SIAM J. NUMER. ANAL.
Vol. 43, No. 4, pp. 1504–1535

NUMERICAL RECONSTRUCTION OF HEAT FLUXES*

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Abstract. This paper studies the reconstruction of heat fluxes on an inner boundary of a heat conductive system when the measurement of temperature in a small subregion near the outer boundary of the physical domain is available. We will first consider two different regularization formulations for this severely ill-posed inverse problem and justify their well-posedness; then we will propose two fully discrete finite element methods to approximate the resultant nonlinear minimization problems. The existence and uniqueness of the discrete minimizers and convergence of the finite element solution are rigorously demonstrated. A conjugate gradient method is formulated to solve the nonlinear finite element optimization problems. Numerical experiments are given to demonstrate the stability and effectiveness of the proposed reconstruction methods.

Key words. numerical reconstruction, heat flux, regularization, optimization, finite elements

AMS subject classifications. 65M30, 35R25, 65M60

DOI. 10.1137/030602551

1. Introduction.

Consider a heat conductive system which occupies an open bounded domain Ω with an outer boundary Γ_o and an inner boundary Γ_i; see Figure 1. We are interested in a heat conductive system which can be modeled by the parabolic equation

$$\frac{\partial u}{\partial t} = \nabla \cdot (\alpha(x,t)\nabla u) \qquad \text{in} \quad \Omega \times (0,T), \tag{1.1}$$

assuming the initial condition

$$u(x,0) = u_0(x) \qquad \text{in} \quad \Omega \tag{1.2}$$

and the heat flux exchanges through the outer and inner boundaries Γ_o and Γ_i as follows:

$$-\alpha(x,t)\frac{\partial u}{\partial n} = c(x,t)(u(x,t) - u_a(x,t)) \qquad \text{on} \quad \Gamma_o \times (0,T), \tag{1.3}$$

$$-\alpha(x,t)\frac{\partial u}{\partial n} = q(x,t) \qquad \text{on} \quad \Gamma_i \times (0,T). \tag{1.4}$$

Here $\alpha(x,t)$ is the heat conductivity, $c(x,t)$ and $u_a(x,t)$ are specified functions, and $q(x,t)$ is the heat flux on the inner boundary Γ_i.

The forward initial-boundary value problem (1.1)–(1.4) has been well studied. The focus of this paper is on a physically more interesting and challenging inverse problem: Is it possible to effectively reconstruct the heat flux $q(x,t)$ on the inner boundary Γ_i for all time $t \in [0,T]$ when Γ_i is inaccessible?

*Received by the editors December 18, 2003; accepted for publication (in revised form) March 4, 2005; published electronically October 31, 2005.

http://www.siam.org/journals/sinum/43-4/60255.html

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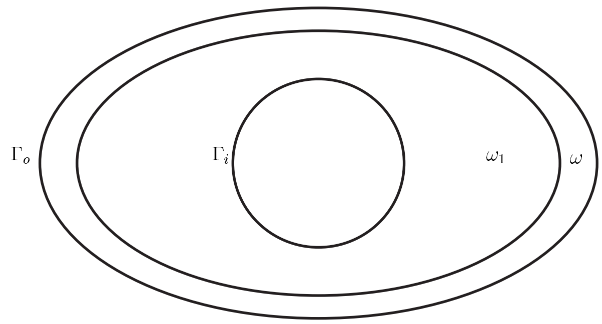

FIG. 1. *Physical domain* $\Omega = \omega_1 \cup (\bar{\omega} \setminus \Gamma_o)$.

In order to possibly reconstruct the heat flux $q(x,t)$, some extra information on the temperature $u(x,t)$ is needed. One choice is to assume the temperature data available in a small subregion ω near the outer boundary Γ_o (see Figure 1). Some high furnaces in steel companies are such examples, where special small devices are installed inside the furnaces but near the outer boundary to measure temperature.

This reconstruction problem is known to be a severely ill-posed inverse problem. One of the main difficulties in the reconstruction comes from both the space and time dependence of the heat flux $q(x,t)$ and the fact that the inner boundary is away from the small measurement subregion. The most severe instability of an inverse problem is triggered when the reconstruction involves some profile at the initial time and on some large boundary portion of a physical domain [7], [17], [19], [20], as is the case encountered here. As far as ill-posed inverse problems are concerned, not much work is found in the literature addressing numerical reconstructions of some physical profiles of both space and time; even less work can be found on convergence and stability analysis for numerical reconstruction methods. We refer readers to [1], [2], [3], [8], [9], [18], and the references therein for numerical reconstructions of profiles of some time-independent parameters in parabolic and elliptic systems.

The aim of this paper is to justify both theoretically and numerically the validation and effectiveness of two regularization formulations for solving the aforementioned severely ill-posed inverse problem of heat flux reconstruction. Indeed, as will be seen from the theory, numerical analysis, and simulations developed in what follows, the regularization methods are very stable and effective in numerical reconstruction of heat fluxes, without any constraints enforced on the search space of heat fluxes if appropriate regularizations are selected. In particular, the resulting nonlinear finite element minimization systems can be efficiently solved by conjugate gradient method.

The rest of this paper is organized as follows. In section 2, we investigate the first formulation with an L^2-regularization of both space and time for the heat flux and validate the "true" well-posedness of the formulation under no constraints on the search space of heat fluxes. In section 3, we study the ill-posedness of heat flux reconstruction and the stability of the regularization. In section 4, we study an alternative formulation of the inverse problem, which uses an L^2-regularization in space and H^1-regularization in time. As will be seen, this formulation turns out to be able to demonstrate much more satisfactory reconstructions. Regarding the approximation of the regularized nonlinear minimization systems, it is very tricky and essential to decide how to effectively discretize in both time and space the nonlinear optimizations and the associated parabolic equation so that the resulting fully discrete schemes converge. For this purpose, two fully discrete finite element approximations are proposed in sections 5 and 6, and the unique existence of discrete minimizers and their convergence to the continuous minimizer are rigorously demonstrated. For solving the nonlinear finite element minimization systems involved in the formulations, a conjugate gradient method is formulated in section 7, and the numerical experiments are presented in section 8 to verify the effectiveness of the proposed reconstruction methods.

We end this section with some useful notation. We define

$$H^m(0,T;B) = \Big\{u(t) \in B \text{ for a.e. } t \in (0,T) \text{ and } \|u\|_{H^m(0,T;B)} < \infty\Big\}$$

for a Banach space B and $m \geq 0$, with its norm given by

$$\|u\|_{H^m(0,T;B)} = \left\{\sum_{k=0}^{m} \int_0^T \|u^{(k)}(t)\|_B^2 dt\right\}^{1/2}.$$

For a given domain $\mathcal{O}$, $H^m(\mathcal{O})$ stands for the standard Sobolev space of mth order for any $m \geq 0$. The norms and seminorms of $H^m(\mathcal{O})$ are denoted by $\|\cdot\|_{m,\mathcal{O}}$ and $|\cdot|_{m,\mathcal{O}}$, respectively. When $m = 0$, we write $L^2(\mathcal{O}) = H^0(\mathcal{O})$ with the norm $\|\cdot\|_{0,\mathcal{O}}$. The domain $\mathcal{O}$ in the subindex will be dropped if $\mathcal{O} = \Omega$.

Further, C is frequently used to denote a generic constant, which depends only on the given data such as domain Ω and coefficients in (1.1)–(1.4) and is independent of unknown functions involved and the discrete time step τ and mesh size h.

2. First regularization formulation. Recall that the inverse problem of interest here is to reconstruct the heat flux $q(x,t)$ in (1.4) on the inner boundary Γ_i, given the temperature measurement $z(x,t) \approx u(x,t)$ in the small subdomain ω (cf. Figure 1). The first approach we will study for solving the inverse problem is to formulate it into the following constrained minimizing process with L^2-regularization in both space and time for possible heat fluxes:

$$\min J(q) = \frac{1}{2}\int_0^T\!\!\int_\omega (u(q) - z)^2 dxdt + \frac{\beta}{2}\int_0^T\!\!\int_{\Gamma_i} q^2 dsdt \tag{2.1}$$

subject to $q \in L^2(0,T;L^2(\Gamma_i))$ and $u(q) \equiv u(q)(\cdot,t) \in H^1(\Omega)$ satisfying

$$u(x,0) = u_0(x) \qquad \text{in} \quad \Omega, \tag{2.2}$$

$$\int_\Omega \frac{\partial u}{\partial t} v dx + \int_\Omega \alpha \nabla u \cdot \nabla v dx + \int_{\Gamma_o} c\, u\, v ds = \int_{\Gamma_o} c\, u_a v ds - \int_{\Gamma_i} q\, v ds \tag{2.3}$$

for all $v \in H^1(\Omega)$ and for a.e. $t \in (0,T)$.

In what follows, we will demonstrate that the inverse problem for reconstruction of heat flux is an ill-posed problem and that the formulation (2.1)–(2.3) is a true regularization of the inverse problem; that is, the minimizer q not only exists uniquely, but also depends on the observation data z continuously.

For the subsequent analysis, we often use the following compactness result (cf. [13]).

LEMMA 2.1. *Suppose that $B_0 \subset B \subset B_1$ are Banach spaces, B_0 and B_1 are reflexive, and B_0 is compactly embedded into B. Let*

$$W = \left\{v;\ v \in L^2(0,T;B_0),\ v' = \frac{dv}{dt} \in L^2(0,T;B_1)\right\},$$

with the norm $\|v\|_W = \|v\|_{L^2(0,T;B_0)} + \|v'\|_{L^2(0,T;B_1)}$. *Then* W *is compactly embedded into* $L^2(0,T;B)$.

Throughout this section, the parameter functions $\alpha(x,t)$, $c(x,t)$, and $u_a(x,t)$ in (1.1)–(1.4) are assumed to satisfy the following natural conditions:

$$\begin{gathered}\alpha(x,t) \geq \alpha_0 > 0 \quad \text{for a.e.} \quad (x,t) \in \Omega \times (0,T), \\ c(x,t) \geq c_0 > 0 \quad \text{for a.e.} \quad (x,t) \in \Gamma_o \times (0,T), \\ \alpha(x,t) \in L^2(0,T;L^2(\Omega)); \quad c(x,t),\ u_a(x,t) \in L^2(0,T;L^2(\Gamma_o)).\end{gathered} \tag{2.4}$$

We start with the following unique existence.

THEOREM 2.2. *There exists a unique minimizer to the optimization problem* (2.1)–(2.3).

Proof. Clearly $\min J(q)$ is finite over $L^2(0,T;L^2(\Gamma_i))$; thus there exists a minimizing sequence $\{q^n\}$ such that

$$\lim_{n\to\infty} J(q^n) = \inf J(q). \tag{2.5}$$

This implies the boundedness of $\{q^n\}$ in $L^2(0,T;L^2(\Gamma_i))$ and thus the existence of such a subsequence, still denoted[1] as q^n, and $\{q^n\}$ converges to q^* weakly in $L^2(0,T;L^2(\Gamma_i))$. We now prove that this q^* is the unique minimizer of (2.1)–(2.3). We divide the proof into four steps.

Step 1. Letting $u^n \equiv u(q^n)(x,t)$, we show that there exists a subsequence of $\{u^n\}$ such that

$$u^n \to u^* \text{ weakly in } L^2(0,T;H^1(\Omega)) \text{ and } L^2(0,T;L^2(\Gamma_o)). \tag{2.6}$$

By the definition of $u(q^n)$ in (2.2)–(2.3), $u^n \in H^1(\Omega)$ satisfies $u^n(x,0) = u_0(x)$, and

$$\int_\Omega \frac{\partial u^n}{\partial t} v dx + \int_\Omega \alpha \nabla u^n \cdot \nabla v dx + \int_{\Gamma_o} c\, u^n v ds = \int_{\Gamma_o} c\, u_a v ds - \int_{\Gamma_i} q^n v ds \tag{2.7}$$

holds for any $v \in H^1(\Omega)$ and a.e. $t \in (0,T)$. Taking $v = u^n$ in (2.7), we obtain

$$\frac{1}{2}\frac{d}{dt}\|u^n\|_0^2 + \int_\Omega \alpha |\nabla u^n|^2 dx + \int_{\Gamma_o} c\,|u^n|^2 ds = \int_{\Gamma_o} c\, u_a u^n ds - \int_{\Gamma_i} q^n u^n ds. \tag{2.8}$$

Integrating over $(0,t)$, we derive

$$\begin{aligned}&\frac{1}{2}\|u^n(\cdot,t)\|_0^2 + \int_0^t\int_\Omega \alpha|\nabla u^n(x,t)|^2 dxdt + \int_0^t\int_{\Gamma_o} c(x,t)|u^n(x,t)|^2 dsdt \\ &= \frac{1}{2}\|u_0\|_0^2 + \int_0^t\int_{\Gamma_o} c(x,t)u_a(x,t)u^n(x,t)dsdt - \int_0^t\int_{\Gamma_i} q^n(x,t)u^n(x,t)dsdt;\end{aligned}$$

then by the Cauchy–Schwarz inequality and assumptions in (2.4), we have

$$\begin{aligned}&\frac{1}{2}\|u^n(\cdot,t)\|_0^2 + \alpha_0\|\nabla u^n\|^2_{L^2(0,t;L^2(\Omega))} + c_0\|u^n\|^2_{L^2(0,t;L^2(\Gamma_o))} \\ &\leq \frac{1}{2}\|u_0\|_0^2 + \|c\, u_a\|_{L^2(0,T;L^2(\Gamma_o))}\|u^n\|_{L^2(0,t;L^2(\Gamma_o))} + \|q^n\|_{L^2(0,T;L^2(\Gamma_i))}\|u^n\|_{L^2(0,t;L^2(\Gamma_i))} \\ &\leq \frac{1}{2}\|u_0\|_0^2 + C\Big(\|u^n\|_{L^2(0,t;L^2(\Gamma_o))} + \|u^n\|_{L^2(0,t;L^2(\Gamma_i))}\Big).\end{aligned}$$

[1]Where no confusion exists, throughout this paper we shall always use the same notation to denote a subsequence taken from some sequence.

