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FINITE ELEMENT METHODS WITH MATCHING AND NONMATCHING MESHES FOR MAXWELL EQUATIONS WITH DISCONTINUOUS COEFFICIENTS*

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Abstract. We investigate the finite element methods for solving time-dependent Maxwell equations with discontinuous coefficients in general three-dimensional Lipschitz polyhedral domains. Both matching and nonmatching finite element meshes on the interfaces are considered, and optimal error estimates for both cases are obtained. The analysis of the latter case is based on an abstract framework for nested saddle point problems, along with a characterization of the trace space for $H(\mathbf{curl}\,; D)$, a new extension theorem for $H(\mathbf{curl}\,; D)$ functions in any Lipschitz domain D, and a novel compactness argument for deriving discrete inf-sup conditions.

Key words. finite element method, Maxwell equations, interface problems, nonmatching meshes, error estimates, saddle point formulation, extension theorem

AMS subject classifications. 65M60, 65N30, 35Q60, 46E35

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1. Introduction.

This paper is concerned with the following Maxwell equations in a dielectric medium:

$$\varepsilon\,\partial_t\mathbf{E} - \mathbf{curl}\,\mathbf{H} = \mathbf{J} \quad \text{in} \quad \Omega\times(0,T), \tag{1.1}$$

$$\mu\,\partial_t\mathbf{H} + \mathbf{curl}\,\mathbf{E} = 0 \quad \text{in} \quad \Omega\times(0,T), \tag{1.2}$$

$$\operatorname{div}(\varepsilon\,\mathbf{E}) = \rho \quad \text{in} \quad \Omega\times(0,T), \tag{1.3}$$

$$\operatorname{div}(\mu\,\mathbf{H}) = 0 \quad \text{in} \quad \Omega\times(0,T). \tag{1.4}$$

Here $\Omega \subset R^3$ is a simply-connected Lipschitz polyhedral domain with connected boundary which is occupied by the dielectric material. $\mathbf{E}$ and $\mathbf{H}$ are the electric and magnetic fields, and $\mathbf{J}$ and ρ are the current density and charge density. We assume that the permeability parameter μ and the permittivity parameter ε of the medium are discontinuous across an interface $\Gamma \subset \Omega$, where Γ is the boundary of a simply-connected Lipschitz polyhedral domain Ω_1 with $\bar\Omega_1 \subset \Omega$ and $\Omega_2 = \Omega \setminus \bar\Omega_1$. Ω_2 is also assumed to be simply-connected which, in turn, implies that Γ is connected. Without loss of generality we consider only the case with ε and μ being two piecewise constant functions in the domain Ω, namely,

$$\varepsilon = \begin{cases} \varepsilon_1 & \text{in} \quad \Omega_1, \\ \varepsilon_2 & \text{in} \quad \Omega_2, \end{cases} \qquad \mu = \begin{cases} \mu_1 & \text{in} \quad \Omega_1, \\ \mu_2 & \text{in} \quad \Omega_2, \end{cases}$$

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and ε_i, μ_i ($i = 1, 2$) are positive constants. It is known that the electric and magnetic fields $\mathbf{E}$ and $\mathbf{H}$ must satisfy the following jump conditions across the interface Γ :

$$[\mathbf{E} \times \mathbf{n}] = 0, \qquad [\varepsilon\, \mathbf{E} \cdot \mathbf{n}] = \rho_{\Gamma} \ , \tag{1.5}$$

$$[\mathbf{H} \times \mathbf{n}] = \mathbf{J}_{\Gamma}, \qquad [\mu\, \mathbf{H} \cdot \mathbf{n}] = 0 \ , \tag{1.6}$$

where $\mathbf{n}$ is the unit outward normal to $\partial\Omega_1$ and ρ_{Γ} and $\mathbf{J}_{\Gamma}$ are the surface charge and current density. Throughout the paper, the jump of any function A across the interface Γ is defined as

$$[A] := A_2|_{\Gamma} - A_1|_{\Gamma}$$

with $A_i = A|_{\Omega_i}$, $i = 1, 2$. We supplement Maxwell equations (1.1)–(1.6) with the initial conditions

$$\mathbf{E}(\mathbf{x}, 0) = \mathbf{E}_0(\mathbf{x}), \qquad \mathbf{H}(\mathbf{x}, 0) = \mathbf{H}_0(\mathbf{x}), \qquad \mathbf{x} \in \Omega \ .$$

Instead of solving the fully coupled Maxwell system (1.1)–(1.6), we are interested in finding the electric and magnetic fields separately. To do so, we first eliminate the magnetic field $\mathbf{H}$ in (1.1)–(1.2) to obtain the electric field equations,

$$\varepsilon\, \partial_{tt}\mathbf{E} + \mathbf{curl}\,(\mu^{-1}\mathbf{curl}\,\mathbf{E}) = \partial_t\mathbf{J} \quad \text{in} \quad \Omega \times (0, T), \tag{1.7}$$

$$\operatorname{div}\,(\varepsilon\, \mathbf{E}) = \rho \quad \text{in} \quad \Omega \times (0, T), \tag{1.8}$$

with the following interface and boundary conditions:

$$[\mathbf{E} \times \mathbf{n}] = 0, \qquad [\varepsilon\mathbf{E} \cdot \mathbf{n}] = \rho_{\Gamma}, \qquad [\mu^{-1} \operatorname{curl} \mathbf{E} \times \mathbf{n}] = -\partial_t \mathbf{J}_{\Gamma}, \tag{1.9}$$

$$\mathbf{E} \times \mathbf{n} = 0 \quad \text{on} \quad \partial\Omega \times (0, T). \tag{1.10}$$

Similarly, we obtain the magnetic field equations from (1.1)–(1.6),

$$\mu\, \partial_{tt}\mathbf{H} + \mathbf{curl}\,(\varepsilon^{-1}\mathbf{curl}\,\mathbf{H}) = -\,\mathbf{curl}\,(\varepsilon^{-1}\mathbf{J}) \quad \text{in} \quad \Omega \times (0, T)\ , \tag{1.11}$$

$$\operatorname{div}\,(\mu\, \mathbf{H}) = 0 \quad \text{in} \quad \Omega \times (0, T)\ , \tag{1.12}$$

with the following interface and boundary conditions:

$$[\mathbf{H} \times \mathbf{n}] = \mathbf{J}_{\Gamma}, \qquad [\mu\, \mathbf{H} \cdot \mathbf{n}] = 0, \qquad [\varepsilon^{-1}\mathbf{curl}\,\mathbf{H} \times \mathbf{n}] = -[\varepsilon^{-1}\mathbf{J} \times \mathbf{n}]\ , \tag{1.13}$$

$$\mathbf{curl}\,\mathbf{H} \times \mathbf{n} = -\mathbf{J} \times \mathbf{n} \quad \text{on} \quad \partial\Omega \times (0, T). \tag{1.14}$$

Due to the practical interests, there has been a great deal of work on the numerical approximation to time-dependent Maxwell equations and also on the convergence analysis of numerical schemes for stationary Maxwell equations and related models; see, for example, [1, 3, 10, 13, 18, 20, 25] and the references therein. On the contrary, not too much work was available in the literature concerning the convergence analysis or error estimates for the fully discrete numerical methods for the time-dependent Maxwell systems. For some recent work in this aspect, we refer readers to [12, 19] and [24] for time-dependent Maxwell systems with continuous coefficients.

In this paper we will study a fully discrete finite element method for the time-dependent Maxwell equations with discontinuous coefficients. There have been numerous studies on the use of finite element methods for elliptic equations having discontinuous coefficients (see, for example, [5, 11, 14, 7]). For the study presented here, we will concentrate on the electric field equations (1.7)–(1.10). There is no essential difficulty for the extension of our numerical analysis here to the case for magnetic field equation (1.11)–(1.14). We will discretize the system (1.7)–(1.10) using an implicit scheme in time and a Nédélec's edge element scheme in space (cf. [21, 22]). In particular, we will investigate finite element methods with both matching and nonmatching grids on the interface Γ. As we will see later, the convergence analysis on the nonmatching case seems to be much more technical than the matching case. For example, a major obstacle is how to find some appropriate finite element spaces which satisfy suitable inf-sup conditions.

