gamma - a) \leqslant 0$. Therefore $\varphi'(\beta) \leqslant 0$, and $\varphi(\beta)$ is decreasing. □

Theorem 4.1 indicates that the solution to the approximate Morozov equation (3.17) is guaranteed only when β_k is sufficiently small. Our numerical experiments also confirm this observation, and we found that only when the starting value β_0 is of the same magnitude as the true solution to the Morozov equation (2.8) does β_1 exist. That is, the two-parameter algorithm II is only a locally convergent algorithm, and moreover, as analysed in theorem 4.1, β_1 is already smaller than the true solution, so it is overestimated and is not regarded as a reasonable regularization parameter in practical applications.

4.1. An improved two-parameter algorithm

Next, we propose an improved two-parameter algorithm, which can preserve the nice properties of the new model function (3.16) but also achieve a global convergence. For this, we replace the function $G_k(\beta)$ by the following relaxation form:

$$\hat{G}_k(\beta) = G_k(\beta) + \alpha_k(G_k(\beta) - G_k(\beta_k)). \tag{4.14}$$

It is important to observe that the sign of the second term in (4.14) is determined by the sign of α_k as $G_k(\beta) \leqslant G_k(\beta_k)$ for $\beta \in [0, \beta_k]$. α_k should lie in a range such that the approximate Morozov equation

$$\hat{G}_k(\beta) = \tfrac{1}{2}\delta^2$$

always has a unique solution, or equivalently, if $\hat{G}_k(0) < \delta^2/2$. This suggests we should choose α_k as follows:

$$\hat{G}_k(0) = \hat{\alpha}\delta^2, \qquad \forall\, \hat{\alpha} \in [0, \tfrac{1}{2}). \tag{4.15}$$

Combining (4.14) with (4.15), we can easily find

$$\alpha_k = \frac{G_k(0) - \hat{\alpha}\,\delta^2}{G_k(\beta_k) - G_k(0)}. \tag{4.16}$$

From the monotonicity of $G_k(\beta)$ we know $1 + \alpha_k = \frac{G(\beta_k) - \hat{\alpha}\,\delta^2}{G(\beta_k) - G_k(0)} > 0$ as long as $G(\beta_k) > \frac{1}{2}\delta^2$, which implies

$$\hat{G}_k'(\beta) > 0. \tag{4.17}$$

So $\hat{G}_k(\beta)$ preserves the monotonicity. With this new function $\hat{G}_k(\beta)$, we are now ready to state our improved two-parameter algorithm.

Two-parameter algorithm III

Given $\beta_0 > 0$ and $\varepsilon > 0$, set $k = 0$.

(1) Solve (2.2) for $f(\beta_k)$ and thus compute $F(\beta_k)$ and $F'(\beta_k) = \frac{1}{2}\|f(\beta_k)\|_X^2$.
Then update T_k and C_k from

$$m_k(\beta_k) = \frac{1}{2}\|z^\delta\|_Y^2 + \frac{C_k}{T_k + \beta_k} = F(\beta_k), \tag{4.18}$$

$$m_k'(\beta_k) = -\frac{C_k}{(T_k + \beta_k)^2} = F'(\beta_k). \tag{4.19}$$

(2) Set the kth model function

$$m_k(\beta) = \frac{1}{2}\|z^\delta\|_Y^2 + \frac{C_k}{T_k + \beta}, \tag{4.20}$$

and solve for β_{k+1} the approximate Morozov's equation

$$\hat{G}_k(\beta) = \tfrac{1}{2}\delta^2. \tag{4.21}$$

(3) STOP If $\hat{G}_k(\beta_k) \leqslant \frac{1}{2}\delta^2$ or $|\beta_{k+1} - \beta_k| \leqslant \varepsilon$; otherwise set $k := k + 1$, GOTO (1).

Theorem 4.2. *If $\hat{G}_0(\beta_0) > \frac{1}{2}\delta^2$, then the sequence $\{\beta_k\}$ generated by the two-parameter algorithm III is well defined. Moreover, the sequence is either finite and terminates at some β_k satisfying $G(\beta_k) \leqslant \frac{1}{2}\delta^2$, or it is infinite and converges to the unique solution β^* of the Morozov equation (2.8) strictly monotone decreasingly with any initial value β_0 lying in $(\beta^*, 1)$.*

Proof. It suffices to show that if $\hat{G}_k(\beta_k) \leqslant \frac{1}{2}\delta^2$ is never reached then $\{\beta_k\}$ converges to β^*. So we assume

$$\hat{G}_k(\beta_k) > \tfrac{1}{2}\delta^2 \qquad \text{for all } k. \tag{4.22}$$

First, we have from (4.21), (4.22) and the monotonicity of $\hat{G}_k(\beta)$ (see (4.17)) that

$$\beta_{k+1} < \beta_k$$

second, the following relation is obvious by using (4.18), (4.19) and (4.14):

$$\hat{G}_k(\beta_k) = G_k(\beta_k) = G(\beta_k), \tag{4.23}$$

this, together with (4.22), implies

$$\beta_k > \beta^* \qquad \text{for all } k. \tag{4.24}$$

Therefore, the convergence of $\{\beta_k\}$ follows from the monotone convergence theorem.