Using the Sobolev trace theorem, we can estimate the above last term as follows, a technique that will be frequently used in the subsequent analysis:

$$\begin{aligned}\|u^n\|^2_{L^2(0,t;L^2(\Gamma_i))} &= \int_0^t \|u^n(\cdot,s)\|^2_{L^2(\Gamma_i)}ds \le \int_0^t \|u^n(\cdot,s)\|^2_{H^{1/2}(\Gamma_i)}ds \\ &\le \int_0^t \|u^n(\cdot,s)\|^2_{H^1(\Omega)}ds \\ &= \int_0^t \|u^n(\cdot,s)\|^2_{L^2(\Omega)}ds + \int_0^t \|\nabla u^n(\cdot,s)\|^2_{L^2(\Omega)}ds \\ &\le \Big(\|u^n\|_{L^2(0,t;L^2(\Omega))} + \|\nabla u^n\|_{L^2(0,t;L^2(\Omega))}\Big)^2.\end{aligned}$$

Taking the square root on both sides, plugging the result into the previous estimate, and then using Young's inequality, we obtain

$$\begin{aligned}\|u^n(\cdot,t)\|_0^2 &\le \|u^n(\cdot,t)\|_0^2 + \alpha_0\|\nabla u^n\|^2_{L^2(0,t;L^2(\Omega))} + c_0\|u^n\|^2_{L^2(0,t;L^2(\Gamma_o))} \\ &\le \|u_0\|_0^2 + C + \int_0^t \|u^n(\cdot,s)\|^2_{L^2(\Omega)}ds.\end{aligned} \tag{2.9}$$

This gives the boundedness of $\{u^n\}$ in $L^\infty(0,T;L^2(\Omega))$ by applying Gronwall's inequality; then using this bound one can get the boundedness of $\{u^n\}$ in $L^2(0,T;H^1(\Omega))$ and $L^2(0,T;L^2(\Gamma_o))$ from the second inequality in (2.9). Now the convergence in (2.6) follows immediately from this boundedness.

Step 2. We prove $u^* = u(q^*)$. Taking any function $\Psi(t) \in C^1[0,T]$ with $\Psi(T)=0$, multiplying both sides of (2.7) by Ψ, and then integrating over $t \in (0,T)$, we get

$$\begin{aligned}&\int_0^T\!\!\int_{\Gamma_o} c\,u_a v\Psi(t)dsdt - \int_0^T\!\!\int_{\Gamma_i} q^n v\Psi(t)dsdt \\ &= -\int_0^T\!\!\int_\Omega u^n v\Psi'(t)dxdt + \int_0^T\!\!\int_\Omega \alpha\nabla u^n\cdot\nabla v\Psi(t)dxdt \\ &\quad - \int_\Omega \Psi(0)u_0(x)vdx + \int_0^T\!\!\int_{\Gamma_o} c\,u^n v\Psi(t)dsdt.\end{aligned}$$

By the weak convergence of q^n and u^n, we deduce from above that

$$\begin{aligned}&\int_0^T\!\!\int_{\Gamma_o} c\,u_a v\Psi(t)dsdt - \int_0^T\!\!\int_{\Gamma_i} q^* v\Psi(t)dsdt \\ &= \int_0^T\!\!\int_\Omega \alpha\nabla u^*\cdot\nabla v\Psi(t)dxdt + \int_0^T\!\!\int_{\Gamma_o} c\,u^* v\Psi(t)dsdt \\ &\quad - \int_\Omega \Psi(0)u_0(x)vdx - \int_0^T\!\!\int_\Omega u^* v\Psi'(t)dxdt.\end{aligned} \tag{2.10}$$

Noting that (2.10) is also true for any $\Psi(t) \in C_0^\infty(0,T)$, by integration by parts over $t \in (0,T)$ for the last term we have

$$\int_\Omega \frac{\partial u^*}{\partial t}vdx + \int_\Omega \alpha\nabla u^*\cdot\nabla vdx + \int_{\Gamma_o} c\,u^* vds = \int_{\Gamma_o} c\,u_a vds - \int_{\Gamma_i} q^* vds \qquad \forall v \in H^1(\Omega)$$

for a.e. $t \in (0,T)$. Using this and integration by parts again for the last term in (2.10) shows that $u^*(x,0) = u_0(x)$. This verifies $u^* = u(q^*)$.

Step 3. We prove the strong convergence

$$\lim_{n\to\infty}\int_0^T\int_\omega |u^n - z|^2 dxdt = \int_0^T\int_\omega |u^* - z|^2 dxdt. \tag{2.11}$$

It suffices to prove the strong convergence of $\{u^n\}$ in $L^2(0,T;L^2(\Omega))$. By Lemma 2.1, we need only show the boundedness of $\{\frac{\partial u^n}{\partial t}\}$ in $L^2(0,T;(H^1(\Omega))')$.

It follows from (2.7) that for any $v \in L^2(0,T;H^1(\Omega))$,

$$\left|\left\langle \frac{\partial u^n}{\partial t}, v\right\rangle\right| \le C(\|u^n\|_{H^1(\Omega)} + \|u^n\|_{L^2(\Gamma_o)} + \|u_a\|_{L^2(\Gamma_o)} + \|q^n\|_{L^2(\Gamma_i)})\|v\|_{H^1(\Omega)}; \tag{2.12}$$

this, along with the boundedness of $\{u^n\}$ proved in Step 1, implies the boundedness of $\{\frac{\partial u^n}{\partial t}\}$ in $L^2(0,T;(H^1(\Omega))')$.

Step 4. We prove q^* is a unique minimizer to the system (2.1)–(2.3). Using the results in Step 3 and the lower semicontinuity of a norm, we have

$$\begin{aligned} J(q^*) &= \frac{1}{2}\int_0^T\int_\omega |u(q^*) - z|^2 dxdt + \frac{\beta}{2}\int_0^T\int_{\Gamma_i} |q^*|^2 dsdt \\ &\le \lim_{n\to\infty}\int_0^T\int_\omega |u(q^n) - z|^2 dxdt + \frac{\beta}{2}\lim_{n\to\infty}\inf\int_0^T\int_{\Gamma_i} |q^n|^2 dsdt \\ &\le \lim_{n\to\infty}\inf J(q^n) = \inf J(q), \end{aligned} \tag{2.13}$$

so q^* is indeed a minimizer. The uniqueness of minimizers is a consequence of the convexity of $u(q)$ and the strict convexity of $J(q)$. ◻

Proposition 2.3. *Assume that $\{q^n\}$, with $q^n \in L^2(0,T;L^2(\Gamma_i))$, is a minimizing sequence of $J(q)$ in* (2.1)*; then $\{q^n\}$ converges to the unique minimizer of $J(q)$ strongly in $L^2(0,T;L^2(\Gamma_i))$.*

Proof. From the proof of Theorem 2.2, we know any subsequence of $\{q^n\}$ has a subsequence converging weakly to the unique minimizer of $J(q)$. Thus the whole sequence $\{q^n\}$ converges weakly to the unique minimizer of $J(q)$. Further, one notices from (2.5), (2.11), and (2.13) that

$$\lim_{n\to\infty}\int_0^T\int_{\Gamma_i} |q^n|^2 dsdt = \int_0^T\int_{\Gamma_i} |q^*|^2 dsdt;$$

thus the weak and norm convergences imply the strong convergence. ◻

3. Ill-posedness of heat flux reconstruction and stability of the regularization.

Next, we study the ill-posedness of heat flux reconstruction and stability of the regularization system (2.1)–(2.3). The following theorem confirms the ill-posedness of the heat flux reconstruction problem (1.1)–(1.4).

Theorem 3.1. *Let $u(q)$ be a mapping from $L^2(0,T;L^2(\Gamma_i))$ to $L^2(0,T;L^2(\omega))$, defined by the system* (2.2)–(2.3) *associated with any given heat flux q in $L^2(0,T;L^2(\Gamma_i))$. Then there exists a sequence $\{q^n\}$ from $L^2(0,T;L^2(\Gamma_i))$ such that $u(q^n) \to 0$ but $\|q^n\|_{L^2(0,T;L^2(\Gamma_i))} \to \infty$, and the inverse of $u(\cdot)$ is unbounded.*

Proof. From the proof of Theorem 2.2, we know for any bounded sequence $\{q^n\}_{n=1}^\infty$ there exists a subsequence $\{q^{n_k}\}_{k=1}^\infty$ such that $\{u(q^{n_k})\}_{k=1}^\infty$ is strongly convergent in $L^2(0,T;L^2(\omega))$. Therefore, as an operator from $L^2(0,T;L^2(\Gamma_i))$ to $L^2(0,T;L^2(\omega))$,

$u(\cdot)$ is compact. On the other hand, one can directly verify that $u(\cdot)$ is a one-to-one mapping and can be decomposed into $u(q) = w(q) + u(0)$, where $w(q)(\cdot, t) \in H^1(\Omega)$ solves the parabolic system (1.1)–(1.4) with $w(q)(x, 0) = 0$ in Ω and $u_a \equiv 0$. The rest of the proof follows the routine procedure; for example, see [10, pp. 13–14]. ∎

The next theorem shows that the solution q to the regularization system (2.1)–(2.3) depends continuously on the observation data z, so system (2.1)–(2.3) is a "true" regularization to the original inverse problem $u(q) = z$. The detailed proof can be found in [14].

THEOREM 3.2. *Let $\{z^n\}$ be a sequence such that*

$$z^n \to z \quad in \quad \mathrm{L}^2(0, \mathrm{T}; \mathrm{L}^2(\omega)) \quad as \quad \mathrm{n} \to \infty, \tag{3.1}$$

and let $\{q^n\}$ be the minimizers of problem (2.1)–(2.3) *with z replaced by z^n. Then the whole sequence $\{q^n\}$ converges in $L^2(0, T; L^2(\Gamma_i))$ to the unique minimizer of* (2.1)–(2.3).

4. An alternative formulation. In this section, we investigate an alternative formulation for reconstruction of heat fluxes in the heat conductive system (1.1)–(1.4), using an L^2-regularization in space and H^1-regularization in time for heat fluxes. As one can see from numerical results in section 8, this new formulation is able to generate more satisfactory reconstructions. This results in the following constrained minimization:

$$\min J(q) = \frac{1}{2}\int_0^T\int_\omega (u(q) - z)^2 dxdt + \frac{\beta}{2}\left(\int_{\Gamma_i} q^2(x, 0)ds + \int_0^T\int_{\Gamma_i} |q_t(x, t)|^2 dsdt\right) \tag{4.1}$$

subject to $q \in H^1(0, T; L^2(\Gamma_i))$ and $u(q) \equiv u(q)(\cdot, t) \in H^1(\Omega)$ satisfying

$$u(x, 0) = u_0(x) \qquad \text{in} \quad \Omega, \tag{4.2}$$

$$\int_\Omega \frac{\partial u}{\partial t} v dx + \int_\Omega \alpha \nabla u \cdot \nabla v dx + \int_{\Gamma_o} c\, u\, v ds = \int_{\Gamma_o} c\, u_a v ds - \int_{\Gamma_i} q\, v ds \tag{4.3}$$

for all $v \in H^1(\Omega)$ and a.e. $t \in (0, T)$.

The following theorem justifies the well-posedness of the system (4.1)–(4.3) and its stability with respect to the observation data.

THEOREM 4.1. *There exists a unique minimizer to the optimization problem* (4.1)–(4.3)*, and the minimizer depends on the observation data z continuously.*

Proof. It is clear that $\min J(q)$ is finite over $H^1(0, T; L^2(\Gamma_i))$; thus there exists a minimizing sequence $\{q^n\}$ such that

$$\lim_{n\to\infty} J(q^n) = \inf J(q).$$

This implies the boundedness of $\{q^n\}$ in $H^1(0, T; L^2(\Gamma_i))$ and the existence of a subsequence, still denoted as $\{q^n\}$, such that

$$\begin{aligned} q^n &\to q^* \text{ weakly in } L^2(0, T; L^2(\Gamma_i)), \\ \frac{\partial q^n}{\partial t} &\to p^* \text{ weakly in } L^2(0, T; L^2(\Gamma_i)), \\ q^n(x, 0) &\to q_0^* \text{ weakly in } L^2(\Gamma_i). \end{aligned}$$

We can show that $p^* = \partial q^*/\partial t$ and $q^*(x,0) = q_0^*$. In fact, taking any function $\varphi(x) \in L^2(\Gamma_i)$ and $\psi(t) \in C_0^\infty(0,T)$, we deduce

$$\int_0^T\int_{\Gamma_i} \frac{\partial}{\partial t} q^n(x,t)\varphi(x)\psi(t)dsdt = -\int_0^T\int_{\Gamma_i} q^n(x,t)\varphi(x)\psi'(t)dsdt.$$

Passing to the limit, we derive

$$\int_0^T\int_{\Gamma_i} p^*(x,t)\varphi(x)\psi(t)dsdt = -\int_0^T\int_{\Gamma_i} q^*(x,t)\varphi(x)\psi'(t)dsdt.$$

This shows $p^* = \partial q^*/\partial t$.

Then letting $\varphi(x) \in L^2(\Gamma_i)$ and $\psi(t) \in C^\infty(0,T)$ with $\psi(T) = 0$ and $\psi(0) = 1$, we obtain

$$\int_0^T\int_{\Gamma_i} \frac{\partial}{\partial t} q^n(x,t)\varphi(x)\psi(t)dsdt = \int_{\Gamma_i} q^n(x,0)\varphi(x)ds - \int_0^T\int_{\Gamma_i} q^n(x,t)\varphi(x)\psi'(t)dsdt.$$

By the weak convergence of $\partial q^n/\partial t$, $q^n(x,0)$, and q^n, we deduce

$$\int_0^T\int_{\Gamma_i} \frac{\partial}{\partial t} q^*(x,t)\varphi(x)\psi(t)dsdt = \int_{\Gamma_i} q_0^*(x)\varphi(x)ds - \int_0^T\int_{\Gamma_i} q^*(x,t)\varphi(x)\psi'(t)dsdt.$$

Integrating by parts the left-hand side, we obtain for any $\varphi(x) \in L^2(\Gamma_i)$ that

$$\int_{\Gamma_i} q_0^*(x)\varphi(x)ds = \int_{\Gamma_i} q^*(x,0)\varphi(x)ds,$$

which implies $q^*(x,0) = q_0^*$. The rest of the proof is similar to those of Theorems 2.2 and 3.2. □

Similarly to Proposition 2.3, we have the following strong convergence (cf. [14]).

PROPOSITION 4.2. *Any minimizing sequence $\{q^n\}$ of $J(q)$ in (4.1) over $H^1(0,T;L^2(\Gamma_i))$ converges to the unique minimizer of $J(q)$ strongly in $H^1(0,T;L^2(\Gamma_i))$.*

5. Finite element approximation of system (2.1)–(2.3) and its convergence. We now propose a fully discrete finite element method for solving the continuous minimization problem (2.1)–(2.3). For the sake of exposition, we study in detail the case where the outer and inner boundaries Γ_o and Γ_i are both circles centered at the origin; see Figure 2. The subsequent results can be extended to more general domains by combining the analysis used here and the finite element analysis for the case when the approximation of the physical domain is involved [4].

Let us start with a triangulation of the domain Ω. To do so, we generate a set of circles all centered at the origin, starting with Γ_i and ending with Γ_o. Next we choose a set of quasi-uniformly distributed points on Γ_o, which are then connected to the origin to yield a set of radial lines, and the intersections of these lines with all the previous generated circles also yield a partition of each circle; see Figure 2. Now the triangulation $\mathcal{T}^h$ of Ω is formed by these sectorial elements. The arc segments on Γ_o and Γ_i generate naturally two triangulations of Γ_o and Γ_i, respectively, denoted by Γ_o^h and Γ_i^h.