The paper is organized as follows. In section 2, we study the stationary model elliptic problem,

$$\mathbf{curl}\,(\alpha\,\mathbf{curl}\,\mathbf{A})+\gamma\beta\mathbf{A}=\mathbf{f}\quad\text{in}\quad\Omega, \tag{1.15}$$

$$\text{div}\,(\beta\,\mathbf{A})=g\quad\text{in}\quad\Omega, \tag{1.16}$$

with the following interface conditions and boundary conditions:

$$[\mathbf{A}\times\mathbf{n}]=0,\qquad[\beta\,\mathbf{A}\cdot\mathbf{n}]=g_{\Gamma},\qquad[\alpha\,\mathbf{curl}\,\mathbf{A}\times\mathbf{n}]=\mathbf{f}_{\Gamma}\quad\text{on}\quad\Gamma, \tag{1.17}$$

$$\mathbf{A}\times\mathbf{n}=0\qquad\text{on}\quad\partial\Omega. \tag{1.18}$$

Here α and β are piecewise constants in Ω; i.e., $\alpha=\alpha_i$, $\beta=\beta_i$ in Ω_i for $i=1,2$, and α_i, β_i are positive constants. For technical reasons, γ is a chosen constant which may be zero in some cases (as in section 3) or strictly positive in some other cases (as in section 4). Such a lower-order term is required to ensure that the corresponding bilinear form induces a full $H(\mathbf{curl}\,;\Omega)$-norm in the latter cases. Finite element methods with matching and nonmatching meshes on the interface will be considered in sections 3 and 4. The results for the above stationary system (1.15)–(1.16) are needed in the convergence analysis of finite element methods for the time-dependent equations (1.7)–(1.10), which will be discussed in section 5. A number of interesting theoretical results concerning the weak formulation and the finite element approximation will be stated there while some technical proofs for an extension theorem and an abstract framework for the convergence analysis will be provided in an appendix at the end of the paper. A few conclusion remarks will be furnished in section 6.

2. Stationary model problem. For the convenience of presentation, we first give some notation that will be used throughout the paper:

$$\begin{aligned}
H(\mathbf{curl}\,;\Omega)&=\{\mathbf{v}\in L^2(\Omega)^3;\ \mathbf{curl}\,\mathbf{v}\in L^2(\Omega)^3\}\,,\\
H^{\alpha}(\mathbf{curl}\,;\Omega)&=\{\mathbf{v}\in H^{\alpha}(\Omega)^3;\ \mathbf{curl}\,\mathbf{v}\in H^{\alpha}(\Omega)^3\}\qquad(\,\alpha>0\,)\,,\\
H_0(\mathbf{curl}\,;\Omega)&=\{\mathbf{v}\in H(\mathbf{curl}\,;\Omega);\ \mathbf{v}\times\mathbf{n}=0\text{ on }\partial\Omega\}\,.
\end{aligned}$$

The spaces $H(\mathbf{curl}\,;\Omega)$ and $H^{\alpha}(\mathbf{curl}\,;\Omega)$ are equipped with the norms

$$\begin{aligned}
||\mathbf{v}||_{\mathbf{curl}\,,\Omega}&=\Big(||\mathbf{v}||^2_{0,\Omega}+||\mathbf{curl}\,\mathbf{v}||^2_{0,\Omega}\Big)^{1/2},\\
||\mathbf{v}||_{\alpha,\mathbf{curl}\,,\Omega}&=\Big(||\mathbf{v}||^2_{\alpha,\Omega}+||\mathbf{curl}\,\mathbf{v}||^2_{\alpha,\Omega}\Big)^{1/2}\,.
\end{aligned}$$

Here and in what follows, $\|\cdot\|_{0,\Omega}$ denotes the $L^2(\Omega)^3$-norm (or the $L^2(\Omega)$-norm for scalar functions) and $\|\cdot\|_{\alpha,\Omega}$ and $|\cdot|_{\alpha,\Omega}$ denote the norm and the seminorm of the Sobolev space $H^{\alpha}(\Omega)^3$ (or $H^{\alpha}(\Omega)$ for scalar functions). Similar definitions are adopted for Ω_1 and Ω_2. If no confusion is caused, we will often omit Ω, Ω_1, and Ω_2 in the subscripts of the norms. The constant C will always represent a generic constant independent of the time step τ and the mesh size h, unless otherwise specified.

We now discuss the variational formulation and the finite element approximation for the stationary model equations (1.15)–(1.18). As usual, we introduce a Lagrangian multiplier p to relax the divergence condition (1.16); hence, we consider the following system:

$$\mathbf{curl}\,(\alpha\,\mathbf{curl}\,\mathbf{A}) + \gamma\beta\mathbf{A} - \beta\,\nabla p = \mathbf{f} \quad \text{in} \quad \Omega, \tag{2.1}$$

$$\operatorname{div}(\beta\mathbf{A}) = g \quad \text{in} \quad \Omega, \tag{2.2}$$

$$[\mathbf{A}\times\mathbf{n}] = 0, \quad [\beta\,\mathbf{A}\cdot\mathbf{n}] = g_\Gamma \quad \text{on} \quad \Gamma, \tag{2.3}$$

$$[p] = 0, \quad [\alpha\,\mathbf{curl}\,\mathbf{A}\times\mathbf{n}] = \mathbf{f}_\Gamma \quad \text{on} \quad \Gamma, \tag{2.4}$$

$$p = 0, \quad \mathbf{A}\times\mathbf{n} = 0 \quad \text{on} \quad \partial\Omega. \tag{2.5}$$

Here γ is a constant which may be taken to be zero unless otherwise stated. Note that if the source terms $\mathbf{f}$ and g satisfy $\operatorname{div}\mathbf{f} = \gamma g$, then we know the solution p of the above system is equal to zero by taking the divergence of (2.1) and using the boundary condition $p = 0$ on $\partial\Omega$. This justifies the introduction of the Lagrangian multiplier p.

To establish an appropriate variational formulation for the system (2.1)–(2.5), we need a new trace space of $H_0(\mathbf{curl}\,;\Omega)$ on the interface Γ. We next introduce this trace space and present some of its properties for the later use.

We know that any $\mathbf{v}\in H_0(\mathbf{curl}\,;\Omega)$ has a tangential trace $\mathbf{v}\times\mathbf{n}$ in $H^{-1/2}(\Gamma)^3$, defined by

$$\langle \mathbf{v}\times\mathbf{n}, \boldsymbol{\varphi}\rangle_\Gamma \;=\; \int_{\Omega_1} \mathbf{v}\cdot\mathbf{curl}\,\boldsymbol{\varphi}\;dx - \int_{\Omega_1} \mathbf{curl}\,\mathbf{v}\cdot\boldsymbol{\varphi}dx \quad \forall\,\boldsymbol{\varphi}\in H^1(\Omega_1)^3 \tag{2.6}$$

or

$$\langle \mathbf{v}\times\mathbf{n}, \boldsymbol{\varphi}\rangle_\Gamma = \int_{\Omega_2} \mathbf{curl}\,\mathbf{v}\cdot\boldsymbol{\varphi}dx - \int_{\Omega_2} \mathbf{v}\cdot\mathbf{curl}\,\boldsymbol{\varphi}dx \quad \forall\boldsymbol{\varphi}\in H^1(\Omega_2)^3\cap H_0(\mathbf{curl}\,;\Omega_2),$$

where $\langle\cdot,\cdot\rangle_\Gamma$ denotes the duality pairing between $H^{-1/2}(\Gamma)^3$ and $H^{1/2}(\Gamma)^3$ (or the duality pairing between $H^{-1/2}(\Gamma)$ and $H^{1/2}(\Gamma)$ for scalar functions) and

$$H_0(\mathbf{curl}\,;\Omega_2) = \{\mathbf{v}\in H(\mathbf{curl}\,;\Omega_2)\,;\ \ \mathbf{v}\times\mathbf{n} = 0 \text{ on } \partial\Omega\}.$$