Let $\lim_{k\to\infty} \beta_k = \bar{\beta}$. We next show that $\bar{\beta}$ is the unique solution β^* to the Morozov equation (2.8), namely $G(\bar{\beta}) = \frac{1}{2}\delta^2$. First, taking limits for T_k and C_k in (3.18), we obtain

$$\lim_{k\to\infty} T_k = \frac{\|Tf(\bar{\beta})\|_Y^2}{\|f(\bar{\beta})\|_X^2} = g(\bar{\beta}), \tag{4.25}$$

$$\lim_{k\to\infty} C_k = -\frac{(\|Tf(\bar{\beta})\|_Y^2 + \bar{\beta}\|f(\bar{\beta})\|_X^2)^2}{2\|f(\bar{\beta})\|_X^2} = -\frac{1}{2}h(\bar{\beta}). \tag{4.26}$$

Then by (4.18) and (4.19) we know

$$G(\beta_{k+1}) = G_{k+1}(\beta_{k+1}),$$

which, with (4.9) and (4.10), gives

$$G(\bar{\beta}) = \lim_{k\to\infty} G(\beta_{k+1}) = \lim_{k\to\infty} G_{k+1}(\beta_{k+1}) = \lim_{k\to\infty} G_k(\beta_{k+1}). \tag{4.27}$$

But it follows from (4.21) and (4.14) that

$$G_k(\beta_{k+1}) + \alpha_k\,(G_k(\beta_{k+1}) - G_k(\beta_k)) = \tfrac{1}{2}\delta^2. \tag{4.28}$$

By the definitions of $G_k(\beta)$ and α_k, (4.25), (4.26) and the convergence of β_k, we see

$$\lim_{k\to\infty} \{G_k(\beta_{k+1}) - G_k(\beta_k)\} = 0,$$

and that α_k is convergent. This, along with (4.27) and (4.28) gives $G(\bar{\beta}) = \frac{1}{2}\delta^2$. □

5. Numerical examples

We now present some numerical experiments to show the effectiveness of the new algorithm, the two-parameter algorithm III. The same three examples as in [11] are tested. All the discretizations used here, e.g., finite element methods, integral quadrature rules, for the forward problems and inverse problems are the same as in [11]. The algorithms will terminate when $|\beta_{k+1} - \beta_k|/\beta_{k+1} \leqslant 10^{-2}$. The initial guesses β_0 for all iterations are always taken to be 0.1, a very rough initial value. In all the tables, β_{opt} stands for the optimal β value which achieves the minimum for $\|f(\beta) - f^*\|_{L^2(\Omega)}$, and is computed as in [11]; β_M stands for the solution of the exact Morozov principle ($\gamma = \infty$ in (2.8), examples 1 and 2) and for the solution of the damped Morozov principle (2.8) (example 3); the rows with TPA(1) and TPA(3) give the number of iterations required by the two-parameter algorithms I and III respectively, the row with Iter(3) contains the β values obtained at the third iterations of TPA(3); the row with

Table 1. β_{opt}, β_M and the number of iterations with $\hat{\alpha} = \frac{1}{4}$.

$\hat{\delta}$	0.01	0.03	0.05	0.07	0.1
β_{opt}	2.77×10^{-6}	9.90×10^{-6}	1.72×10^{-5}	2.60×10^{-5}	4.34×10^{-5}
Iter(3)	3.63×10^{-6}	1.32×10^{-5}	2.40×10^{-5}	3.34×10^{-5}	4.93×10^{-5}
β_M	3.62×10^{-6}	1.25×10^{-5}	2.23×10^{-5}	3.24×10^{-5}	4.74×10^{-5}
TPA(3)	4	6	6	6	6
TPA(1)	9	11	14	15	16
Iter(KZ)	4	4	5	5	5

Iter(KZ) gives the number of iterations required by the combined algorithm proposed in [11], namely the locally convergent quasi-Newton's method with the initial guesses from the second iterates of the two-parameter algorithm I (these numbers also count two iterations from the two-parameter algorithm, so are different from the numbers shown in [11] by 2).