For each sectorial element K, say $K = \{(r\cos\theta, r\sin\theta);\ r_1 \le r \le r_2, \theta_1 \le \theta \le \theta_2\}$, there exists a one-to-one mapping $\hat{F}_K : \hat{K} \to K$ such that $K = \hat{F}_K(\hat{K})$, where $\hat{K}$

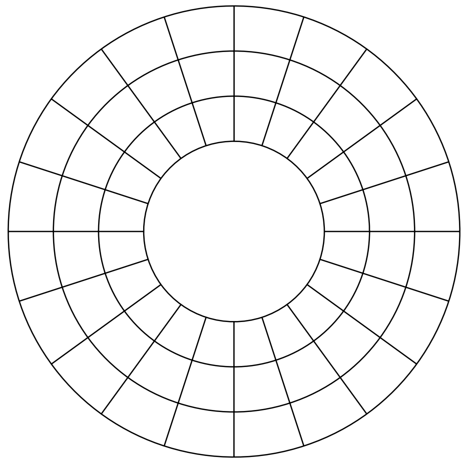

FIG. 2. *Circular partition of* Ω *and partition of each circle.*

is a rectangular reference element. For example, if $\hat{K} = [0,1] \times [0,1]$, we can take $\hat{F}_K$ as

$$\left\{ \begin{array}{l} x = (r\, r_2 + (1-r) r_1) \cos(\theta\, \theta_2 + (1-\theta)\theta_1), \\ y = (r\, r_2 + (1-r) r_1) \sin(\theta\, \theta_2 + (1-\theta)\theta_1). \end{array} \right. \tag{5.1}$$

Now we can define the finite element space V^h to be

$$V^h = \Big\{ v_h \in C(\bar{\Omega});\ v_h(x)\big|_K = \hat{v} \circ \hat{F}_K^{-1}(x)\ \forall \hat{v} \in \mathcal{Q}_1(\hat{K}) \Big\},$$

where $\mathcal{Q}_1(\hat{K})$ is the space of bilinear functions on the reference element $\hat{K}$, and $V^h_{\Gamma_o}$, $V^h_{\Gamma_i}$ are the restrictions of V^h on Γ_o and Γ_i, respectively.

To fully discretize the system (2.1)–(2.3), we also need the time discretization. For this, we divide the time interval $[0,T]$ into M equally spaced subintervals using nodal points

$$\Delta : \quad 0 = t_0 < t_1 < \cdots < t_M = T \tag{5.2}$$

with $t_n = n\tau$, $\tau = T/M$. For a continuous mapping $u : [0,T] \to L^2(\Omega)$, we define $u^n = u(\cdot, t_n)$ for $0 \le n \le M$. For a given sequence $\{u^n\}_{n=0}^M \subset L^2(\Omega)$, we define its difference quotient and the averaging $\bar{u}^n$ of a function $u(\cdot,t)$ as follows:

$$\partial_\tau u^n = \frac{u^n - u^{n-1}}{\tau}, \quad \bar{u}^n = \frac{1}{\tau} \int_{t_{n-1}}^{t_n} u(\cdot, t) dt, \tag{5.3}$$

where for $n = 0$, we let $\bar{u}^0 = u(\cdot, 0)$.

In our subsequent convergence analysis, we need a crucial projection operator Q_h from $L^2(\Omega)$ into V^h defined on sectorial elements, which should possess the following L^2- and H^1-stability and optimal L^2-norm error estimate:

$$\lim_{h \to 0} \|v - Q_h v\|_1 = 0 \quad \forall v \in H^1(\Omega), \tag{5.4}$$

$$\|Q_h v\|_0 \le C \|v\|_0, \quad \|Q_h v\|_1 \le C\|v\|_1 \quad \forall v \in H^1(\Omega), \tag{5.5}$$

$$\|v - Q_h v\|_0 \le C h \|v\|_1 \quad \forall v \in H^1(\Omega). \tag{5.6}$$

Noting that the transform $\hat{F}_K$: $\hat{K} \to K$ is not of polynomial type and that the functions in V^h may not be piecewise polynomials, the standard L^2-projection operator from $L^2(\Omega)$ into V^h (cf. [16]) does not satisfy these properties. Instead, we introduce a novel weighted L^2-projection operator Q_h from $L^2(\Omega)$ into V^h as follows:

$$\sum_{K\in\mathcal{T}^h}\int_K (Q_h w)\, v\, |J_K^{-1}(x,y)|\, dxdy = \sum_{K\in\mathcal{T}^h}\int_K w\, v\, |J_K^{-1}(x,y)|\, dxdy \quad \forall\, w \in L^2(\Omega), v\in V^h,$$

where $J_K(x,y) = \hat{J}_K(r,\theta)$ for all x, y and r, θ defined by (5.1) and $\hat{J}_K(r,\theta)$ is the Jacobian determinant of the transform $\hat{F}_K$. One can show that this weighted operator Q_h is well-defined, and it possesses all the properties (5.4), (5.5), and (5.6). The detailed proof was given in Xie [14].

Now we are ready to formulate the finite element approximation of the minimization (2.1)–(2.3). We approximate the heat flux $q(x,t)$ by a piecewise constant function $q_{h,\tau}(x,t)$ over the time partition Δ in (5.2):

$$q_{h,\tau}(x,t) = \sum_{n=1}^{M} \chi_n(t) q_h^n(x), \tag{5.7}$$

where $q_h^n(x) \in V_{\Gamma_i}^h$ and $\chi_n(t)$ is the characteristic function on the interval (t_{n-1}, t_n).

Using the composite trapezoidal rule for the time discretization of the first integral in (2.1) and the exact time integration for the second term, the fully discrete finite element approximation to problem (2.1)–(2.3) can be formulated as follows:

$$\min J_{h,\tau}(q_{h,\tau}) = \frac{\tau}{2}\sum_{n=0}^{M}\alpha_n \int_\omega (u_h^n - z^n)^2 dx + \frac{\beta\tau}{2}\sum_{n=1}^{M}\int_{\Gamma_i} |q_h^n|^2 ds \tag{5.8}$$

over all $q_h^n \in V_{\Gamma_i}^h$ with $u_h^n \equiv u_h^n(q_{h,\tau}) \in V^h$ satisfying

$$u_h^0 = Q_h u_0(x), \tag{5.9}$$

$$\begin{aligned}&\int_\Omega \partial_\tau u_h^n \phi_h dx + \int_\Omega \bar{\alpha}^n \nabla u_h^n \cdot \nabla\phi_h dx + \int_{\Gamma_o} \bar{c}^n u_h^n \phi_h ds\\ &= \int_{\Gamma_o} \bar{c}^n \bar{u}_a^n \phi_h ds - \int_{\Gamma_i} q_h^n \phi_h ds \quad \forall \phi_h \in V^h\end{aligned} \tag{5.10}$$

for $n = 1, 2, \ldots, M$. Here $\{\alpha_n\}$ are the coefficients of the composite trapezoidal rule, i.e., $\alpha_0 = \alpha_M = \frac{1}{2}$ and $\alpha_n = 1$ for all $n \neq 0, M$.

For convenience, the minimization of $J_{h,\tau}$ also shall be regarded as the minimization over the product space $\prod_{n=1}^{M} V_{\Gamma_i}^h$, and we will often write (5.8) as

$$\min J_{h,\tau}(\{q_h^1, q_h^2, \ldots, q_h^M\}) = \frac{\tau}{2}\sum_{n=0}^{M}\alpha_n \int_\omega (u_h^n - z^n)^2 dx + \frac{\beta\tau}{2}\sum_{n=1}^{M}\int_{\Gamma_i} |q_h^n|^2 ds. \tag{5.11}$$

Before verifying the existence of a unique minimizer to the finite element minimization (5.8)–(5.10), we first derive some useful a priori estimates on the discrete solutions u_h^n to the system (5.9)–(5.10).

In the rest of this section, we assume on the functions $\alpha(x,t)$ and $c(x,t)$ in (1.1)–(1.4) that

$$\alpha \in H^1(0,T;L^\infty(\Omega)) \quad \text{and} \quad c \in H^1(0,T;L^\infty(\Gamma_o))$$

and introduce two related constants

$$C_1 = \|\alpha\|_{H^1(0,T;L^\infty(\Omega))}, \quad C_2 = \|c\|_{H^1(0,T;L^\infty(\Gamma_o))}.$$

The following auxiliary lemma (cf. [14]) will be needed in the subsequent analysis.

Lemma 5.1. *For any*

$$f \in H^1(0,T;L^\infty(\Omega)) \quad \textit{and} \quad g \in L^2(0,T;L^\infty(\Omega)),$$

we have the estimates

$$\|\bar{f}^n - \bar{f}^{n-1}\|_{L^\infty(\Omega)} \le \sqrt{\tau}\|f_t\|_{L^2(t_{n-2},t_n;L^\infty(\Omega))}, \tag{5.12}$$

$$\|\bar{f}^n\bar{g}^n - \overline{fg}^n\|_{L^2(\Omega)} \le \frac{2}{3}\|f_t\|_{L^2(t_{n-1},t_n;L^\infty(\Omega))}\,\|g\|_{L^2(t_{n-1},t_n;L^2(\Omega))}. \tag{5.13}$$

Lemma 5.2. *Assume that u_h^n is the solution of the finite element system* (5.9)–(5.10) *corresponding to $q_{h,\tau}$. Then we have the following stability estimates:*

$$\begin{aligned}
&\max_{1\le n\le M}\|u_h^n\|_0^2 + \tau\sum_{n=1}^M\|\nabla u_h^n\|_0^2 + \tau\sum_{n=1}^M\|u_h^n\|_{0,\Gamma_o}^2\\
&\le C\,(\|u_0\|_0^2 + C_2^2\|u_a\|^2_{L^2(0,T;L^2(\Gamma_o))} + \|q_{h,\tau}\|^2_{L^2(0,T;L^2(\Gamma_i))}),
\end{aligned} \tag{5.14}$$

$$\begin{aligned}
&\max_{1\le n\le M}\|\nabla u_h^n\|_0^2 + \max_{1\le n\le M}\|u_h^n\|_{0,\Gamma_o}^2 + \tau\sum_{n=1}^M\|\partial_\tau u_h^n\|_0^2\\
&\le C\,\tau^{-1}(\|u_0\|_1^2 + C_2^2\|u_a\|^2_{L^2(0,T;L^2(\Gamma_o))} + \|q_{h,\tau}\|^2_{L^2(0,T;L^2(\Gamma_i))}),
\end{aligned} \tag{5.15}$$

$$\begin{aligned}
&\tau\sum_{n=1}^M\|\partial_\tau u_h^n\|^2_{(H^1(\Omega))'}\\
&\le C\,(C_1^2 + C_2^2 + \tau^{-1}h^2)\,(\|u_0\|_0^2 + C_2^2\|u_a\|^2_{L^2(0,T;L^2(\Gamma_o))} + \|q_{h,\tau}\|^2_{L^2(0,T;L^2(\Gamma_i))}).
\end{aligned} \tag{5.16}$$

Proof. The proof of (5.14) follows directly by taking $\phi_h = \tau u_h^n$ in (5.10) and then applying the Sobolev trace theorem and Young's and Gronwall's inequalities.

Next, we show (5.15). Taking $\phi_h = \tau\partial_\tau u_h^n = u_h^n - u_h^{n-1}$ in (5.10), we obtain

$$\begin{aligned}
&\tau\|\partial_\tau u_h^n\|_0^2 + \int_\Omega \bar{\alpha}^n\nabla u_h^n\cdot\nabla(u_h^n - u_h^{n-1})dx + \int_{\Gamma_o}\bar{c}^n u_h^n(u_h^n - u_h^{n-1})ds\\
&= \int_{\Gamma_o}\bar{c}^n\bar{u}_a^n(u_h^n - u_h^{n-1})ds - \int_{\Gamma_i} q_h^n(u_h^n - u_h^{n-1})ds.
\end{aligned}$$

Summing up the above equation over $n = 1, 2, \dots, k \le M$, we obtain

$$\tau \sum_{n=1}^{k} \|\partial_\tau u_h^n\|_0^2 + \frac{1}{2}\sum_{n=1}^{k}\int_\Omega \bar{\alpha}^n(|\nabla u_h^n|^2 - |\nabla u_h^{n-1}|^2)dx$$
$$+ \frac{1}{2}\sum_{n=1}^{k}\int_{\Gamma_o} \bar{c}^n(|u_h^n|^2 - |u_h^{n-1}|^2)ds$$
$$\le \sum_{n=1}^{k}\int_{\Gamma_o} \bar{c}^n\bar{u}_a^n(u_h^n - u_h^{n-1})ds - \sum_{n=1}^{k}\int_{\Gamma_i} q_h^n(u_h^n - u_h^{n-1})ds.$$

Then using the discrete integration by parts formula

$$\sum_{n=1}^{k}(a_n - a_{n-1})b_n = a_k b_k - a_0 b_0 - \sum_{n=1}^{k} a_{n-1}(b_n - b_{n-1}), \tag{5.17}$$

where b_0 appearing on the right-hand side can be any real number, we derive

$$\tau \sum_{n=1}^{k} \|\partial_\tau u_h^n\|_0^2 + \frac{1}{2}\alpha_0\|\nabla u_h^k\|_0^2 + \frac{1}{2}c_0\|u_h^k\|_{0,\Gamma_o}^2$$
$$\le \frac{1}{2}\int_\Omega \bar{\alpha}^0|\nabla u_h^0|^2dx + \frac{1}{2}\sum_{n=1}^{k}\int_\Omega(\bar{\alpha}^n - \bar{\alpha}^{n-1})|\nabla u_h^{n-1}|^2dx$$
$$+\frac{1}{2}\int_{\Gamma_o}\bar{c}^0|u_h^0|^2ds + \frac{1}{2}\sum_{n=1}^{k}\int_{\Gamma_o}(\bar{c}^n - \bar{c}^{n-1})|u_h^{n-1}|^2ds$$
$$+\int_{\Gamma_o}\bar{c}^k\bar{u}_a^k u_h^k ds - \sum_{n=1}^{k}\int_{\Gamma_o}(\bar{c}^n\bar{u}_a^n - \bar{c}^{n-1}\bar{u}_a^{n-1})u_h^{n-1}ds$$
$$-\int_{\Gamma_i} q_h^k u_h^k ds + \sum_{n=1}^{k}\int_{\Gamma_i}(q_h^n - q_h^{n-1})u_h^{n-1}ds,$$

where $\bar{u}_a^0$ and q_h^0 are taken to be 0. We now estimate the terms on the right-hand side of the above inequality. First, for those terms without summation, we can deduce by using the properties of Q_h and the Sobolev trace theorem that

$$\frac{1}{2}\int_\Omega \bar{\alpha}^0|\nabla u_h^0|^2dx + \frac{1}{2}\int_{\Gamma_o}\bar{c}^0|u_h^0|^2ds \le C\,(C_1 + C_2)\|u_0\|_1^2,$$

$$\int_{\Gamma_o}\bar{c}^k\bar{u}_a^k u_h^k ds \le \frac{1}{2}\|\bar{c}^k\bar{u}_a^k\|_{0,\Gamma_o}^2 + \frac{1}{2}\|u_h^k\|_{0,\Gamma_o}^2 \le \tau^{-1}\left(\tau\sum_{n=1}^{k}\|\bar{c}^n\bar{u}_a^n\|_{0,\Gamma_o}^2 + \tau\sum_{n=1}^{M}\|u_h^n\|_{0,\Gamma_o}^2\right),$$

$$\int_{\Gamma_i} q_h^k u_h^k ds \le \frac{1}{2}\|q_h^k\|_{0,\Gamma_i}^2 + \frac{1}{2}\|u_h^k\|_1^2 \le \tau^{-1}\left(\|q_{h,\tau}\|_{L^2(0,T;L^2(\Gamma_i))}^2 + \tau\sum_{n=1}^{M}\|u_h^n\|_1^2\right).$$