However, this characterization of the trace $\mathbf{v}\times\mathbf{n}$ is rather inconvenient in applications since this trace mapping from $H_0(\mathbf{curl}\,;\Omega)$ to $H^{-1/2}(\Gamma)^3$ is not onto. To overcome this difficulty, we introduce the following trace space on Γ for functions in $H_0(\mathbf{curl}\,;\Omega)$:

$$T(\Gamma) = \{\mathbf{s}\in H^{-1/2}(\Gamma)^3;\ \exists\,\mathbf{v}\in H_0(\mathbf{curl}\,;\Omega) \text{ such that } \ \mathbf{v}\times\mathbf{n} = \mathbf{s} \text{ on } \Gamma\}.$$

It is not difficult to see that $T(\Gamma)$ is a Banach space under the norm:

$$\|\mathbf{s}\|_{T(\Gamma)} = \inf\{\|\mathbf{v}\|_{\mathbf{curl}\,,\Omega};\mathbf{v}\in H_0(\mathbf{curl}\,;\Omega) \ \text{ and } \ \mathbf{v}\times\mathbf{n} = \mathbf{s} \text{ on } \Gamma\}. \tag{2.7}$$

We note that for domains with smooth boundary, $T(\Gamma)$ is well known, and it is often denoted by $H^{-1/2}(\operatorname{div},\Gamma)$ (cf. [8]). For domains with a Lipschitz boundary, we next provide a new characterization of this space.

For convenience, we denote $X_1 = H(\mathbf{curl}\,;\Omega_1)$ and $X_2 = H_0(\mathbf{curl}\,;\Omega_2)$. Note that the left-hand side of (2.6) does not make sense if $\boldsymbol{\varphi}$ belongs to X_1, but the right-hand side of (2.6) is well-defined for all $\mathbf{v}$ and $\boldsymbol{\varphi}$ in X_1; thus, for any $\mathbf{s} \in T(\Gamma)$, instead of (2.6) we can define

$$\langle\!\langle \mathbf{s}, \boldsymbol{\varphi} \rangle\!\rangle_{1,\Gamma} = \int_{\Omega_1} \mathbf{v} \cdot \mathbf{curl}\,\boldsymbol{\varphi}\, dx - \int_{\Omega_1} \mathbf{curl}\,\mathbf{v} \cdot \boldsymbol{\varphi}\, dx \quad \forall\, \boldsymbol{\varphi} \in X_1, \tag{2.8}$$

where $\mathbf{v} \in H_0(\mathbf{curl}\,,\Omega)$ such that $\mathbf{v} \times \mathbf{n} = \mathbf{s}$ on Γ. Using the Green's formula (2.6) and the density of $H^1(\Omega_1)$ in X_1, we know that (2.8) is independent of the choice of $\mathbf{v} \in H(\mathbf{curl}\,;\Omega)$ such that $\mathbf{v} \times \mathbf{n} = \mathbf{s}$ on Γ. Thus (2.8) is well-defined. Similarly, we can define

$$\langle\!\langle \mathbf{s}, \boldsymbol{\varphi} \rangle\!\rangle_{2,\Gamma} = \int_{\Omega_2} \mathbf{curl}\,\mathbf{v} \cdot \boldsymbol{\varphi} dx - \int_{\Omega_2} \mathbf{v} \cdot \mathbf{curl}\,\boldsymbol{\varphi} dx \quad \forall \boldsymbol{\varphi} \in X_2, \tag{2.9}$$

where $\mathbf{v} \in H_0(\mathbf{curl}\,;\Omega)$ such that $\mathbf{s} = \mathbf{v} \times \mathbf{n}$ on Γ. It is clear from the definitions (2.7)–(2.9) that

$$\langle\!\langle \mathbf{s}, \boldsymbol{\varphi} \rangle\!\rangle_{1,\Gamma} - \langle\!\langle \mathbf{s}, \boldsymbol{\varphi} \rangle\!\rangle_{2,\Gamma} \le \|\mathbf{s}\|_{T(\Gamma)} \|\boldsymbol{\varphi}\|_{X_1 \times X_2} \quad \forall\, \boldsymbol{\varphi} \in X_1 \times X_2, \tag{2.10}$$

which implies that $\langle\!\langle \mathbf{s}, \cdot \rangle\!\rangle_{1,\Gamma} - \langle\!\langle \mathbf{s}, \cdot \rangle\!\rangle_{2,\Gamma}$ defines a continuous linear functional in $X_1 \times X_2$. Hence by Riesz representation theorem, there exists a $\mathbf{Q} \in X_1 \times X_2$ satisfying

$$\int_{\Omega_1 \cup \Omega_2} (\mathbf{curl}\,\mathbf{Q} \cdot \mathbf{curl}\,\boldsymbol{\varphi} + \mathbf{Q} \cdot \boldsymbol{\varphi})\, dx = \langle\!\langle \mathbf{s}, \boldsymbol{\varphi} \rangle\!\rangle_{1,\Gamma} - \langle\!\langle \mathbf{s}, \boldsymbol{\varphi} \rangle\!\rangle_{2,\Gamma} \quad \forall\, \boldsymbol{\varphi} \in X_1 \times X_2 \tag{2.11}$$

and the following identity holds:

$$\|\mathbf{Q}\|_{X_1 \times X_2} = \sup_{\boldsymbol{\varphi} \in X_1 \times X_2} \frac{\langle\!\langle \mathbf{s}, \boldsymbol{\varphi} \rangle\!\rangle_{1,\Gamma} - \langle\!\langle \mathbf{s}, \boldsymbol{\varphi} \rangle\!\rangle_{2,\Gamma}}{\|\boldsymbol{\varphi}\|_{X_1 \times X_2}}. \tag{2.12}$$

Now applying the Green's formula (2.6) to (2.11) with test functions $\boldsymbol{\varphi} \in H_0^1(\Omega_i)^3$ and $\boldsymbol{\varphi} \in H^1(\Omega_i)^3$, respectively, we obtain

$$\begin{aligned} \mathbf{curl}\,\mathbf{curl}\,\mathbf{Q} + \mathbf{Q} &= 0 \quad \text{in} \quad \Omega_1 \cup \Omega_2, \\ \mathbf{curl}\,\mathbf{Q} \times \mathbf{n} &= 0 \quad \text{on} \quad \partial\Omega, \\ \mathbf{curl}\,\mathbf{Q} \times \mathbf{n} &= \mathbf{s} \quad \text{on} \quad \Gamma \end{aligned}$$

where the last two relations are understood in the sense of $H^{-1/2}(\partial\Omega)^3$ and $H^{-1/2}(\Gamma)^3$, respectively. This yields $\mathbf{curl}\,\mathbf{Q} \in H_0(\mathbf{curl}\,;\Omega)$ and

$$\|\mathbf{s}\|_{T(\Gamma)} \le \|\mathbf{curl}\,\mathbf{Q}\|_{\mathbf{curl}\,,\Omega} = \|\mathbf{Q}\|_{X_1 \times X_2}. \tag{2.13}$$

Combining (2.10)–(2.13) we have proved the following lemma.

LEMMA 2.1. *For any* $\mathbf{s} \in T(\Gamma)$, *we have the equality*

$$\|\mathbf{s}\|_{T(\Gamma)} = \sup_{\boldsymbol{\varphi} \in X_1 \times X_2} \frac{\langle\!\langle \mathbf{s}, \boldsymbol{\varphi} \rangle\!\rangle_{1,\Gamma} - \langle\!\langle \mathbf{s}, \boldsymbol{\varphi} \rangle\!\rangle_{2,\Gamma}}{\| \boldsymbol{\varphi} \|_{X_1 \times X_2}}\ .$$

A direct consequence of this lemma is that $T(\Gamma)$ is indeed a Hilbert space. In fact, if

$\mathbf{s}_i \in T(\Gamma)$ and $\mathbf{Q}_i \in X_1 \times X_2$ $(i = 1, 2)$ is the corresponding solution of (2.11), then we can define an inner product $(\!(\cdot,\cdot)\!)_\Gamma$ as follows:

$$(\!(\mathbf{s}_1, \mathbf{s}_2)\!)_\Gamma = \int_{\Omega_1 \cup \Omega_2} (\mathbf{curl}\, \mathbf{Q}_1 \cdot \mathbf{curl}\, \mathbf{Q}_2 + \mathbf{Q}_1 \cdot \mathbf{Q}_2) dx\,.$$

Clearly, $\|\,\mathbf{s}\,\|_{T(\Gamma)} = (\!(\mathbf{s}, \mathbf{s})\!)_\Gamma^{1/2}$.