Example 1. Consider the following two-point boundary value problem:

$$-(q(x)u_x)_x = f(x) \qquad \text{in } (0,1), \text{ with } u(0) = u(1) = 0. \tag{5.1}$$

We take the coefficient function $q(x)$ and the observation data z of u as

$$q(x) = \mathrm{e}^{1+x^2}, \qquad z = u(f^*) = \mathrm{e}^{-x}\sin(\pi x),$$

then the exact source term $f(x)$ which is to be recovered can be obtained from (5.1):

$$f^* = -q_x \mathrm{e}^{-x}\{\pi\cos(\pi x) - \sin(\pi x)\} + q\mathrm{e}^{-x}\{2\pi\cos(\pi x) + (\pi^2 - 1)\sin(\pi x)\}.$$

For the convenience of readers' numerical verifications, we take the following observation data with the sinusoidal noise:

$$z^\delta(x) = z(x) + \hat{\delta}\sin(1.5\pi(2x-1)).$$

The results are listed in table 1, starting with a very poor initial value $\beta_0 = 0.1$. We can see from the table that the new two-parameter algorithm TPA(3) converges much faster than the two-parameter algorithm TPA(1), and with almost the same speed as the Iter(KZ) method. In fact, from the results in the row with Iter(3), one can find that the β values obtained at the third iterations of TPA(3) are already acceptable.

Example 2. Consider the following two-dimensional elliptic problem:

$$-\nabla \cdot (q(x,y)\nabla u) + c(x,y)u = f(x,y) \qquad \text{in } \Omega, \tag{5.2}$$

$$\frac{\partial u}{\partial n} = 0 \qquad \text{on} \quad \partial\Omega. \tag{5.3}$$

We take the coefficient functions $q(x,y)$, $c(x,y)$ and the unperturbed observation data as

$$q(x,y) = \mathrm{e}^{x+y}, \qquad c(x,y) = \mathrm{e}^{1+x^2+y^2}, \qquad u(f^*) = \cos(\pi x)\cos(\pi y).$$

For the convenience of readers' numerical verifications, the noisy data are taken to be of the sinusoidal form

$$z^\delta(x,y) = u(x,y) + \hat{\delta}\sin(1.5\pi(2x-1))\sin(1.5\pi(2y-1)).$$

The exact source term $f(x,y)$ to be recovered is the right-hand side function of equation (5.2) using the given coefficients $q(x,y)$, $c(x,y)$ and the exact observation $u(x,y)$. The numerical results are shown in table 2, starting with a very poor initial value $\beta_0 = 0.1$. We can see from the table that the new algorithm TPA(3) converges much faster than TPA(1), and even faster than the Iter(KZ) method. In fact, from the results in the row with Iter(2), one can find that the β values obtained at the second iterations are already acceptable.

Table 2. β_{opt}, β_M and the number of iterations using $\hat{\alpha} = \frac{1}{4}$.

$\hat{\delta}$	0.01	0.03	0.05	0.07	0.1
β_{opt}	1.21×10^{-6}	3.81×10^{-6}	6.85×10^{-6}	0.98×10^{-5}	1.39×10^{-5}
Iter(2)	0.99×10^{-6}	3.70×10^{-6}	6.93×10^{-6}	1.05×10^{-5}	1.64×10^{-5}
β_M	1.02×10^{-6}	3.27×10^{-6}	5.55×10^{-6}	0.78×10^{-5}	1.11×10^{-5}
TPA(3)	2	3	3	3	4
TPA(1)	5	6	7	8	8
Iter(KZ)	5	5	6	5	5

Table 3. β_{opt}, β_M and the number of iterations with $\gamma = 1.3$.

$\hat{\delta}$	0.01	0.03	0.05	0.07	0.1
β_{opt}	4.30×10^{-5}	1.34×10^{-4}	2.30×10^{-4}	3.33×10^{-4}	5.02×10^{-4}
β_M	3.36×10^{-5}	1.81×10^{-4}	3.95×10^{-4}	6.59×10^{-4}	1.14×10^{-3}
Iter	3	3	3	3	3

Table 4. The number of iterations with different γ values.

$\gamma\backslash\hat{\delta}$	0.01	0.03	0.05	0.07	0.1
1.0	2	2	2	2	2
1.3	3	3	3	3	3
1.5	3	3	3	3	3
2.0	4	4	4	4	4
∞	4	4	4	4	4

Example 3. Consider the integral equation arising from the irrigation canal problem [7]

$$z(h) = 2 \int_0^h \sqrt{2g(h-y)}\, f(y)\, \mathrm{d}y. \tag{5.4}$$

We want to reconstruct $f(y)$ from the measurements of $z(h)$. We take the true function

$$f(y) = \mathrm{e}^{-y}(2\pi\cos(\pi y) + (\pi^2 - 1)\sin(\pi y))$$

and the observation function $z(h), h \in [0, 1]$, was computed using formula (5.4). Then we add noise to the observation data as follows:

$$z^{\delta}(h) = z(h) + \hat{\delta}\sin(3\pi h).$$

The numerical results of the two-parameter algorithm III are shown in table 3 when the damped Morozov principle with $\gamma = 1.3$ is used; the number of iterations required for different γ values are summarized in table 4.

More numerical results regarding the three examples discussed in this section can be found in [11], including some results and algorithms regarding the prediction of the noise level δ.

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