Using (5.12) we have the following estimates:

$$\sum_{n=1}^{k}\int_\Omega(\bar{\alpha}^n - \bar{\alpha}^{n-1})|\nabla u_h^{n-1}|^2dx \le \frac{4}{3}C_1\sqrt{\tau}\sum_{n=1}^{k}\|\nabla u_h^{n-1}\|_0^2,$$
$$\sum_{n=1}^{k}\int_{\Gamma_o}(\bar{c}^n - \bar{c}^{n-1})|u_h^{n-1}|^2ds \le \frac{4}{3}C_2\sqrt{\tau}\sum_{n=1}^{k}\|u_h^{n-1}\|_{0,\Gamma_o}^2.$$

Applying the Cauchy–Schwarz inequality and the Sobolev trace theorem, we have

$$\sum_{n=1}^{k}\int_{\Gamma_o}(\bar{c}^n\bar{u}_a^n-\bar{c}^{n-1}\bar{u}_a^{n-1})u_h^{n-1}ds\le\sum_{n=1}^{k}\|\bar{c}^n\bar{u}_a^n\|_{0,\Gamma_o}^2+\sum_{n=1}^{k}\|u_h^{n-1}\|_{0,\Gamma_o}^2,$$
$$\sum_{n=1}^{k}\int_{\Gamma_i}(q_h^n-q_h^{n-1})u_h^{n-1}ds\le\sum_{n=1}^{k}\|q_h^n\|_{0,\Gamma_i}^2+\sum_{n=1}^{k}\|u_h^{n-1}\|_1^2.$$

Combining all these estimates with (5.14), we obtain (5.15).

It remains to show (5.16). For any $\phi\in H^1(\Omega)$, taking $\phi_h=Q_h\phi$ in (5.10), we have

$$\int_\Omega\partial_\tau u_h^nQ_h\phi dx+\int_\Omega\bar{\alpha}^n\nabla u_h^n\nabla Q_h\phi dx+\int_{\Gamma_o}\bar{c}^nu_h^nQ_h\phi ds=\int_{\Gamma_o}\bar{c}^n\bar{u}_a^nQ_h\phi ds-\int_{\Gamma_i}q_h^nQ_h\phi ds.$$

Using the property of Q_h in (5.5) and the Cauchy–Schwarz inequality, we derive

$$\left|\int_\Omega\partial_\tau u_h^nQ_h\phi dx\right|\le C\left(C_1\|\nabla u_h^n\|_0+C_2\|u_h^n\|_{0,\Gamma_o}+C_2\|\bar{u}_a^n\|_{0,\Gamma_o}+\|q_h^n\|_{0,\Gamma_i}\right)\|\phi\|_1.$$

On the other hand, applying the Cauchy–Schwarz inequality and the property of Q_h in (5.6), we obtain

$$\left|\int_\Omega\partial_\tau u_h^n(\phi-Q_h\phi)dx\right|\le Ch\|\partial_\tau u_h^n\|_0\|\phi\|_1.$$

It follows from the above two inequalities that for any $\phi\in H^1(\Omega)$,

$$\begin{aligned}\left|\int_\Omega\partial_\tau u_h^n\phi dx\right|\le C\,(C_1\|\nabla u_h^n\|_0&+C_2\|u_h^n\|_{0,\Gamma_o}+C_2\|\bar{u}_a^n\|_{0,\Gamma_o}\\&+\|q_h^n\|_{0,\Gamma_i}+h\|\partial_\tau u_h^n\|_0)\|\phi\|_1,\end{aligned}$$

which implies

$$\|\partial_\tau u_h^n\|_{(H^1(\Omega))'}\le C\,(C_1\|\nabla u_h^n\|_0+C_2\|u_h^n\|_{0,\Gamma_o}+C_2\|\bar{u}_a^n\|_{0,\Gamma_o}+\|q_h^n\|_{0,\Gamma_i}+h\|\partial_\tau u_h^n\|_0).$$

Taking squares on both sides and adding up over $n=1,\dots,M$, (5.16) then follows from (5.14) and (5.15). ∎

Remark 5.3. Fortunately, the unbounded factor τ^{-1} in the estimate (5.15) can be cancelled in the subsequent convergence analysis; see (5.28) and the last estimate in the proof of Lemma 5.5.

Based on the stability estimates (5.14)–(5.16), we are now ready to show the existence and uniqueness of minimizers to the finite element system (5.9)–(5.11).

THEOREM 5.4. *There exists a unique minimizer to the finite element system* (5.9)–(5.11).

Proof. By the stability estimates of Lemma 5.2 and the same argument as in Theorem 2.2, we know there exists a minimizing sequence $\{q_h^{1,k},q_h^{2,k},\dots,q_h^{M,k}\}_{k=1}^\infty$ such that

$$\lim_{k\to\infty}J_{h,\tau}(\{q_h^{1,k},q_h^{2,k},\dots,q_h^{M,k}\})=\inf_{\{q_h^n\}_{n=1}^M\in V_\Gamma^h}J_{h,\tau}(\{q_h^1,q_h^2,\dots,q_h^M\}),$$

and

$$q_h^{n,k} \to q_h^{n,*} \quad \text{in any norm for} \quad n = 1, 2, \ldots, M \quad \text{as} \quad k \to \infty.$$

Next, we prove $\{q_h^{1,*}, q_h^{2,*}, \ldots, q_h^{M,*}\}$ is the unique minimizer of (5.9)–(5.11). Let $q_{h,\tau}^k$ and $q_{h,\tau}^*$ be the functions defined in (5.7) by $\{q_h^{n,k}\}_{n=1}^M$ and $\{q_h^{n,*}\}_{n=1}^M$, respectively; then $u_h^n(q_{h,\tau}^k)$ and $u_h^n(q_{h,\tau}^*)$ are the finite element solutions to (5.9)–(5.10) corresponding to $q_{h,\tau}^k$ and $q_{h,\tau}^*$, respectively.

Let $w_h^{n,k} = u_h^n(q_{h,\tau}^k) - u_h^n(q_{h,\tau}^*)$; then $w_h^{0,k} = 0$ and for $n = 1, 2, \ldots, M$, $w_h^{n,k}$ solves

$$\int_\Omega \partial_\tau w_h^{n,k} \phi_h dx + \int_\Omega \bar{\alpha}^n \nabla w_h^{n,k} \cdot \nabla \phi_h dx + \int_{\Gamma_o} \bar{c}^n w_h^{n,k} \phi_h ds$$
$$= \int_{\Gamma_i} (q_h^{n,*} - q_h^{n,k}) \phi_h ds \quad \forall \phi_h \in V^h.$$

Taking $\phi_h = \tau w_h^{n,k}$ in the above equation, one can directly show by Gronwall's inequality that

$$\max_{1 \le n \le M} \|w_h^{n,k}\|_0^2 \le C\tau \sum_{n=1}^M \|q_h^{n,*} - q_h^{n,k}\|_{0,\Gamma_i}^2. \tag{5.18}$$

This proves $w_h^{n,k} \to 0$, and so we have $u_h^n(q_{h,\tau}^k) \to u_h^n(q_{h,\tau}^*)$ as $k \to \infty$.

The rest of the proof is basically the same as that of Theorem 2.2. □

The remaining part of this section is devoted to one of the central issues of our interest: Will the discrete minimizer of the system (5.8)–(5.10) converge to the global minimizer of the continuous problem (2.1)–(2.3)? If yes, is the convergence only weak or can it be strong in some norm? To answer this question, we need some preparations.

For a given function $f \in C([0,T]; X)$, with X being a Banach space, we define a step function approximation, based on the time partition (5.2):

$$S_\Delta f(x,t) = \sum_{n=1}^M \chi_n(t) f(x, t_n). \tag{5.19}$$

We know (cf. [21]) that

$$\lim_{\tau \to 0} \int_0^T \|S_\Delta f(\cdot, t) - f(\cdot, t)\|_X^2 dt = 0. \tag{5.20}$$

Next, we shall demonstrate a most important and technical result in the paper: for any weakly convergent sequence $q_{h,\tau}$ in $L^2(0,T; L^2(\Gamma_i))$ with respect to h and τ, the corresponding finite element solution $u_h^n(q_{h,\tau})$ defined in (5.9)–(5.10) will converge strongly in $L^2(0,T; L^2(\omega))$. More accurately, we have the following lemma.

LEMMA 5.5. *If $q_{h,\tau}$ converges to some q weakly in $L^2(0,T; L^2(\Gamma_i))$ as h and τ tend to* 0*, then*

$$\tau \sum_{n=0}^M \alpha_n \int_\omega (u_h^n(q_{h,\tau}) - z^n)^2 dx \to \int_0^T \int_\omega (u(q) - z)^2 dx dt.$$

Proof. For $1 \le n \le M$, we shall use the following notation:

$$u_h^n = u_h^n(q_{h,\tau}), \quad u^n = u(q)(\cdot, t_n).$$

By (5.20), we can directly verify

$$\lim_{\tau \to 0} \tau \sum_{n=0}^{M} \alpha_n \int_\omega (u^n - z^n)^2 dx = \int_0^T \int_\omega (u(q) - z)^2 dx dt.$$

Therefore it suffices to show

$$\lim_{\substack{h \to 0 \\ \tau \to 0}} \tau \sum_{n=0}^{M} \alpha_n \int_\omega (u_h^n - z^n)^2 dx = \lim_{\substack{h \to 0 \\ \tau \to 0}} \tau \sum_{n=0}^{M} \alpha_n \int_\omega (u^n - z^n)^2 dx$$

or, equivalently,

$$\lim_{\substack{h \to 0 \\ \tau \to 0}} \tau \sum_{n=0}^{M} \int_\omega (u_h^n - u^n)^2 dx = 0. \tag{5.21}$$

For this, we construct two interpolations based on $\{u_h^n\}$: the first one is the piecewise linear interpolation over the time partition (5.2),

$$u_{h,\tau}(x,t) = \frac{t - t_{n-1}}{\tau} u_h^n + \frac{t_n - t}{\tau} u_h^{n-1}, \qquad t \in (t_{n-1}, t_n),$$

while the second one is the piecewise constant interpolation

$$\tilde{u}_{h,\tau}(x,t) = \sum_{n=1}^{M} \chi_n(t) u_h^n(x).$$

By straightforward computations, we have

$$\|\tilde{u}_{h,\tau}\|^2_{L^2(0,T;H^1(\Omega))} = \tau \sum_{n=1}^{M} \|u_h^n\|_1^2, \quad \left\| \frac{\partial}{\partial t} u_{h,\tau} \right\|^2_{L^2(0,T;(H^1(\Omega))')} = \tau \sum_{n=1}^{M} \|\partial_\tau u_h^n\|^2_{(H^1(\Omega))'}$$

and

$$\begin{aligned}
&\|u_{h,\tau}\|^2_{L^2(0,T;H^1(\Omega))} \\
&= \frac{\tau}{3} \sum_{n=1}^{M} \int_\Omega (|u_h^n|^2 + |u_h^{n-1}|^2 + u_h^n u_h^{n-1} + |\nabla u_h^n|^2 + |\nabla u_h^{n-1}|^2 + \nabla u_h^n \cdot \nabla u_h^{n-1}) dx \\
&\le \tau \sum_{n=0}^{M} \|u_h^n\|_1^2.
\end{aligned}$$

These, together with the stability estimates (5.14)–(5.16), indicate that both $\{u_{h,\tau}\}$ and $\{\tilde{u}_{h,\tau}\}$ are bounded in $L^2(0,T;H^1(\Omega))$ and $\{\frac{\partial}{\partial t} u_{h,\tau}\}$ is bounded in $L^2(0,T;(H^1(\Omega))')$. So by Lemma 2.1 there exist a subsequence of $\{u_{h,\tau}\}$ such that

$$u_{h,\tau} \to u^* \text{ weakly in } L^2(0,T;H^1(\Omega)) \text{ and strongly in } L^2(0,T;L^2(\Omega)), \tag{5.22}$$

$$\frac{\partial}{\partial t} u_{h,\tau} \to v^* \text{ weakly in } L^2(0,T;(H^1(\Omega))'), \tag{5.23}$$

and a subsequence of $\{\tilde{u}_{h,\tau}\}$ such that

$$\tilde{u}_{h,\tau} \to \tilde{u}^* \text{ weakly in } L^2(0,T;H^1(\Omega)) \tag{5.24}$$

for some $u^*, \tilde{u}^* \in L^2(0,T;H^1(\Omega))$ and $v^* \in L^2(0,T;(H^1(\Omega))')$.

From (5.23), we know for any $\varphi(x) \in H^1(\Omega)$ and $\psi(t) \in C_0^\infty(0,T)$,

$$\lim_{\substack{h\to 0\\ \tau\to 0}} \int_0^T\int_\Omega \frac{\partial u_{h,\tau}(x,t)}{\partial t}\varphi(x)\psi(t)dxdt = \int_0^T\int_\Omega v^*(x,t)\varphi(x)\psi(t)dxdt. \tag{5.25}$$

Integrating by parts the left-hand side and using (5.22), we obtain

$$-\int_0^T\int_\Omega u^*(x,t)\varphi(x)\psi'(t)dxdt = \int_0^T\int_\Omega v^*(x,t)\varphi(x)\psi(t)dxdt,$$

which gives

$$v^*(x,t) = \frac{\partial u^*(x,t)}{\partial t}. \tag{5.26}$$

Next, taking any $\varphi(x) \in H^1(\Omega)$ and $\psi(t) \in C^1[0,T]$ with $\psi(T) = 0$, integrating by parts to both sides of (5.25), and noting (5.26), we get

$$\begin{aligned}&\lim_{\substack{h\to 0\\ \tau\to 0}} \left\{-\int_\Omega Q_h u_0(x)\varphi(x)\psi(0)dx - \int_0^T\int_\Omega u_{h,\tau}(x,t)\varphi(x)\psi'(t)dxdt\right\}\\ &= -\int_\Omega u^*(x,0)\varphi(x)\psi(0)dx - \int_0^T\int_\Omega u^*(x,t)\varphi(x)\psi'(t)dxdt.\end{aligned}$$

By the convergence property of Q_h and (5.22) we obtain

$$u^*(x,0) = u_0(x). \tag{5.27}$$

Next, we show $u^*(x,t) = \tilde{u}^*(x,t)$. In fact, by direct computing and (5.15), we obtain

$$\int_0^T \|u_{h,\tau}(\cdot,t) - \tilde{u}_{h,\tau}(\cdot,t)\|_0^2 dt = \frac{\tau^3}{3}\sum_{n=1}^M \|\partial_\tau u_h^n\|_0^2 \le C\tau; \tag{5.28}$$

this with (5.22) proves that $\tilde{u}_{h,\tau}$ converges to $\tilde{u}^*$ strongly in $L^2(0,T;L^2(\Omega))$ and $u^*(x,t) = \tilde{u}^*(x,t)$.