In practice, Lemma 2.1 is rather inconvenient as it uses information from both Ω_1 and Ω_2 to define the norm on $T(\Gamma)$. To overcome the inconvenience, we note that

$$(2.14)\qquad |\!|\!|\mathbf{s}|\!|\!|_{1,\Gamma} = \sup_{\boldsymbol{\varphi} \in X_1} \frac{\langle\!\langle \mathbf{s}, \boldsymbol{\varphi} \rangle\!\rangle_{1,\Gamma}}{\|\,\boldsymbol{\varphi}\,\|_{X_1}}, \qquad |\!|\!|\mathbf{s}|\!|\!|_{2,\Gamma} = \sup_{\boldsymbol{\varphi} \in X_2} \frac{\langle\!\langle \mathbf{s}, \boldsymbol{\varphi} \rangle\!\rangle_{2,\Gamma}}{\|\,\boldsymbol{\varphi}\,\|_{X_2}}$$

are also norms of $T(\Gamma)$. It is clear that

$$|\!|\!|\mathbf{s}|\!|\!|_{1,\Gamma} \le \|\,\mathbf{s}\,\|_{T(\Gamma)}, \qquad |\!|\!|\mathbf{s}|\!|\!|_{2,\Gamma} \le \|\,\mathbf{s}\,\|_{T(\Gamma)} \quad \forall \mathbf{s} \in T(\Gamma).$$

Both $|\!|\!|\cdot|\!|\!|_{1,\Gamma}$ and $|\!|\!|\cdot|\!|\!|_{2,\Gamma}$ are, in fact, equivalent norms on $T(\Gamma)$ to $\|\cdot\|_{T(\Gamma)}$. To show this, we need the following extension theorem which may be of independent interest.

LEMMA 2.2 (extension theorem). *Assume U is a bounded domain in R^3 with a Lipschitz boundary ∂U. Let $U \subset\subset D$. Then there exists a bounded linear operator*

$$E : H(\mathbf{curl}\,; U) \to H(\mathbf{curl}\,; R^3)$$

such that $E\mathbf{v} = \mathbf{v}$ on U and $\mathrm{supp}(E\mathbf{v}) \subset D\ \forall\, \mathbf{v} \in H(\mathbf{curl}\,; U)$.

Because the proof of Lemma 2.2 has not been seen in the literature and the full proof of the result is somewhat involved, we will describe it in more detail in the appendix, section A1.

LEMMA 2.3. *The norms $|\!|\!|\cdot|\!|\!|_{1,\Gamma}$ and $|\!|\!|\cdot|\!|\!|_{2,\Gamma}$ are equivalent to $\|\cdot\|_{T(\Gamma)}$.*

Proof. We need only to prove that

$$|\!|\!|\mathbf{s}|\!|\!|_{1,\Gamma} \le C|\!|\!|\mathbf{s}|\!|\!|_{2,\Gamma} \quad \forall \mathbf{s} \in T(\Gamma).$$

For any $\boldsymbol{\varphi}_1 \in X_1$, we use Lemma 2.2 to get a function $\boldsymbol{\varphi} \in H_0(\mathbf{curl}\,; \Omega)$ such that $\boldsymbol{\varphi} = \boldsymbol{\varphi}_1$ on Ω_1 and

$$\|\,\boldsymbol{\varphi}\,\|_{X_2} \le \|\,\boldsymbol{\varphi}\,\|_{\mathbf{curl}\,;\Omega} \le C^* \|\,\boldsymbol{\varphi}_1\,\|_{X_1}.$$

Note that $\langle\!\langle \mathbf{s}, \boldsymbol{\varphi} \rangle\!\rangle_{1,\Gamma} = \langle\!\langle \mathbf{s}, \boldsymbol{\varphi} \rangle\!\rangle_{2,\Gamma}$ for any $\boldsymbol{\varphi} \in H_0(\mathbf{curl}\,; \Omega)$ by the Green's formula. Thus, we have

$$\begin{aligned} |\!|\!|\mathbf{s}|\!|\!|_{1,\Gamma} &= \sup_{\boldsymbol{\varphi}_1 \in X_1} \frac{\langle\!\langle \mathbf{s}, \boldsymbol{\varphi}_1 \rangle\!\rangle_{1,\Gamma}}{\|\,\boldsymbol{\varphi}_1\,\|_{X_1}} = \sup_{\boldsymbol{\varphi}_1 \in X_1} \frac{\langle\!\langle \mathbf{s}, \boldsymbol{\varphi} \rangle\!\rangle_{2,\Gamma}}{\|\,\boldsymbol{\varphi}_1\,\|_{X_1}} \\ &\le C^* \sup_{\boldsymbol{\varphi} \in H_0(\mathbf{curl}\,;\Omega)} \frac{\langle\!\langle \mathbf{s}, \boldsymbol{\varphi} \rangle\!\rangle_{2,\Gamma}}{\|\,\boldsymbol{\varphi}\,\|_{\mathbf{curl}\,;\Omega}} \le C^* \sup_{\boldsymbol{\varphi} \in X_2} \frac{\langle\!\langle \mathbf{s}, \boldsymbol{\varphi} \rangle\!\rangle_{2,\Gamma}}{\|\,\boldsymbol{\varphi}\,\|_{X_2}} = |\!|\!|\mathbf{s}|\!|\!|_{2,\Gamma}. \end{aligned}$$

This completes the proof. ∎

3. Finite element methods with a matching grid. In this section we discuss the finite element method based on a matching finite element grid on the interface Γ; i.e., the restrictions from two finite element triangulations in Ω_1 and Ω_2 match with each other on Γ. So both triangulations from Ω_1 and Ω_2 are combined into a standard triangulation of the whole domain Ω. In this case, we set $\gamma = 0$ in the system (2.1)–(2.5) and we require that $\mathbf{A} \in H(\mathbf{curl}\,;\Omega)$ and $p \in H_0^1(\Omega)$ (conforming across the interface). This leads to the following weak formulation for the system (2.1)–(2.5) with $\gamma = 0$.

Find $(\mathbf{A}, p) \in H_0(\mathbf{curl}\,;\Omega) \times H_0^1(\Omega)$ such that

$$a(\mathbf{A},\mathbf{B}) + b(\mathbf{B},p) = \int_\Omega \mathbf{f}\cdot\mathbf{B}dx - \langle\!\langle \mathbf{f}_\Gamma, \mathbf{B}\rangle\!\rangle_{2,\Gamma} \quad \forall\, \mathbf{B} \in H_0(\mathbf{curl}\,;\Omega), \tag{3.1}$$

$$b(\mathbf{A},q) = \int_\Omega g\,q dx + \langle g_\Gamma, q\rangle_\Gamma \quad \forall\, q \in \mathbf{H}_0^1(\Omega), \tag{3.2}$$

where we assume $\mathbf{f} \in L^2(\Omega)^3$, $\mathbf{g} \in L^2(\Omega)$, $\mathbf{f}_\Gamma \in T(\Gamma)$ and $g_\Gamma \in H^{-1/2}(\Gamma)$. The two bilinear forms $a(\cdot,\cdot)$ and $b(\cdot,\cdot)$ are defined by

$$a(\mathbf{B},\mathbf{D}) = \int_\Omega \alpha\,\mathbf{curl}\,\mathbf{B}\cdot\mathbf{curl}\,\mathbf{D}dx \quad \forall\, \mathbf{B},\mathbf{D} \in X,$$
$$b(\mathbf{B},q) = -\int_\Omega \beta\,\mathbf{B}\cdot\nabla q dx \quad \forall\, \mathbf{B} \in X,\ q \in Q,$$

with $X = H_0(\mathbf{curl}\,;\Omega)$ and $Q = H_0^1(\Omega)$.

The following compactness result is due to Weber [28].

LEMMA 3.1. *Let β be a piecewise constant function in Ω and $\{\mathbf{E}_n\}$ be a sequence in $L^2(\Omega)^3$ satisfying*

$$\|\mathbf{E}_n\|_{0,\Omega} \le C, \qquad \|\mathbf{curl}\,\mathbf{E}_n\|_{0,\Omega} \le C, \qquad \|\mathrm{div}\,\beta\,\mathbf{E}_n\|_{0,\Omega} \le C, \qquad \mathbf{E}_n \times \mathbf{n}|_{\partial\Omega} = 0$$

for some constant C; then $\{\mathbf{E}_n\}$ has a convergent subsequence in $L^2(\Omega)^3$.

With the above lemma, we may get the following theorem.

THEOREM 3.2. *There exists a unique solution $(\mathbf{A}, p) \in X \times Q$ to* (3.1)–(3.2).

The proof is omitted as it following from a standard argument (see, for example, [21]) based on the Babuska–Brezzi theory.

We next discuss the finite element discretization of (3.1)–(3.2). We note that for the matching grid case, studies of finite element discretization for the Maxwell systems have been given in [2, 20]. The main error estimates given in this section are, in fact, similar to the existing ones in the literature. We include the discussion here for the sake of completeness, and some of the technical results are also useful for the later discussion on nonmatching grids.

Let $\mathcal{T}^h$ be a shape regular triangulation of Ω which matches with the interface Γ, i.e.,

$$\overset{\circ}{K} \cap \Gamma = \emptyset \quad \forall\, K \in \mathcal{T}^h.$$

Here $\overset{\circ}{K}$ is the interior of K. Let S_h be the standard continuous piecewise linear finite element space and V_h be the Nédélec $H(\mathbf{curl}\,,\Omega)$-conforming edge element space defined by

$$V_h = \{\mathbf{v}_h \in H(\mathbf{curl}\,;\Omega);\ \mathbf{v}_h = \mathbf{a}_K + \mathbf{b}_K \times \mathbf{x} \quad \text{on } K \quad \forall\, K \in \mathcal{T}^h\}\,, \tag{3.3}$$

where $\mathbf{a}_K$ and $\mathbf{b}_K$ are two constant vectors. It is known (cf. [21]) that any function $\mathbf{v}_h \in V_h$ is uniquely determined by the degrees of freedom in the set $M_E(\mathbf{v})$ of the moments on each element $K \in \mathcal{T}^h$, which is given by

$$M_E(\mathbf{v}) = \left\{ \int_e \mathbf{v}\cdot\tau\, ds\,; \quad e \text{ is an edge of } K \right\}.$$

Here τ is the unit vector along the edge e. For any $K \in \mathcal{T}^h$ and $p > 2$, we denote

$$Y_p(K) = \{\mathbf{v} \in L^p(K)^3; \mathbf{curl}\,\mathbf{v} \in L^p(K)^3;\ \text{and } \mathbf{v} \times \mathbf{n} \in L^p(\partial K)^3\}.$$

We know from [2, Lemma 4.7] that the integrals required in the definition of $M_E(\mathbf{v})$ make sense for any $\mathbf{v} \in Y_p(K)$. Thus for any $\mathbf{v} \in H^s(\Omega)^3$ with $\mathbf{curl}\,\mathbf{v} \in L^p(\Omega)^3$, where $s > 1/2$ and $p > 2$, we can define an interpolation $\pi_h \mathbf{v}$ such that $\pi_h \mathbf{v} \in V_h$, and $\pi_h \mathbf{v}$ has the same degrees of freedom as $\mathbf{v}$ on each K in $\mathcal{T}^h$.

Following the same argument as the one used in [12], we can prove the following interpolation error estimate.

LEMMA 3.3. *For $1/2 < s \le 1$, we have*

$$\|\mathbf{v} - \pi_h\,\mathbf{v}\|_{0,K} + \|\mathbf{curl}\,(\mathbf{v} - \pi_h\,\mathbf{v})\|_{0,K} \le C\,h_K^s \|\mathbf{v}\|_{s,\mathbf{curl},K} \quad \forall\,\mathbf{v} \in H^s(\mathbf{curl}\,; K),$$

where h_K is the diameter of $K \in \mathcal{T}^h$.

Furthermore, under the assumptions on the domain Ω given in section 1, the interpolation operator π_h has the following property [21].

LEMMA 3.4. *Let $\mathbf{v} = \nabla p$ with $p \in H_0^1(\Omega)$. Then, if $\mathbf{v}$ is regular enough to ensure the existence of $\pi_h \mathbf{v}$, we have $\pi_h \mathbf{v} = \nabla p_h$ for some $p_h \in S_h \cap H_0^1(\Omega)$.*

Now define two finite element subspaces of V_h and S_h:

$$X_h = V_h \cap H_0(\mathbf{curl}\,;\Omega), \qquad Q_h = S_h \cap H_0^1(\Omega).$$

Then the finite element approximation of (3.1)–(3.2) can be formulated as follows.

Find $(\mathbf{A}_h, p_h) \in X_h \times Q_h$ such that

$$a(\mathbf{A}_h, \mathbf{B}_h) + b(\mathbf{B}_h, p_h) = \int_\Omega \mathbf{f} \cdot \mathbf{B}_h dx - \langle\!\langle \mathbf{f}_\Gamma, \mathbf{B}_h \rangle\!\rangle_{2,\Gamma} \quad \forall\,\mathbf{B}_h \in X_h, \tag{3.4}$$

$$b(\mathbf{A}_h, q_h) = \int_\Omega g\, q_h dx + \langle g_\Gamma, q_h \rangle_\Gamma \quad \forall\, q_h \in Q_h. \tag{3.5}$$

The following theorem is the main result of this section.

THEOREM 3.5. *Assume the solution $(\mathbf{A}, p)$ of problems (3.1)–(3.2) has the following regularity: for some $1/2 < s \le 1$,*

$$\mathbf{A} \in H^s(\mathbf{curl}\,;\Omega_i), \qquad p \in H^{1+s}(\Omega_i), \qquad i = 1, 2.$$

Then there exists a unique solution $(\mathbf{A}_h, p_h) \in X_h \times Q_h$ to the discrete problem (3.4)–(3.5), and we have the error estimate

$$\|\mathbf{A} - \mathbf{A}_h\|_{\mathbf{curl},\Omega} + \|p - p_h\|_{1,\Omega} \le C\,h^s \sum_{i=1}^{2} \{\|\mathbf{A}\|_{s,\mathbf{curl},\Omega_i} + \|p\|_{1+s,\Omega_i}\}. \tag{3.6}$$

To show the theorem, we need the following embedding result proved in [2] for any polyhedral domains.

LEMMA 3.6. *Let $X_N(\Omega)$ and $X_T(\Omega)$ be the following subspaces of $L^2(\Omega)^3$:*

$$X_N(\Omega) = \{\mathbf{v} \in L^2(\Omega)^3;\ \mathbf{curl}\,\mathbf{v} \in L^2(\Omega)^3,\ \operatorname{div}\mathbf{v} \in L^2(\Omega),\ \mathbf{v} \times \mathbf{n} = 0 \quad \text{on} \quad \partial\Omega\},$$
$$X_T(\Omega) = \{\mathbf{v} \in L^2(\Omega)^3;\ \mathbf{curl}\,\mathbf{v} \in L^2(\Omega)^3,\ \operatorname{div}\mathbf{v} \in L^2(\Omega),\ \mathbf{v} \cdot \mathbf{n} = 0 \quad \text{on} \quad \partial\Omega\}.$$

If the domain Ω is a Lipschitz polyhedral domain in R^3, then there exists a real number $r > 1/2$ such that $X_T(\Omega)$ and $X_N(\Omega)$ are continuously embedded into $H^r(\Omega)^3$.