Below we will show $u^* = u(q)$. For any $\varphi(x) \in H^1(\Omega)$ and $\psi(t) \in C_0^\infty(0,T)$, let $\phi(x,t) = \varphi(x)\psi(t)$ and $\phi_{h,\tau}(x,t) = \sum_{n=1}^M \chi_n(t)Q_h\phi(x,t_n)$. Then we have

$$\begin{aligned}&\int_0^T \|\phi(\cdot,t) - \phi_{h,\tau}(\cdot,t)\|_1^2 dt\\ &\le 2\int_0^T \|\phi(\cdot,t) - S_\Delta\phi(\cdot,t)\|_1^2 dt + 2\int_0^T \|S_\Delta\phi(\cdot,t) - \phi_{h,\tau}(\cdot,t)\|_1^2 dt\\ &\le 2\int_0^T \|\phi(\cdot,t) - S_\Delta\phi(\cdot,t)\|_1^2 dt + 2\,T\max_{0\le t\le T}|\psi(t)|^2\,\|Q_h\varphi(\cdot) - \varphi(\cdot)\|_1^2.\end{aligned}$$

Therefore by (5.20) and the convergence property of Q_h, we deduce

$$\phi_{h,\tau} \text{ converges to } \phi \text{ strongly in } L^2(0,T;H^1(\Omega)). \tag{5.29}$$

By direct computations we have the following equalities:

$$\int_0^T \int_\Omega \frac{\partial}{\partial t} u_{h,\tau}(x,t)\phi_{h,\tau}(x,t)dxdt = \tau \sum_{n=1}^M \int_\Omega \partial_\tau u_h^n Q_h\phi(x,t_n)dx,$$

$$\int_0^T \int_\Omega \alpha(x,t)\nabla\tilde{u}_{h,\tau}(x,t)\nabla\phi_{h,\tau}(x,t)dxdt = \tau \sum_{n=1}^M \int_\Omega \bar{\alpha}^n \nabla u_h^n \nabla Q_h\phi(x,t_n)dx,$$

$$\int_0^T \int_{\Gamma_o} c(x,t)\tilde{u}_{h,\tau}(x,t)\phi_{h,\tau}(x,t)dsdt = \tau \sum_{n=1}^M \int_{\Gamma_o} \bar{c}^n u_h^n Q_h\phi(x,t_n)ds,$$

$$-\int_0^T \int_{\Gamma_o} c(x,t)u_a(x,t)\phi_{h,\tau}(x,t)dsdt = -\tau \sum_{n=1}^M \int_{\Gamma_o} \overline{cu}_a^n Q_h\phi(x,t_n)ds,$$

$$\int_0^T \int_{\Gamma_i} q_{h,\tau}(x,t)\phi_{h,\tau}(x,t)dsdt = \tau \sum_{n=1}^M \int_{\Gamma_i} q_h^n Q_h\phi(x,t_n)ds;$$

adding them together and using the discrete parabolic equation (5.10), we obtain

$$\begin{aligned}&\int_0^T\int_\Omega \frac{\partial}{\partial t} u_{h,\tau}(x,t)\phi_{h,\tau}(x,t)dxdt + \int_0^T\int_\Omega \alpha(x,t)\nabla\tilde{u}_{h,\tau}(x,t)\nabla\phi_{h,\tau}(x,t)dxdt\\ &+\int_0^T\int_{\Gamma_o} c(x,t)\tilde{u}_{h,\tau}(x,t)\phi_{h,\tau}(x,t)dsdt - \int_0^T\int_{\Gamma_o} c(x,t)u_a(x,t)\phi_{h,\tau}(x,t)dsdt\\ &= -\int_0^T\int_{\Gamma_i} q_{h,\tau}(x,t)\phi_{h,\tau}(x,t)dsdt + \tau\sum_{n=1}^M \int_{\Gamma_o} (\bar{c}^n\bar{u}_a^n - \overline{cu}_a^n)Q_h\phi(x,t_n)ds.\end{aligned} \tag{5.30}$$

Taking the limit as h and τ tend to 0 and using the convergence (5.22)–(5.24) and (5.29), we derive that for any $\varphi(x) \in H^1(\Omega)$ and $\psi(t) \in C_0^\infty(0,T)$

$$\begin{aligned}&\int_0^T\int_\Omega \frac{\partial u^*}{\partial t}\varphi(x)\psi(t)dxdt + \int_0^T\int_\Omega \alpha\nabla u^*\cdot\nabla\varphi(x)\psi(t)dxdt + \int_0^T\int_{\Gamma_o} c\, u^*\varphi(x)\psi(t)dsdt\\ &= \int_0^T\int_{\Gamma_o} c\, u_a\varphi(x)\psi(t)dsdt - \int_0^T\int_{\Gamma_i} q\,\varphi(x)\psi(t)dsdt,\end{aligned} \tag{5.31}$$

where we have used the limit

$$\lim_{\substack{h\to 0\\ \tau\to 0}} \tau\sum_{n=1}^M \int_{\Gamma_o} (\bar{c}^n\bar{u}_a^n - \overline{cu}_a^n)Q_h\phi(x,t_n)ds = 0. \tag{5.32}$$

To see this, it follows from (5.13), the trace theorem, and the Cauchy–Schwarz inequality that

$$\begin{aligned}&\tau\sum_{n=1}^M \int_{\Gamma_o} (\bar{c}^n\bar{u}_a^n - \overline{cu}_a^n)Q_h\phi(x,t_n)ds\\ &\le C\,\tau \max_{0\le t\le T}|\psi(t)|\,\|Q_h\varphi\|_1 \sum_{n=1}^M \left\|\bar{c}^n\bar{u}_a^n - \overline{cu}_a^n\right\|_{0,\Gamma_o}\\ &\le C\,\tau \max_{0\le t\le T}|\psi(t)|\,\|\varphi\|_1 \left(\sum_{n=1}^M \|c_t\|^2_{L^2(t_{n-1},t_n;L^\infty(\Gamma_o))}\right)^{\frac{1}{2}} \left(\sum_{n=1}^M \|u_a\|^2_{L^2(t_{n-1},t_n;L^2(\Gamma_o))}\right)^{\frac{1}{2}}\\ &\le C\,\tau \max_{0\le t\le T}|\psi(t)|\,\|\varphi\|_1\,\|c_t\|_{L^2(0,T;L^\infty(\Gamma_o))}\,\|u_a\|_{L^2(0,T;L^2(\Gamma_o))}.\end{aligned}$$

Clearly, the fact that $u^* = u(q)$ follows then from (5.31).

Now we can show the desired relation (5.21). For this, setting $f(x,t) = u_{h,\tau}(x,t) - u(x,t)$, we can write and estimate using Lemma 5.2 as follows:

$$\begin{aligned}
&\tau \sum_{n=1}^{M} \int_\Omega (u_h^n - u^n)^2 dx - \int_0^T \|u_{h,\tau}(\cdot,t) - u(\cdot,t)\|_0^2 dt \\
&= \sum_{n=1}^{M} \int_{t_{n-1}}^{t_n} \int_\Omega \Big(|f(x,t_n)|^2 - |f(x,t)|^2 \Big) dx dt \\
&\le \left\{ \sum_{n=1}^{M} \int_{t_{n-1}}^{t_n} \|f(\cdot,t_n) + f(\cdot,t)\|_0^2 dt \right\}^{\frac{1}{2}} \left\{ \sum_{n=1}^{M} \int_{t_{n-1}}^{t_n} \|f(\cdot,t_n) - f(\cdot,t)\|_0^2 dt \right\}^{\frac{1}{2}} \\
&\le C \left\{ \sum_{n=1}^{M} \int_{t_{n-1}}^{t_n} \|f(\cdot,t_n) - f(\cdot,t)\|_0^2 dt \right\}^{\frac{1}{2}}.
\end{aligned}$$

By (5.22), the second term at the left-hand side of the above inequality tends to 0 as $h, \tau \to 0$. But the last term can be estimated as follows:

$$\begin{aligned}
&\sum_{n=1}^{M} \int_{t_{n-1}}^{t_n} \|f(\cdot,t_n) - f(\cdot,t)\|_0^2 dt \\
&= \sum_{n=1}^{M} \int_{t_{n-1}}^{t_n} \|u - u^n + (t_n - t)\partial_\tau u_h^n\|_0^2 dt \\
&\le 2 \sum_{n=1}^{M} \int_{t_{n-1}}^{t_n} \|u - u^n\|_0^2 dt + 2 \sum_{n=1}^{M} \int_{t_{n-1}}^{t_n} \int_\Omega (t_n - t)^2 |\partial_\tau u_h^n|^2 dx dt \\
&= 2 \sum_{n=1}^{M} \int_{t_{n-1}}^{t_n} \|u - u^n\|_0^2 dt + \frac{2}{3}\tau^3 \sum_{n=1}^{M} \|\partial_\tau u_h^n\|_0^2.
\end{aligned}$$

From (5.20) and (5.15), the last two terms both tend to 0, and (5.21) follows. □

Finally, we are ready to show the main convergence results of this section.

THEOREM 5.6. *Let $\{q_{h,\tau}^*\}$ be a sequence of minimizers to the finite element minimization problem* (5.8)–(5.10)*; then as h and τ tend to* 0*, the whole sequence $\{q_{h,\tau}^*\}$ converges strongly in $L^2(0,T;L^2(\Gamma_i))$ to the unique minimizer of the continuous problem* (2.1)–(2.3).

Proof. Using the stability estimate (5.14), it is easy to know that $J_{h,\tau}(q_{h,\tau}^*) \le C$ for some constant C independent of h and τ. This implies that $\{q_{h,\tau}^*\}$ is bounded in $L^2(0,T;L^2(\Gamma_i))$ and there exists a subsequence of $\{q_{h,\tau}^*\}$, still denoted as $\{q_{h,\tau}^*\}$, such that $q_{h,\tau}^* \to q^*$ weakly in $L^2(0,T;L^2(\Gamma_i))$ as $h, \tau \to 0$.

Now for any $q \in L^2(0,T;L^2(\Gamma_i))$ and any fixed $\varepsilon > 0$, by the density results there exists a $q_\varepsilon \in H^1(0,T;H^{1/2}(\Gamma_i))$ such that

$$\|q - q_\varepsilon\|_{L^2(0,T;L^2(\Gamma_i))} \le \varepsilon.$$

Then we define an extension $\tilde{q}_\varepsilon$ of q_ε as follows: $\tilde{q}_\varepsilon \in H^1(\Omega)$ solves

$$-\Delta \tilde{q}_\varepsilon = 0 \ \text{ in } \ \Omega, \quad \tilde{q}_\varepsilon = q_\varepsilon \ \text{ on } \ \Gamma_i, \quad \tilde{q}_\varepsilon = 0 \ \text{ on } \ \Gamma_o.$$

One can verify that $\tilde{q}_\varepsilon \in H^1(0,T;H^1(\Omega))$ and $\|\tilde{q}_\varepsilon\|_{H^1(0,T;H^1(\Omega))} \le C\|q_\varepsilon\|_{H^1(0,T;H^{1/2}(\Gamma_i))}$. Define

$$\tilde{q}_\varepsilon^{h,\tau}(x,t) = \sum_{n=1}^{M} \chi_n(t) Q_h \tilde{q}_\varepsilon(x,t_n).$$

Let $q_\varepsilon^{h,\tau}$ be the restriction of $\tilde{q}_\varepsilon^{h,\tau}$ on Γ_i; then $q_\varepsilon^{h,\tau} \in V_{\Gamma_i}^h$ and for any $\varepsilon > 0$,

$$\begin{aligned}
&\|q_\varepsilon^{h,\tau} - q_\varepsilon\|^2_{L^2(0,T;L^2(\Gamma_i))} \le \|q_\varepsilon^{h,\tau} - q_\varepsilon\|^2_{L^2(0,T;H^{1/2}(\Gamma_i))} \le C\|\tilde{q}_\varepsilon^{h,\tau} - \tilde{q}_\varepsilon\|^2_{L^2(0,T;H^1(\Omega))}\\
&= C\sum_{n=1}^{M}\int_{t_{n-1}}^{t_n} \|Q_h\tilde{q}_\varepsilon(\cdot,t_n) - \tilde{q}_\varepsilon(\cdot,t)\|_1^2 dt\\
&\le C\sum_{n=1}^{M}\int_{t_{n-1}}^{t_n} \|Q_h\tilde{q}_\varepsilon(\cdot,t_n) - Q_h\tilde{q}_\varepsilon(\cdot,t) + Q_h\tilde{q}_\varepsilon(\cdot,t) - \tilde{q}_\varepsilon(\cdot,t)\|_1^2 dt\\
&\le C\int_0^T \|S_\Delta\tilde{q}_\varepsilon(\cdot,t) - \tilde{q}_\varepsilon(\cdot,t)\|_1^2 dt + C\int_0^T \|Q_h\tilde{q}_\varepsilon(\cdot,t) - \tilde{q}_\varepsilon(\cdot,t)\|_1^2 dt.
\end{aligned}$$

Thus $q_\varepsilon^{h,\tau} \to q_\varepsilon$ in $L^2(0,T;L^2(\Gamma_i))$ as $h,\tau \to 0$. Using this and Lemma 5.5, we can derive

$$\begin{aligned}
J(q^*) &\le \lim_{\substack{h\to 0\\ \tau\to 0}} \frac{\tau}{2}\sum_{n=0}^{M}\alpha_n\int_\omega (u_h^n(q_{h,\tau}^*) - z^n)^2 dx + \frac{\beta}{2}\liminf_{\substack{h\to 0\\ \tau\to 0}} \int_0^T\int_{\Gamma_i} |q_{h,\tau}^*|^2 ds dt\\
&\le \liminf_{\substack{h\to 0\\ \tau\to 0}} J_{h,\tau}(q_{h,\tau}^*) \le \liminf_{\substack{h\to 0\\ \tau\to 0}} J_{h,\tau}(q_\varepsilon^{h,\tau})\\
&= \frac{1}{2}\int_0^T\int_\omega (u(q_\varepsilon) - z)^2 dx dt + \frac{\beta}{2}\int_0^T\int_{\Gamma_i} q_\varepsilon^2 ds dt\\
&= J(q_\varepsilon).
\end{aligned}$$

Letting $\varepsilon \to 0$, we deduce

$$J(q^*) \le J(q) \qquad \forall\, q \in L^2(0,T;L^2(\Gamma_i)), \tag{5.33}$$

which indicates that q^* is the unique minimizer of the continuous problem (2.1)–(2.3).

The strong convergence follows by the same trick as used in Proposition 2.3. ∎

Remark 5.7. All the results obtained in this paper can be naturally extended to a three-dimensional domain Ω with every two-dimensional cross-section being the domain as in Figure 2.

6. Finite element approximation of system (4.1)–(4.3) and its convergence.

Next, we shall discuss the discretization of system (4.1)–(4.3). As we did for system (2.1)–(2.3), we use the composite trapezoidal rule for the time discretization of the first integral in (4.1) and the exact time integration for the second term. But as the time derivative of the identifying parameter $q(x,t)$ is involved in the regularization term now, we cannot ensure the convergence of the resultant fully discrete scheme for the entire system (4.1)–(4.3) if the backward Euler scheme is still used for approximating the parabolic problem (4.3). Instead we shall adopt the Crank–Nicolson scheme. This results in the following finite element approximation of (4.1)–(4.3):

(6.1)

$$\min J_{h,\tau}(q_{h,\tau}) = \frac{\tau}{2}\sum_{n=0}^{M}\alpha_n\int_\omega (u_h^n - z^n)^2 dx + \frac{\beta}{2}\left(\int_{\Gamma_i}|q_h^0|^2 ds + \tau\sum_{n=1}^{M}\int_{\Gamma_i}|\partial_\tau q_h^n|^2 ds\right)$$

over all $q_h^n \in V_{\Gamma_i}^h$ with $u_h^n \equiv u_h^n(q_{h,\tau}) \in V^h$ satisfying $u_h^0 = Q_h u_0$ in Ω and

$$\int_\Omega \partial_\tau u_h^n \phi_h dx + \int_\Omega \bar{\alpha}^n \nabla \frac{u_h^n + u_h^{n-1}}{2} \cdot \nabla \phi_h dx + \int_{\Gamma_o} \bar{c}^n \frac{u_h^n + u_h^{n-1}}{2} \phi_h ds$$
$$= \int_{\Gamma_o} \bar{c}^n \bar{u}_a^n \phi_h ds - \int_{\Gamma_i} \frac{q_h^n + q_h^{n-1}}{2} \phi_h ds \quad \forall \phi_h \in V^h \tag{6.2}$$

for $n = 1, 2, \dots, M$. Here $\{\alpha_n\}$ are the coefficients of the composite trapezoidal rule: $\alpha_0 = \alpha_M = \frac{1}{2}$ and $\alpha_n = 1$ for all $n \neq 0, M$. The heat flux q is approximated by $q_{h,\tau}$, a piecewise linear interpolation based on $\{q_h^n\}$ over the time partition Δ in (5.2):

$$q_{h,\tau}(x,t) = \frac{t - t_{n-1}}{\tau} q_h^n + \frac{t_n - t}{\tau} q_h^{n-1}, \qquad t \in (t_{n-1}, t_n). \tag{6.3}$$

For the fully discrete finite element scheme (6.1)–(6.2), we can show (cf. [14]) the following theorem.

Theorem 6.1. *There exists a unique minimizer to the finite element problem* (6.1)–(6.2).

In the rest of this section, we study the convergence of the discrete minimizer of (6.1)–(6.2) to the global minimizer of the continuous problem (4.1)–(4.3). For this purpose, we assume on functions $\alpha(x,t)$, $c(x,t)$, and $u_a(x,t)$ in (1.1)–(1.4) that

$$\alpha \in W^{1,\infty}(0,T; L^\infty(\Omega)) \quad \text{and} \quad c,\ u_a \in W^{1,\infty}(0,T; L^\infty(\Gamma_o)) \tag{6.4}$$

and introduce three related constants:

$$C_1 = \|\alpha\|_{W^{1,\infty}(0,T;L^\infty(\Omega))}, \quad C_2 = \|c\|_{W^{1,\infty}(0,T;L^\infty(\Gamma_o))}, \quad C_3 = \|u_a\|_{W^{1,\infty}(0,T;L^\infty(\Gamma_o))}. \tag{6.5}$$

Using these constants, we can derive the following estimates (cf. [14]):

$$\|\bar{\alpha}^n - \bar{\alpha}^{n-1}\|_{L^\infty(\Omega)} \le C_1 \tau, \quad \|\bar{c}^n \bar{u}_a^n - \bar{c}^{n-1} \bar{u}_a^{n-1}\|_{L^\infty(\Gamma_o)} \le \frac{5}{3} C_2 C_3 \tau. \tag{6.6}$$

For the convergence analysis, we first establish some stability estimates of the finite element solution to (6.2).

Lemma 6.2. *Let u_h^n be the finite element solution of system* (6.2) *corresponding to the given heat flux $\{q_h^n\}_{n=0}^M$; then we have the following stability estimates:*

$$\max_{1\le n\le M} \|u_h^n\|_0^2 + \tau \sum_{n=1}^M \left\| \nabla \frac{u_h^n + u_h^{n-1}}{2} \right\|_0^2 + \tau \sum_{n=1}^M \left\| \frac{u_h^n + u_h^{n-1}}{2} \right\|_{0,\Gamma_o}^2$$
$$\le C \left(\|u_0\|_0^2 + C_2^2 C_3^2 + \tau \sum_{n=0}^M \|q_h^n\|_{0,\Gamma_i}^2 \right), \tag{6.7}$$

$$\tau \sum_{n=1}^M \|\partial_\tau u_h^n\|_0^2 + \max_{1\le n\le M} \|\nabla u_h^n\|_0^2 + \max_{1\le n\le M} \|u_h^n\|_{0,\Gamma_o}^2$$
$$\le C \left(\|u_0\|_1^2 + C_2^2 C_3^2 + \max_{1\le n\le k} \|q_h^n\|_{0,\Gamma_i}^2 + \tau \sum_{n=1}^M \|\partial_\tau q_h^n\|_{0,\Gamma_i}^2 + \tau \sum_{n=0}^M \|q_h^n\|_{0,\Gamma_i}^2 \right). \tag{6.8}$$

Proof. Taking $\phi_h = \tau\frac{u_h^n+u_h^{n-1}}{2}$ in (6.2), we have

$$
\begin{aligned}
&\frac{1}{2}\|u_h^n\|_0^2 - \frac{1}{2}\|u_h^{n-1}\|_0^2 + \alpha_0\tau\left\|\nabla\frac{u_h^n+u_h^{n-1}}{2}\right\|_0^2 + c_0\tau\left\|\frac{u_h^n+u_h^{n-1}}{2}\right\|_{0,\Gamma_o}^2\\
&\le \tau\int_{\Gamma_o}\bar{c}^n\bar{u}_a^n\frac{u_h^n+u_h^{n-1}}{2}ds - \tau\int_{\Gamma_i}\frac{q_h^n+q_h^{n-1}}{2}\frac{u_h^n+u_h^{n-1}}{2}ds.
\end{aligned}
$$

Summing up the above equation over $n = 1, 2, \dots, k \le M$, we derive

$$
\begin{aligned}
&\frac{1}{2}\|u_h^k\|_0^2 - \frac{1}{2}\|u_h^0\|_0^2 + \alpha_0\tau\sum_{n=1}^k\left\|\nabla\frac{u_h^n+u_h^{n-1}}{2}\right\|_0^2 + c_0\tau\sum_{n=1}^k\left\|\frac{u_h^n+u_h^{n-1}}{2}\right\|_{0,\Gamma_o}^2\\
&\le \tau\sum_{n=1}^k\int_{\Gamma_o}\bar{c}^n\bar{u}_a^n\frac{u_h^n+u_h^{n-1}}{2}ds - \tau\sum_{n=1}^k\int_{\Gamma_i}\frac{q_h^n+q_h^{n-1}}{2}\,\frac{u_h^n+u_h^{n-1}}{2}ds;
\end{aligned}
$$

then (6.7) follows by applying the trace theorem and Young's and Gronwall's inequalities.

Next, taking $\phi_h = \tau\partial_\tau u_h^n$ in (6.2), we have

$$
\begin{aligned}
&\tau\|\partial_\tau u_h^n\|_0^2 + \frac{1}{2}\int_\Omega\bar{\alpha}^n(|\nabla u_h^n|^2 - |\nabla u_h^{n-1}|^2)dx + \frac{1}{2}\int_{\Gamma_o}\bar{c}^n(|u_h^n|^2 - |u_h^{n-1}|^2)ds\\
&= \int_{\Gamma_o}\bar{c}^n\bar{u}_a^n(u_h^n - u_h^{n-1})ds - \int_{\Gamma_i}\frac{q_h^n+q_h^{n-1}}{2}(u_h^n - u_h^{n-1})ds.
\end{aligned}
$$

Summing up the above equation over $n = 1, 2, \dots, k \le M$ and using the formula (5.17), we deduce

$$
\begin{aligned}
&\tau\sum_{n=1}^k\|\partial_\tau u_h^n\|_0^2 + \frac{1}{2}\alpha_0\|\nabla u_h^k\|_0^2 + \frac{1}{2}c_0\|u_h^k\|_{0,\Gamma_o}^2\\
&\le \frac{1}{2}\int_\Omega\bar{\alpha}^0|\nabla u_h^0|^2dx + \frac{1}{2}\sum_{n=1}^k\int_\Omega(\bar{\alpha}^n - \bar{\alpha}^{n-1})|\nabla u_h^{n-1}|^2dx\\
&\quad+\frac{1}{2}\int_{\Gamma_o}\bar{c}^0|u_h^0|^2ds + \frac{1}{2}\sum_{n=1}^k\int_{\Gamma_o}(\bar{c}^n - \bar{c}^{n-1})|u_h^{n-1}|^2ds\\
&\quad+\int_{\Gamma_o}\bar{c}^k\bar{u}_a^ku_h^kds - \int_{\Gamma_o}\bar{c}^0\bar{u}_a^0u_h^0ds - \sum_{n=1}^k\int_{\Gamma_o}(\bar{c}^n\bar{u}_a^n - \bar{c}^{n-1}\bar{u}_a^{n-1})u_h^{n-1}ds\\
&\quad-\frac{1}{2}\int_{\Gamma_i}(q_h^k + q_h^{k-1})u_h^kds + \frac{1}{2}\int_{\Gamma_i}q_h^0u_h^0ds\\
&\quad+\sum_{n=1}^k\int_{\Gamma_i}\left(\frac{q_h^n+q_h^{n-1}}{2} - \frac{q_h^{n-1}+q_h^{n-2}}{2}\right)u_h^{n-1}ds.
\end{aligned}
$$

We now estimate all the terms on the right-hand side above. First, for those terms without summation, we can easily deduce

$$
\begin{aligned}
&\int_\Omega \bar{\alpha}^0 |\nabla u_h^0|^2 dx + \int_{\Gamma_o} \bar{c}^0 |u_h^0|^2 ds \le C\,(C_1 + C_2)\|u_0\|_1^2,\\
&\int_{\Gamma_o} \bar{c}^0 \bar{u}_a^0 u_h^0 ds + \frac{1}{2}\int_{\Gamma_i} q_h^0 u_h^0 ds \le C\,(C_2 C_3 + \|q_h^0\|_{0,\Gamma_i})\|u_0\|_1,\\
&\int_{\Gamma_o} \bar{c}^k \bar{u}_a^k u_h^k ds \le \frac{1}{4} c_0 \|u_h^k\|_{0,\Gamma_o}^2 + C\, C_2^2 C_3^2,\\
&-\frac{1}{2}\int_{\Gamma_i} (q_h^k + q_h^{k-1}) u_h^k ds \le \frac{1}{4}\alpha_0 \|\nabla u_h^k\|_0^2 + C\left(\max_{1\le n\le M}\|u_h^n\|_0^2 + \max_{1\le n\le M}\|q_h^n\|_{0,\Gamma_i}^2\right).
\end{aligned}
$$

Using (6.6), we obtain the following estimates:

$$
\begin{aligned}
\frac{1}{2}\sum_{n=1}^k \int_\Omega (\bar{\alpha}^n - \bar{\alpha}^{n-1})|\nabla u_h^{n-1}|^2 dx &\le \frac{1}{2} C_1\,\tau \sum_{n=1}^k \|\nabla u_h^{n-1}\|_0^2,\\
\frac{1}{2}\sum_{n=1}^k \int_{\Gamma_o} (\bar{c}^n - \bar{c}^{n-1})|u_h^{n-1}|^2 ds &\le \frac{1}{2} C_2\,\tau \sum_{n=1}^k \|u_h^{n-1}\|_{0,\Gamma_o}^2,\\
-\sum_{n=1}^k \int_{\Gamma_o} (\bar{c}^n \bar{u}_a^n - \bar{c}^{n-1}\bar{u}_a^{n-1}) u_h^{n-1} ds &\le C\,\tau \sum_{n=1}^k \|u_h^{n-1}\|_{0,\Gamma_o}^2 + C.
\end{aligned}
$$

For the last term, we use the Cauchy–Schwarz inequality to obtain

$$
\begin{aligned}
&\sum_{n=1}^k \int_{\Gamma_i} \left(\frac{q_h^n + q_h^{n-1}}{2} - \frac{q_h^{n-1} + q_h^{n-2}}{2}\right) u_h^{n-1} ds\\
&\le \frac{1}{2}\tau \sum_{n=1}^k \|u_h^{n-1}\|_{0,\Gamma_i}^2 + \frac{1}{8\tau}\sum_{n=1}^k \left\|(q_h^n + q_h^{n-1}) - (q_h^{n-1} + q_h^{n-2})\right\|_{0,\Gamma_i}^2\\
&\le C\,\tau \sum_{n=1}^k \left(\|\nabla u_h^{n-1}\|_0^2 + \|u_h^{n-1}\|_0^2 + \|\partial_\tau q_h^n\|_{0,\Gamma_i}^2\right).
\end{aligned}
$$

Now (6.8) follows by combining all of the above estimates and using Gronwall's inequality. ◻

As we did for the finite element system (5.8)–(5.10), we need the following crucial technical result for the convergence of the finite element approximation (6.1)–(6.2).

LEMMA 6.3. *If $q_{h,\tau}$ converges to some q weakly in $H^1(0,T;L^2(\Gamma_i))$ as h and τ tend to* 0, *then*

$$
\lim_{\substack{h\to 0\\ \tau\to 0}} \tau \sum_{n=0}^M \alpha_n \int_\omega (u_h^n(q_h^n) - z^n)^2 dx = \int_0^T\!\!\int_\omega (u(q) - z)^2 dx dt.
$$

Proof. As in the proof of Lemma 5.5, it suffices to show (5.21).