Proof of Theorem 3.5. Let $N_h = \{\mathbf{B}_h \in X_h;\ b(\mathbf{B}_h, q_h) = 0 \quad \forall\, q_h \in Q_h\}$. Then Theorem 3.5 follows immediately from Lemma 3.3, from the coercivity, and from the inf-sup condition below by applying the standard Babuska–Brezzi theory (for example, cf. [6]):

$$a(\mathbf{B}_h, \mathbf{B}_h) \geq C_0 \, \|\mathbf{B}_h\|^2_{\mathbf{curl};\ \Omega} \quad \forall\, \mathbf{B}_h \in N_h, \tag{3.7}$$

$$\sup_{\mathbf{B}_h \in X_h} \frac{b(\mathbf{B}_h, q_h)}{\|\mathbf{B}_h\|_{\mathbf{curl},\Omega}} \geq C_1 \|q_h\|_{1,\Omega} \quad \forall\, q_h \in Q_h\ . \tag{3.8}$$

The inf-sup condition (3.8) is a consequence of the Poincáre inequality by taking $\mathbf{B}_h = \nabla q_h \in X_h$ for any fixed $q_h \in Q_h$. The coercivity (3.7) was proved in [21] under the assumption that the finite element triangulation $\mathcal{T}^h$ is quasi-uniform. We now give a proof assuming only that $\mathcal{T}^h$ is shape regular. For any $\mathbf{B}_h \in N_h$, let $\boldsymbol{\varphi} \in H^1_0(\Omega)$ be the solution of the following problem:

$$\int_\Omega \nabla \boldsymbol{\varphi} \cdot \nabla v dx = \int_\Omega \mathbf{B}_h \cdot \nabla v dx \quad \forall\, v \in H^1_0(\Omega)\ .$$

Set $\mathbf{w} = \mathbf{B}_h - \nabla \boldsymbol{\varphi}$; then it is obvious to see

$$\mathbf{curl}\,\mathbf{w} = \mathbf{curl}\,\mathbf{B}_h, \qquad \operatorname{div} \mathbf{w} = 0 \quad \text{in } \Omega; \qquad \mathbf{w} \times \mathbf{n} = 0 \quad \text{on } \partial\Omega\ . \tag{3.9}$$

Hence $\mathbf{w} \in X_N(\Omega)$ and so $\mathbf{w} \in H^r(\Omega)$ for some $r > 1/2$ by Lemma 3.6. Now since $\mathbf{curl}\,\mathbf{w} = \mathbf{curl}\,\mathbf{B}_h \in L^\infty(\Omega)^3$, we know that $\pi_h \mathbf{w}$ is well-defined. Note that $\pi_h \mathbf{B}_h = \mathbf{B}_h$; we obtain by Lemma 3.4 that

$$\mathbf{B}_h = \pi_h \mathbf{w} + \nabla \varphi_h \quad \text{for some} \quad \varphi_h \in Q_h. \tag{3.10}$$

On the other hand, by using Lemma 3.3, (3.9), and the local inverse estimate, we have

$$\begin{aligned} \|\mathbf{w} - \pi_h \mathbf{w}\|_{0,K} &\leq C h_K^r (\|\mathbf{w}\|_{r,K} + \|\mathbf{curl}\,\mathbf{w}\|_{r,K}) \\ &= C h_K^r (\|\mathbf{w}\|_{r,K} + \|\mathbf{curl}\,\mathbf{B}_h\|_{r,K}) \\ &\leq C\|\mathbf{w}\|_{r,K} + C\|\mathbf{curl}\,\mathbf{B}_h\|_{0,K}, \end{aligned}$$

which yields, by using Lemma 3.6 and (3.9),

$$\begin{aligned} \|\pi_h \mathbf{w}\|_{0,\Omega} &\leq C\,(\|\mathbf{w}\|_{0,\Omega} + \|\mathbf{w}\|_{r,\Omega} + \|\mathbf{curl}\,\mathbf{B}_h\|_{0,\Omega}) \\ &\leq C\,(\|\mathbf{w}\|_{0,\Omega} + \|\mathbf{curl}\,\mathbf{w}\|_{0,\Omega} + \|\mathbf{curl}\,\mathbf{B}_h\|_{0,\Omega}) \\ &\leq C\,(\|\mathbf{w}\|_{0,\Omega} + \|\mathbf{curl}\,\mathbf{B}_h\|_{0,\Omega}) \\ &\leq C\|\mathbf{curl}\,\mathbf{B}_h\|_{0,\Omega}. \end{aligned} \tag{3.11}$$

Here we have used Lemma 3.1 to conclude that

$$\|\mathbf{w}\|_{0,\Omega} \leq C\|\mathbf{curl}\,\mathbf{w}\|_{0,\Omega} = C\|\mathbf{curl}\,\mathbf{B}_h\|_{0,\Omega}.$$

Now the theorem follows by multiplying (3.10) by $\beta\,\mathbf{B}_h$, then using (3.11) and the fact that $\mathbf{B}_h \in N_h$. ◻

4. Finite element method with a nonmatching grid. The finite element method discussed in section 3 for solving the system (2.1)–(2.5) is based on a matching finite element mesh on the interface Γ. This imposes a serious restriction, especially in three dimensions, on the triangulations in Ω_1 and Ω_2: both must match with each other on Γ. We are now going to relax this restriction and consider a nonmatching mesh on the interface that allows the two triangulations in Ω_1 and Ω_2 to be generated independently. This advantage, however, brings some difficulty to the convergence analysis since the resulting finite element spaces will be nonconforming for both the unknown $\mathbf{A}$ and Lagrangian multiplier p. Similarly to the treatment of the divergence condition, we will also deal with the constraints $[\mathbf{A} \times \mathbf{n}] = 0$ and $[p] = 0$ on Γ by a Lagrangian multiplier approach. As we shall see, the introduction of these new multipliers leads to a nested saddle point problem, for which we need to first generalize the standard saddle point theory (for example, cf. [6, 17]).

4.1. Abstract framework. Let X, Q, M be three real Hilbert spaces and $a : X \times X \to R$, $b : X \times Q \to R$, $c : Q \times M \to R$ be three continuous bilinear functionals. Given $f \in X', g \in Q'$, and $\chi \in M'$, we consider the following problem: Find $(u, p, \lambda) \in X \times Q \times M$ such that

$$a(u,v) + b(v,p) = \langle f, v\rangle \quad \forall\, v \in X, \tag{4.1}$$

$$b(u,q) + c(q,\lambda) = \langle g, q\rangle \quad \forall\, q \in Q, \tag{4.2}$$

$$c(p,\mu) = \langle \chi, \mu\rangle \quad \forall\, \mu \in M. \tag{4.3}$$

In the above we have used the same notation $\langle\cdot,\cdot\rangle$ to denote all three duality pairings. This saddle point problem and its discretization were considered earlier in [29]. The results to be shown below are more general than the ones in [29] in the sense that our statement on the coercivity and the inf-sup condition for both continuous and discrete cases is more precise and much less restrictive. We will give all the proofs in the appendix, section A2.

We first introduce two subspaces $N_1 \subset Q$ and $N_2 \subset X$ as follows:

$$N_1 = \{q \in Q;\ \ c(q,\mu) = 0 \quad \forall\, \mu \in M\}, \tag{4.4}$$

$$N_2 = \{v \in X;\ \ b(v,q) = 0 \quad \forall\, q \in N_1\}. \tag{4.5}$$

Note that in the definition of N_2, the test function q is required only to be in N_1, which naturally provides a relaxed condition on the spaces than that given in [29].

We have the following result on the existence and uniqueness of the solution for the system (4.1)–(4.3).