We first construct two interpolations based on $\{u_h^n\}$: one is a piecewise linear interpolation over the time partition Δ,

$$
u_{h,\tau}(x,t) = \frac{t - t_{n-1}}{\tau} u_h^n + \frac{t_n - t}{\tau} u_h^{n-1}, \quad t \in (t_{n-1}, t_n) \quad \text{for } \; n = 1, 2, \dots, M,
$$

and the other is a piecewise constant interpolation,

$$\tilde{u}_{h,\tau}(x,t) = \frac{1}{2}(u_h^n + u_h^{n-1}), \quad t \in (t_{n-1}, t_n) \quad \text{for} \quad n = 1, 2, \dots, M.$$

Using the definition of $q_{h,\tau}$ in (6.3) and the simple identity

$$q_h^n = \tau \sum_{k=1}^{n} \partial_\tau q_h^k + q_h^0,$$

we can directly see that

$$\tau \sum_{n=1}^{M} \|\partial_\tau q_h^n\|_{0,\Gamma_i}^2 = \tau \sum_{n=1}^{M} \left\| \frac{\partial}{\partial t} q_{h,\tau} \right\|_{0,\Gamma_i}^2 = \left\| \frac{\partial}{\partial t} q_{h,\tau} \right\|_{L^2(0,T;L^2(\Gamma_i))}^2,$$

$$\|q_h^n\|_{0,\Gamma_i}^2 \le 2\tau T \sum_{k=1}^{n} \|\partial_\tau q_h^k\|_{0,\Gamma_i}^2 + 2\|q_h^0\|_{0,\Gamma_i}^2.$$

With these relations, the assumption on $q_{h,\tau}$, and the stability estimates (6.7)–(6.8), we can easily check that both $\{u_{h,\tau}\}$ and $\{\tilde{u}_{h,\tau}\}$ are bounded in $L^2(0,T;H^1(\Omega))$ and that $\{\frac{\partial}{\partial t} u_{h,\tau}\}$ is bounded in $L^2(0,T;L^2(\Omega))$. So there exist a subsequence $\{u_{h,\tau}\}$ such that

$$u_{h,\tau} \to u^* \text{ weakly in } L^2(0,T;H^1(\Omega)) \text{ and strongly in } L^2(0,T;L^2(\Omega)),$$
$$\frac{\partial}{\partial t} u_{h,\tau} \to \frac{\partial u^*}{\partial t} \text{ weakly in } L^2(0,T;L^2(\Omega)),$$

and a subsequence $\{\tilde{u}_{h,\tau}\}$ such that

$$\tilde{u}_{h,\tau} \to \tilde{u}^* \text{ weakly in } L^2(0,T;H^1(\Omega)) \tag{6.9}$$

for some $u^* \in H^1(0,T;L^2(\Omega))$ and $\tilde{u}^* \in L^2(0,T;H^1(\Omega))$. We can further show that $u^*(x,0) = u_0(x)$ and $u^* = \tilde{u}^*$ using the fact that

$$\int_0^T \|u_{h,\tau}(\cdot,t) - \tilde{u}_{h,\tau}(\cdot,t)\|_0^2 dt = \frac{\tau^3}{12} \sum_{n=1}^{M} \|\partial_\tau u_h^n\|_0^2 \to 0. \tag{6.10}$$

Next, we show $u^* = u(q)$. Let $\phi_{h,\tau}(x,t)$ be defined as in the proof of Lemma 5.5. By simple computations we have the following equalities:

$$\int_0^T \int_\Omega \frac{\partial}{\partial t} u_{h,\tau}(x,t) \phi_{h,\tau}(x,t) dx dt = \tau \sum_{n=1}^{M} \int_\Omega \partial_\tau u_h^n Q_h \phi(x,t_n) dx,$$

$$\int_0^T \int_\Omega \alpha(x,t) \nabla \tilde{u}_{h,\tau}(x,t) \nabla \phi_{h,\tau}(x,t) dx dt = \tau \sum_{n=1}^{M} \int_\Omega \bar{\alpha}^n \nabla \frac{u_h^n + u_h^{n-1}}{2} \nabla Q_h \phi(x,t_n) dx,$$

$$\int_0^T \int_{\Gamma_o} c(x,t) \tilde{u}_{h,\tau}(x,t) \phi_{h,\tau}(x,t) ds dt = \tau \sum_{n=1}^{M} \int_{\Gamma_o} \bar{c}^n \frac{u_h^n + u_h^{n-1}}{2} Q_h \phi(x,t_n) ds,$$

$$\int_0^T \int_{\Gamma_o} c(x,t) u_a(x,t) \phi_{h,\tau}(x,t) ds dt = \tau \sum_{n=1}^{M} \int_{\Gamma_o} \overline{cu}_a^n Q_h \phi(x,t_n) ds,$$

$$\int_0^T \int_{\Gamma_i} q_{h,\tau}(x,t) \phi_{h,\tau}(x,t) ds dt = \tau \sum_{n=1}^{M} \int_{\Gamma_i} \frac{q_h^n + q_h^{n-1}}{2} Q_h \phi(x,t_n) ds.$$

Adding them together and using (6.2), we obtain a similar equation to (5.30). The rest of the proof is basically the same as that in Lemma 5.5. □

By virtue of Lemma 6.3, we can show the convergence of the finite element approximation (6.1)–(6.2), following the same lines as for Theorem 5.6; see Xie [14].

THEOREM 6.4. *Let* $\{q^*_{h,\tau}\}$ *be a sequence of minimizers to the discrete minimization problem* (6.1)–(6.2)*; then as* h *and* τ *tend to* 0*, the whole sequence* $\{q^*_{h,\tau}\}$ *converges strongly in* $H^1(0,T;L^2(\Gamma_i))$ *to the unique minimizer of the continuous problem* (4.1)–(4.3).

7. Solutions of finite element minimizaton problems. In this section, we shall formulate a conjugate gradient algorithm to solve the nonlinear finite element minimization problems (5.8)–(5.10) and (6.1)–(6.2). We present details only for system (6.1)–(6.2), while the algorithm for system (5.8)–(5.10) can be formulated similarly; for details of the latter system, we refer to Xie [14].

We first derive the Gateaux derivative of the cost functional $J_{h,\tau}(q_{h,\tau})$ in (6.1), or the form $J_{h,\tau}(\{q_h^0,\dots,q_h^M\})$. Let $N=\dim(V^h_{\Gamma_i})$, and let $\{\psi_i\}_{i=1}^N$ be the basis of $V^h_{\Gamma_i}$. For any element from space $V^h_{\Gamma_i}\times\cdots\times V^h_{\Gamma_i}$, say $\{q_h^0,\dots,q_h^M\}$, let $\mathcal{U}_h^n\equiv u_h^n(q_{h,\tau})'p_{h,\tau}$ be the Gateaux derivative of solution $u_h^n(q_{h,\tau})$ to (6.1)–(6.2) in the direction $p_{h,\tau}$, or $\{p_h^0,\dots,p_h^M\}$. We easily see that $\mathcal{U}_h^0=0$, and for $n=1,2,\dots,M$ and any $\phi_h\in V^h$, the derivative $\mathcal{U}_h^n\in V^h$ satisfies

$$\begin{aligned}&\int_\Omega \partial_\tau\mathcal{U}_h^n\phi_h dx+\int_\Omega \bar\alpha^n\nabla\frac{\mathcal{U}_h^n+\mathcal{U}_h^{n-1}}{2}\cdot\nabla\phi_h dx+\int_{\Gamma_o}\bar c^n\frac{\mathcal{U}_h^n+\mathcal{U}_h^{n-1}}{2}\phi_h ds\\&\quad=-\int_{\Gamma_i}\frac{p_h^n+p_h^{n-1}}{2}\phi_h ds.\end{aligned}$$

This enables us to derive the first and second derivatives of the cost functional $J_{h,\tau}$ in (6.1):

$$\begin{aligned}J_{h,\tau}(q_{h,\tau})'p_{h,\tau}&=\tau\sum_{n=1}^M\alpha_n\int_\omega(u_h^n-z^n)\mathcal{U}_h^n dx\\&\quad+\beta\left(\int_{\Gamma_i}q_h^0p_h^0ds+\tau\sum_{n=1}^M\int_{\Gamma_i}\partial_\tau q_h^n\,\partial_\tau p_h^n ds\right),\end{aligned}\tag{7.1}$$

$$\begin{aligned}J_{h,\tau}(q_{h,\tau})''p_{h,\tau}r_{h,\tau}&=\tau\sum_{n=1}^M\alpha_n\int_\omega\Big(u_h^n(q_{h,\tau})'p_{h,\tau}\Big)\Big(u_h^n(q_{h,\tau})'r_{h,\tau}\Big)dx\\&\quad+\beta\left(\int_{\Gamma_i}p_h^0r_h^0ds+\tau\sum_{n=1}^M\int_{\Gamma_i}\partial_\tau p_h^n\,\partial_\tau r_h^n ds\right).\end{aligned}\tag{7.2}$$

Clearly, evaluating the derivatives of $J_{h,\tau}$ at a given point $q_{h,\tau}$ using formula (7.1) is extremely expensive. To reduce the cost, we introduce an adjoint equation for the Crank–Nicolson scheme (6.2), which seems to have not been studied in the literature before. This needs to be done carefully in order to meet our final goal. A discrete sequence $\{w_h^n\}_{n=0}^M$ is defined in such a way that $w_h^M=0$ and $w_h^n\in V^h$ for $n\neq M$ solves

$$\begin{aligned}&-\int_\Omega\frac{w_h^n-w_h^{n-1}}{\tau}\phi_h dx+\int_\Omega\frac{\bar\alpha^{n+1}\nabla w_h^n+\bar\alpha^n\nabla w_h^{n-1}}{2}\cdot\nabla\phi_h dx\\&+\int_{\Gamma_o}\frac{\bar c^{n+1}w_h^n+\bar c^n w_h^{n-1}}{2}\phi_h ds=\alpha_n\int_\omega(u_h^n-z^n)\phi_h dx\quad\forall\phi_h\in V^h.\end{aligned}\tag{7.3}$$

Now taking $\phi_h = \mathcal{U}_h^n$ in (7.3), we can rewrite the first term in $J_{h,\tau}(q_{h,\tau})' p_{h,\tau}$ as

$$J_{h,\tau}^1 = -\sum_{n=1}^M \int_\Omega \frac{w_h^n - w_h^{n-1}}{\tau} \mathcal{U}_h^n dx + \sum_{n=1}^M \int_\Omega \frac{\bar\alpha^{n+1} \nabla w_h^n + \bar\alpha^n \nabla w_h^{n-1}}{2} \cdot \nabla \mathcal{U}_h^n dx + \sum_{n=1}^M \int_{\Gamma_o} \frac{\bar c^{n+1} w_h^n + \bar c^n w_h^{n-1}}{2} \mathcal{U}_h^n ds.$$

Then using formula (5.17), the identity[2]

$$\sum_{n=1}^k (a_n + a_{n-1}) b_n = a_k b_k - a_0 b_0 + \sum_{n=1}^k a_{n-1}(b_n + b_{n-1}),$$

and the equation for $\mathcal{U}_h^n$, we obtain

$$\begin{aligned} J_{h,\tau}^1 &= \sum_{n=1}^M \int_\Omega \frac{\mathcal{U}_h^n - \mathcal{U}_h^{n-1}}{\tau} w_h^{n-1} dx + \sum_{n=1}^M \int_\Omega \bar\alpha^n \nabla \frac{\mathcal{U}_h^n + \mathcal{U}_h^{n-1}}{2} \cdot \nabla w_h^{n-1} dx \\ &\quad + \sum_{n=1}^M \int_{\Gamma_o} \bar c^n \frac{\mathcal{U}_h^n + \mathcal{U}_h^{n-1}}{2} w_h^{n-1} ds \\ &= -\sum_{n=1}^M \int_{\Gamma_i} \frac{p_h^n + p_h^{n-1}}{2} w_h^{n-1} ds. \end{aligned}$$

This, along with (7.1), leads to a very simple formula for evaluating the derivative of $J_{h,\tau}$:

$$J_{h,\tau}(q_{h,\tau})' p_{h,\tau} = \int_{\Gamma_i} \left(\beta q_h^0 p_h^0 + \sum_{n=1}^M \left\{ \frac{\beta (q_h^n - q_h^{n-1})(p_h^n - p_h^{n-1})}{\tau} - \frac{\tau (p_h^n + p_h^{n-1}) w_h^{n-1}}{2} \right\} \right) ds. \tag{7.4}$$

Next, we are going to formulate the conjugate gradient method for the nonlinear minimization (6.1). Let us first establish one-to-one correspondences between finite element functions and their coefficient vectors. For any $q_h^j \in V_{\Gamma_i}^h$, we write its representation in terms of the basis $\{\psi_i\}_{i=1}^N$ as

$$q_h^j = \sum_{i=1}^N q_i^j \psi_i.$$

Then each finite element function $q_{h,\tau}$ or $\{q_h^0, q_h^1, \ldots, q_h^M\}$ corresponds uniquely to an $(M+1)N$-dimensional vector

$$\mathbf{q} = (q_1^0, \ldots, q_N^0, q_1^1, \ldots, q_N^1, q_1^2, \ldots, q_N^2, \ldots, q_1^M, \ldots, q_N^M)^T.$$

[2]This crucial identity has not been seen in the literature before and has no continuous counterpart, unlike the widely used identity (5.17) that is known as the discrete integration by parts formula.

Letting $f(\mathbf{q}) = J_{h,\tau}(q_{h,\tau})$, one can directly verify the relation for the first derivatives of $f(\mathbf{q})$,

$$\frac{\partial f(\mathbf{q})}{\partial q_i^j} = J_{h,\tau}(\{q_h^0, q_h^1, \dots, q_h^M\})'(\{0, \dots, \psi_i, \dots, 0\}),$$

and the relation for the Hessian $H = (h_{ij})$,

$$h_{ik} :\equiv \frac{\partial^2 f(\mathbf{q})}{\partial q_i^j \partial q_k^l} = J_{h,\tau}(\{q_h^0, q_h^1, \dots, q_h^M\})''(\{0, \dots, \psi_i, \dots, 0\})(\{0, \dots, \psi_k, \dots, 0\}).$$

This leads to the following expression:

$$f(\mathbf{q}) = \frac{1}{2}\mathbf{q}^T H \mathbf{q} + \nabla f(\mathbf{0})^T \mathbf{q} + f(\mathbf{0}).$$

We see that the evaluation of the Hessian H is extremely expensive. Fortunately, only its products with vectors are needed in the conjugate gradient method, and such products can be done with much less cost by noting the identity that $H\mathbf{q} = \nabla f(\mathbf{q}) - \nabla f(\mathbf{0})$ and the simple formula (7.4).

We are now ready to state the conjugate gradient algorithm for solving the discrete minimization problem (6.1)–(6.2). We shall use $(J_{h,\tau}(q_{h,\tau}))'$ for $\nabla f(\mathbf{q})$ to emphasize the dependence of the first order derivatives on mesh size h and time step τ.

Conjugate Gradient Algorithm

Step 1. Given a tolerance ε, compute $\tilde{\mathbf{g}}_0 = (J_{h,\tau}(0))'$.

Step 2. Given an initial guess $q_{h,\tau}^{(0)}$, solve the direct problem (6.2) and the adjoint equation (7.3), then compute $\mathbf{g}_0 = (J_{h,\tau}(q_{h,\tau}^{(0)}))'$ by using (7.4). Set $\mathbf{d}_0 := -\mathbf{g}_0$ and $k := 0$.

Step 3. Solve the one-dimensional problem

$$J_{h,\tau}(q_{h,\tau}^{(k)} + \alpha_k d_{h,\tau}^{(k)}) = \min_{\alpha} J_{h,\tau}(q_{h,\tau}^{(k)} + \alpha d_{h,\tau}^{(k)})$$

by computing

$$\alpha_k = -\frac{\mathbf{g}_k^T \mathbf{d}_k}{\mathbf{d}_k^T((J_{h,\tau}(d_{h,\tau}^{(k)}))' - \tilde{\mathbf{g}}_0)}.$$

Set $q_{h,\tau}^{(k+1)} := q_{h,\tau}^{(k)} + \alpha_k d_{h,\tau}^{(k)}$ and $k := k+1$. Compute

$$\mathbf{g}_k = (J_{h,\tau}(q_{h,\tau}^{(k)}))',$$
$$\beta_k = \frac{\mathbf{g}_k^T \mathbf{g}_k}{\mathbf{g}_{k-1}^T \mathbf{g}_{k-1}},$$
$$\mathbf{d}_k = -\mathbf{g}_k + \beta_k \mathbf{d}_{k-1}.$$

Step 4. If $\|\mathbf{g}_k\| \le \varepsilon \|\mathbf{g}_0\|$, stop; otherwise goto Step 3.

8. Numerical experiments. In this section we show some numerical experiments on heat flux reconstructions using the two regularization methods (5.8)–(5.10) and (6.1)–(6.2). The physical domain Ω is taken to be $\Omega = \{(x, y);\ (\frac{1}{2})^2 \le x^2 + y^2 \le 1\}$. The domain Ω is triangulated as in Figure 2 using sectorial elements, with each

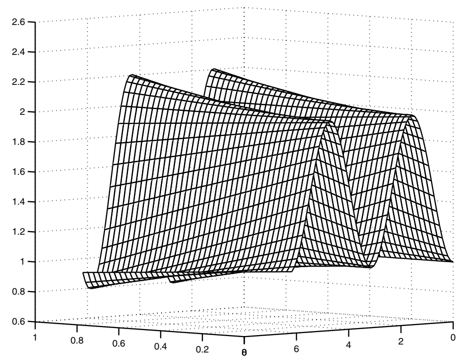

FIG. 3. *Exact* q.

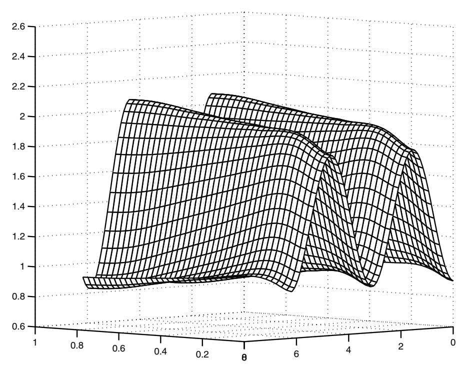

FIG. 4. *Method* (6.1) *used,* $\beta = 7 \times 10^{-6}$*, iter* $= 21$*, error* $= 4.19 \times 10^{-2}$.

circle divided into 60 arcs of equal length. The time interval $[0, 1]$ is divided into 40 equally spaced subintervals. For the conjugate gradient method, we take the tolerance $\epsilon = 10^{-4}$, and the initial guess $q_{h,\tau}^{(0)}$ of the heat flux is taken to be a constant zero everywhere in the whole space-time domain. In all three-dimensional figures shown below, the x-axis stands for the time interval varying from 0 to 1 and the y-axis stands for the inner boundary $\Gamma_i = \{(x, y);\ x^2 + y^2 = (\frac{1}{2})^2\}$ represented by the polar coordinate θ varying from 0 to 2π, while the z-axis shows the magnitude of the heat flux at each point (t, θ). The errors listed under each figure are the relative L^2-norm errors between the exact and numerically reconstructed heat fluxes.

In our simulations, the coefficients α, c, and u_a in (1.1) and (1.3) are taken to be $\alpha(x, t) = 1, c(x, t) = 1$, and $u_a(x, t) = 0$. In order to select more general and difficult profiles of heat fluxes for our tests, we add a source term $f(x, t)$ in (1.1). As our first example, we try the exact solution $u(x, y, t)$ and the heat flux $q(x, y, t)$ to be reconstructed as follows:

$$u(x, y, t) = x^2 + 2y^2 + t + \sin(xyt), \quad q(x, y, t) = 4x^2 + 8y^2 + 4xyt\cos(xyt).$$

Instead of the exact data $u(x, y, t)$, we use the perturbed data of the form $z(x, y, t) = u(x, y, t) + \delta\, u(x, y, t)$ as the measurement data, with the noise level $\delta = 1\%$ (1% relative noise pointwise). We first test the case when the measurement region is taken to be $\omega = \{(x, y);\ (\frac{3}{4})^2 \le x^2 + y^2 \le 1\}$. Figure 3 plots the exact heat flux, while Figure 4 shows the numerically reconstructed heat flux using the finite element method

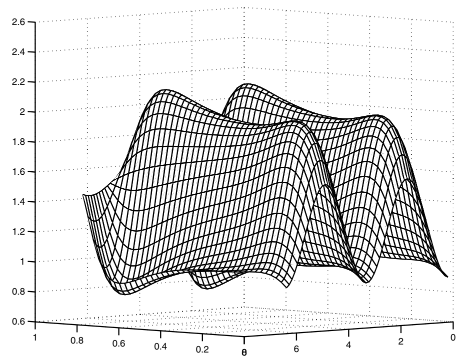

FIG. 5. *Method* (5.8) *used,* $\beta = 2 \times 10^{-6}$*, iter* $= 5$*, error* $= 7.73 \times 10^{-2}$.

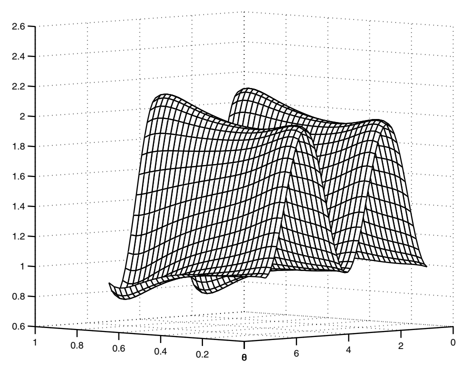

FIG. 6. *Method* (5.8) *used, plots from the initial* 4 *and last* 5 *time points removed, error* $= 3.09 \times 10^{-2}$.

(6.1) with L^2-regularization in space but H^1-regularization in time for heat fluxes. From Figure 4 we see that the numerical reconstruction works very well, considering the difficult oscillation of the heat flux in space. Also the conjugate gradient iteration performs very stably for such an oscillating heat flux, starting with a very bad initial guess of constant zero everywhere in the space-time domain. Figure 5 presents the numerical reconstruction using the finite element method (5.8) with L^2-regularization in both space and time for heat fluxes. One finds that the quality of reconstruction is far from satisfactory compared to the result we have seen in Figure 4 using the finite element method (6.1); the reconstruction is especially bad near the initial and terminal time. But interestingly, when we remove the bad reconstruction at a few initial and terminal time points, the remaining reconstruction seems very satisfactory again; see Figure 6.

We have also tried to see the effects of the measurement region. When the measurement region is reduced to a smaller subdomain $\omega = \{(x, y);\ (\frac{4}{5})^2 \le x^2 + y^2 \le 1\}$, the numerical reconstructions are not affected too much; see Figures 7 and 8.

As our second example, we take the exact solutions $u(x, y, t)$ and $q(x, y, t)$ in (1.1) and (1.4) as the following functions:

$$u(x, y, t) = \sin \pi t (x \cos \pi y + y \sin \pi x),$$
$$q(x, y, t) = 2 \sin \pi t (\pi x y (\cos \pi x - \sin \pi y) + x \cos \pi y + y \sin \pi x).$$

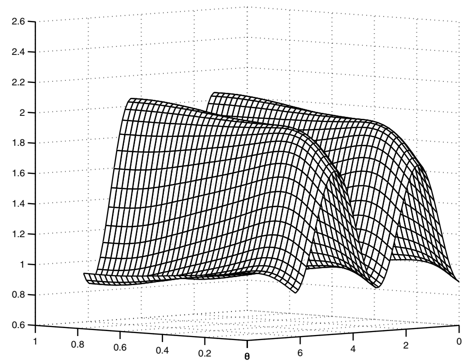

FIG. 7. *Method* (6.1) *used,* $\beta = 10^{-5}$*, iter* $= 23$*, error* $= 5.25 \times 10^{-2}$.

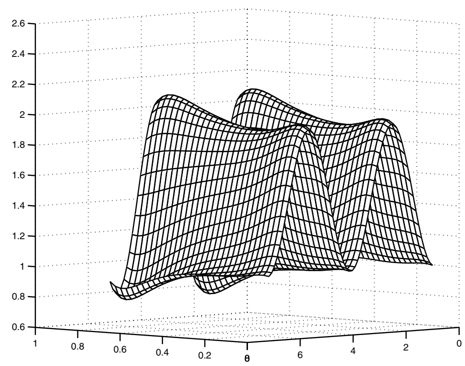

FIG. 8. *Method* (5.8) *used,* $\beta = 6 \times 10^{-6}$*, iter* $= 13$*, plots from the initial* 4 *and last* 5 *time points removed, error* $= 3.57 \times 10^{-2}$.

Again, the perturbed data $z(x, y, t) = u(x, y, t) + \delta\, u(x, y, t)$, with 1% noise pointwise, is taken to be the measurement data in ω. We first test the case when the measurement region is taken to be $\omega = \{(x, y);\, (\frac{3}{4})^2 \le x^2 + y^2 \le 1\}$. Figure 9 plots the exact heat flux q, which appears to be very challenging for numerical reconstruction as it oscillates widely in both time and space direction. Figure 10 shows the numerically reconstructed heat flux using the finite element method (6.1) with L^2-regularization in space but H^1-regularization in time for heat fluxes. This demonstrates very satisfactory performance of the numerical reconstruction algorithm, especially the stability and effectiveness of the conjugate gradient iteration, considering that it is such an oscillating heat flux and that it starts with a very bad initial guess of constant zero everywhere in the space-time domain. Figure 11 presents the numerical reconstruction using the finite element method (5.8) with L^2-regularization in both space and time for heat fluxes. Again its quality of reconstruction is not as good as the one obtained using the finite element method (6.1), and the accuracy is much worse.

When the measurement subregion is reduced to a smaller subdomain $\omega = \{(x, y);\, (\frac{4}{5})^2 \le x^2 + y^2 \le 1\}$, again the numerical reconstructions have not been affected much, as we have seen in the first example.

9. Concluding remarks. The inverse problem of reconstructing profiles of both time- and space-dependent heat fluxes on an inner boundary of a heat conductive

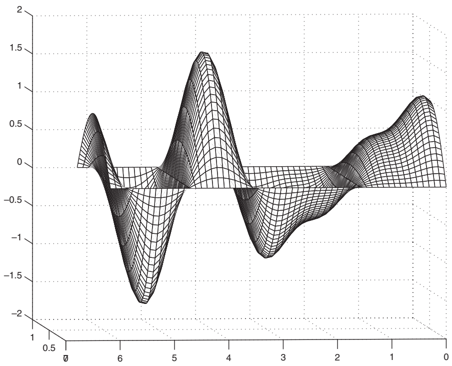

FIG. 9. *The exact* q.

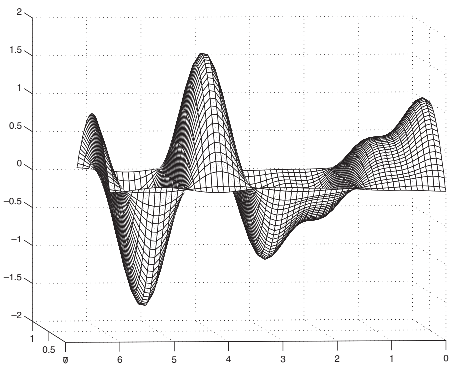

FIG. 10. *Method* (6.1) *used,* $\beta = 10^{-9}$, $iter = 30$, $error = 3.68 \times 10^{-2}$.

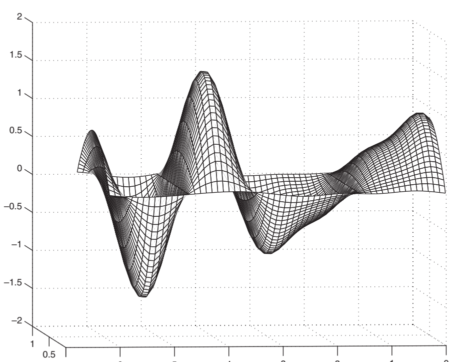

FIG. 11. *Method* (5.8) *used,* $\beta = 4 \times 10^{-8}$, $iter = 29$, $error = 11.59 \times 10^{-2}$.

system is investigated. The reconstruction problem is severely ill-posed as it involves the heat flux profile at the initial time and on the inner boundary. Validation and effectiveness of two regularization formulations are justified both theoretically and numerically for the reconstruction, without any constraints enforced on the search spaces of heat fluxes when appropriate regularizations are selected. Regarding the approximation of the regularized nonlinear minimization systems, it is very tricky and essential to decide how to effectively discretize in both time and space the nonlinear optimizations and the associated parabolic equation and its adjoint so that the resulting fully discrete schemes converge. Two such discrete approaches are proposed to approximate two nonlinear minimization formulations: the first uses the backward Euler scheme in time, while the second requires the Crank–Nicolson scheme, with both adopting piecewise linear finite elements for space approximation and the trapezoidal and midpoint rules for discretization of the cost functionals. A novel weighted discrete projection operator Q_h is introduced which possesses both L^2- and H^1-stability and L^2-optimal error estimate, crucial to the success of convergence analysis of two fully discrete schemes. The resulting nonlinear finite element minimization systems are shown to be well suited for the solutions by conjugate gradient method. Numerical experiments have demonstrated the stability and effectiveness of the reconstruction algorithms.

There exists little work on numerical reconstruction of both time- and space-dependent physical profiles, and even less on convergence analysis of numerical reconstruction methods. As we have seen, the convergence analyses of the fully discrete schemes are much more difficult and trickier than the cases with only space-dependent profiles. This paper provides a relatively complete study on reconstruction of both time- and space-dependent heat fluxes, including well-posedness of the regularized systems, convergence of fully discrete approximations, numerical algorithms for solving discrete nonlinear minimizations, and numerical experiments. Most technical tools should be useful in theoretical and numerical analysis of regularization methods for other inverse problems.

Acknowledgment. The authors wish to thank an anonymous referee for his constructive comments.

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