LEMMA 4.1. *Assume that $a(\cdot,\cdot)$ is N_2-coercive, i.e.,*

$$a(v,v) \ge a_0\|v\|_X^2 \quad \forall\, v \in N_2, \tag{4.6}$$

and the following inf-sup conditions hold:

$$\inf_{q\in N_1}\sup_{v\in X}\frac{b(v,q)}{\|v\|_X\|q\|_Q} \ge b_0, \tag{4.7}$$

$$\inf_{\mu\in M}\sup_{q\in Q}\frac{c(q,\mu)}{\|q\|_Q\|\mu\|_M} \ge c_0 \tag{4.8}$$

for some positive constants a_0, b_0, c_0. Then the problem (4.1)–(4.3) *has a unique solution (u,p,λ) in $X \times Q \times M$.*

Next let $X_h \subset X$, $Q_h \subset Q$, and $M_h \subset M$ be three finite dimensional subspaces. We introduce the corresponding approximation of (4.1)–(4.3) as follows.

Find $(u_h, p_h, \lambda_h) \in X_h \times Q_h \times M_h$ such that

$$a(u_h, v_h) + b(v_h, p_h) = \langle f, v_h \rangle \quad \forall\, v_h \in X_h, \tag{4.9}$$

$$b(u_h, q_h) + c(q_h, \lambda_h) = \langle g, q_h \rangle \quad \forall\, q_h \in Q_h, \tag{4.10}$$

$$c(p_h, \mu_h) = \langle \chi, \mu_h \rangle \quad \forall\, \mu_h \in M_h. \tag{4.11}$$

Corresponding to (4.4)–(4.5) we set

$$N_{1h} = \{q_h \in Q_h;\ c(q_h, \mu_h) = 0 \quad \forall\, \mu_h \in M_h\}, \tag{4.12}$$

$$N_{2h} = \{v_h \in X_h;\ b(v_h, q_h) = 0 \quad \forall\, q_h \in N_{1h}\}. \tag{4.13}$$

We have the following result concerning the discrete problem (4.9)–(4.11).

LEMMA 4.2. *Assume the following conditions hold.*

(i) $a(\cdot,\cdot)$ *is N_{2h}-coercive; i.e., there exists a constant $a^* > 0$ such that*

$$a(v_h, v_h) \geq a^* \|v_h\|_X^2 \quad \forall\, v_h \in N_{2h}. \tag{4.14}$$

(ii) *There exists a constant $b^* > 0$ such that*

$$\inf_{q_h \in N_{1h}} \sup_{v_h \in X_h} \frac{b(v_h, q_h)}{\|v_h\|_X \|q_h\|_Q} \geq b^*. \tag{4.15}$$

(iii) *There exists a constant $c^* > 0$ such that*

$$\inf_{\mu_h \in M_h} \sup_{q_h \in Q_h} \frac{c(q_h, \mu_h)}{\|q_h\|_Q \|\mu_h\|_M} \geq c^*. \tag{4.16}$$

Then the discrete problem (4.9)–(4.11) has a unique solution (u_h, p_h, λ_h) in the space $X_h \times Q_h \times M_h$ with the following error estimate,

$$\begin{aligned} &\|u - u_h\|_X + \|p - p_h\|_Q + \|\lambda - \lambda_h\|_M \\ &\leq C\Big\{ \inf_{v_h \in X_h} \|u - v_h\|_X + \inf_{q_h \in Q_h} \|p - q_h\|_Q + \inf_{\mu_h \in M_h} \|\lambda - \mu_h\|_M \Big\}, \end{aligned}$$

where the constant C depends only on a^*, b^*, c^*, and on the operator norms $\|a\|$, $\|b\|$, and $\|c\|$ of the bilinear functional $a(\cdot,\cdot), b(\cdot,\cdot)$, and $c(\cdot,\cdot)$, respectively.

4.2. The variational formulation. In this subsection we introduce a new weak formulation for the system (2.1)–(2.5) with the constant coefficient γ for the lower-order term being strictly positive. Note that the results of section 3 for matching finite element grids are applicable to both stationary and time-dependent Maxwell equations even in the absence of the lower-order term. For nonmatching finite element grids, however, without this additional term, we have some technical difficulty in the verification of the coercivity of the bilinear form corresponding to the term $\mathbf{curl}\,(\alpha\,\mathbf{curl}\,\mathbf{A})$. Though such verification has been made in the matching grid case; see Lemma 4.7.

Let us now introduce the following spaces:

$$\begin{aligned} X_1 &= H(\mathbf{curl}\,, \Omega_1), &\quad X_2 &= \{\mathbf{v} \in H(\mathbf{curl}\,, \Omega_2);\ \mathbf{v} \times \mathbf{n} = 0 \text{ on } \partial\Omega\}, \\ Q_1 &= \quad H^1(\Omega_1), &\quad Q_2 &= \{v \in H^1(\Omega_2);\ v = 0 \text{ on } \partial\Omega\}. \end{aligned}$$

Then set

$$X = X_1 \times X_2, \qquad Q = Q_1 \times Q_2 \times T(\Gamma), \qquad M = H^{-1/2}(\Gamma)$$

and define three bilinear forms $a : X \times X \to R$, as follows:

$$a(\mathbf{A}, \mathbf{B}) = \int_\Omega \gamma\beta\, \mathbf{A} \cdot \mathbf{B}\, dx + \sum_{i=1}^{2} \int_{\Omega_i} \alpha_i\, \mathbf{curl}\, \mathbf{A}_i \cdot \mathbf{curl}\, \mathbf{B}_i\; dx \quad \forall\, \mathbf{A}, \mathbf{B} \in X,$$

$$b(\mathbf{B}, (q, \mathbf{s})) = -\sum_{i=1}^{2} \int_{\Omega_i} \beta_i\, \nabla q_i \cdot \mathbf{B}_i\, dx + \langle\!\langle \mathbf{s}, \mathbf{B}_2 \rangle\!\rangle_{2,\Gamma} - \langle\!\langle \mathbf{s}, \mathbf{B}_1 \rangle\!\rangle_{1,\Gamma} \ \ \forall\, \mathbf{B} \in X,\ (q, \mathbf{s}) \in Q,$$

$$c((q, \mathbf{s}), \mu) = \langle q_1 - q_2, \mu \rangle_\Gamma \quad \forall\, (q, \mathbf{s}) \in Q,\ \mu \in M.$$

Now applying the standard technique of integration by parts leads immediately to the following weak formulation of problem (2.1)–(2.5).

Problem (P). Given $(\mathbf{f}, g) \in L^2(\Omega)^4$, find $(\mathbf{A}, (p, \mathbf{t}), \lambda) \in X \times Q \times M$ such that

$$a(\mathbf{A}, \mathbf{B}) + b(\mathbf{B}, (p, \mathbf{t})) = \sum_{i=1}^{2} \int_{\Omega_i} \mathbf{f} \cdot \mathbf{B}_i\, dx - \langle\!\langle \mathbf{f}_\Gamma, \mathbf{B}_2 \rangle\!\rangle_{2,\Gamma} \ \ \forall\, \mathbf{B} \in X, \tag{4.17}$$

$$b(\mathbf{A}, (q, \mathbf{s})) + c((q, \mathbf{s}), \lambda) = \sum_{i=1}^{2} \int_{\Omega_i} g\, q_i dx + \langle g_\Gamma, q_2 \rangle_\Gamma \ \ \forall\, (q, \mathbf{s}) \in Q, \tag{4.18}$$

$$c((p, \mathbf{t}), \mu) = 0 \ \ \forall\, \mu \in M. \tag{4.19}$$

The above weak formulation is different from the one used in section 3; it is more consistent with the finite element discretization on a nonmatching grid on the interface Γ. Note that the above system can also be derived based on an optimal control formulation of the interface problem (1.15)–(1.18). Similar approaches have been studied for the Poisson-type equations in [14]. The above formulation can be used as a basis for the further development of nonoverlapping domain decomposition methods.

First, we have the following result on the existence and uniqueness of solutions to the Problem (P).

THEOREM 4.3. *There exists a unique solution* $(\mathbf{A}, (p, t), \lambda) \in X \times Q \times M$ *to the system* (4.17)–(4.19). *Moreover,* $(\mathbf{A}, p) \in H_0(\mathbf{curl}\,, \Omega) \times H_0^1(\Omega)$ *satisfies* (2.1)–(2.2) *in the sense of distribution, and the Lagrangian multiplier* $(\mathbf{t}, \lambda) \in T(\Gamma) \times H^{-1/2}(\Gamma)$ *satisfies the following relations:*

$$\begin{array}{lll} \mathbf{t} = \alpha_1\, \mathbf{curl}\, \mathbf{A}_1 \times \mathbf{n} = \alpha_2\, \mathbf{curl}\, \mathbf{A}_2 \times \mathbf{n} - \mathbf{f}_\Gamma & \text{in} & T(\Gamma), \\ \lambda = \beta_1\, \mathbf{A}_1 \cdot \mathbf{n} = \beta_2\, \mathbf{A}_2 \cdot \mathbf{n} - g_\Gamma & \text{in} & H^{-1/2}(\Gamma). \end{array}$$

Proof. We only prove that the system (4.17)–(4.19) has a unique solution; the rest of the theorem can be easily obtained by an appropriate application of the Green's formula. For the proof, we need to verify the three conditions in Lemma 4.1.

First, it is obvious that the bilinear form $a : X \times X \to R$ is N_2-elliptic; i.e.,

$$a(\mathbf{B}, \mathbf{B}) \ge a_0\, \|\mathbf{B}\|_X^2 \quad \forall\, \mathbf{B} \in N_2 \tag{4.20}$$

for some constant $a_0 > 0$.

Second, we show that there exists a constant $b_0 > 0$ such that

$$\sup_{\mathbf{B} \in X} \frac{b(\mathbf{B}, (q, \mathbf{s}))}{\|\mathbf{B}\|_X} \ge b_0 \|(q, \mathbf{s})\|_Q \quad \forall\, (q, \mathbf{s}) \in N_1 \times T(\Gamma). \tag{4.21}$$

We note that $N_1 = H_0^1(\Omega)$. For any given $\mathbf{s} \in T(\Gamma)$, let $\mathbf{Q} \in H(\mathbf{curl}\,, \Omega_1)$ be the solution of the following problem:

$$\int_{\Omega_1} (\mathbf{curl}\, \mathbf{Q} \cdot \mathbf{curl}\, \mathbf{B} + \mathbf{Q} \cdot \mathbf{B}) dx = \langle\!\langle \mathbf{s}, \mathbf{B} \rangle\!\rangle_{1,\Gamma} \quad \forall\, \mathbf{B} \in H(\mathbf{curl}\,; \Omega_1). \tag{4.22}$$

It is easy to see that

$$\|\|\mathbf{s}\|\|_{1,\Gamma} = \|\mathbf{Q}\|_{\mathbf{curl},\Omega_1}. \tag{4.23}$$

Now let $u \in H_0^1(\Omega)$ be the solution of the following elliptic interface problem:

$$\int_\Omega \beta \nabla u \cdot \nabla v \, dx = \int_{\Omega_1} \beta_1 \mathbf{Q} \cdot \nabla v \, dx \quad \forall v \in H_0^1(\Omega). \tag{4.24}$$

It is clear that (4.24) has a unique solution $u \in H_0^1(\Omega)$ which satisfies

$$\|\nabla u\|_{0,\Omega} \le C\|\mathbf{Q}\|_{0,\Omega_1}. \tag{4.25}$$

With these preparations, for any $q \in H_0^1(\Omega)$ and $\mathbf{s} \in T(\Gamma)$, we define

$$\tilde{\mathbf{B}} = \begin{cases} -\nabla q_1 + \nabla u_1 - \mathbf{Q} & \text{in } \Omega_1, \\ -\nabla q_2 + \nabla u_2 & \text{in } \Omega_2, \end{cases}$$

where $q_i = q|_{\Omega_i}$, $u_i = u|_{\Omega_i}$, $i = 1, 2$. It is obvious that $\tilde{\mathbf{B}} \in X$ and

$$\begin{aligned} \|\tilde{\mathbf{B}}\|_X &= \| - \nabla q_1 + \nabla u_1 - \mathbf{Q}\|_{0,\Omega_1} + \|\mathbf{curl}\,\mathbf{Q}\|_{0,\Omega_1} + \| - \nabla q_2 + \nabla u_2\|_{0,\Omega_2} \\ &\le C \left(\|\nabla q\|_{0,\Omega} + \|\mathbf{Q}\|_{\mathbf{curl},\Omega_1}\right) \end{aligned} \tag{4.26}$$

where we have used (4.25). Note that since $q_1 = q_2$, $u_1 = u_2$ on Γ, we have

$$\nabla(q_1 - q_2) \times \mathbf{n} = 0, \qquad \nabla(u_1 - u_2) \times \mathbf{n} = 0 \quad \text{on} \quad \Gamma.$$

Thus, by using (4.24) and (4.22), we are able to obtain

$$\begin{aligned} b(\tilde{\mathbf{B}}, (q, \mathbf{s})) &= -\int_\Omega \beta \nabla q \cdot (-\nabla q + \nabla u) dx + \int_{\Omega_1} \beta_1 \nabla q_1 \cdot \mathbf{Q}\, dx + \langle\langle \mathbf{s}, \mathbf{Q}\rangle\rangle_{1,\Gamma} \\ &= \int_\Omega \beta |\nabla q|^2 dx + \langle\langle \mathbf{s}, \mathbf{Q}\rangle\rangle_{1,\Gamma} \\ &= \int_\Omega \beta |\nabla q|^2 dx + \|\mathbf{Q}\|^2_{\mathbf{curl},\Omega_1}, \end{aligned}$$

which yields, together with (4.26), (4.23), and Lemma 2.3,

$$\frac{b(\tilde{\mathbf{B}}, (q, \mathbf{s}))}{\|\tilde{\mathbf{B}}\|_X} \ge C \left(\|\nabla q\|_{0,\Omega} + \|\mathbf{Q}\|_{\mathbf{curl},\Omega_1}\right) \ge C \left(\|q\|_{1,\Omega} + \|\mathbf{s}\|_{T(\Gamma)}\right).$$

This completes the proof of (4.21).

Finally, we verify that there exists a constant $c_0 > 0$ such that

$$\sup_{(q,\mathbf{s}) \in Q} \frac{c((q, \mathbf{s}), \mu)}{\|(q, \mathbf{s})\|_Q} \ge c_0 \|\mu\|_M \quad \forall \mu \in M. \tag{4.27}$$

This follows immediately from the trace theorem

$$\sup_{(q,\mathbf{s}) \in Q} \frac{c((q, \mathbf{s}), \mu)}{\|(q, \mathbf{s})\|_Q} \ge \sup_{q_1 \in H^1(\Omega_1)} \frac{\langle q_1, \mu\rangle_\Gamma}{\|q_1\|_{1,\Omega_1}} = \|\mu\|_M.$$

From (4.20), (4.21), (4.27), and Lemma 4.1 we conclude that the system (4.17)–(4.19) has a unique solution $(\mathbf{A}, (p, \mathbf{t}), \lambda) \in X \times Q \times M$. $\square$

4.3. Finite element discretization. In this subsection we propose a finite element method for solving the problem (4.17)–(4.19), which allows a nonmatching finite element grid on the interface Γ. Let $\mathcal{T}^{h_1}$ and $\mathcal{T}^{h_2}$ be a shape regular triangulation of Ω_1 and Ω_2, respectively. They induce naturally two finite element triangulations Γ_{h_1} and Γ_{h_2} on the interface Γ. Let Γ_{h_0} be an another shape regular triangulation over Γ. Note that Γ_{h_i}, $i = 0, 1, 2$, are allowed to be different from each other. However, we make the following reasonable assumption:

(H1) Each triangle in Γ_{h_1} and Γ_{h_2} must be contained in some triangle of Γ_{h_0}.

Let $X_{h_i} \subset X_i, i = 1, 2$, be the Nédélec edge element space defined over $\mathcal{T}^{h_i}$ (cf. (3.3)), let $Q_{h_i} \subset Q_i, i = 1, 2$, be the standard piecewise linear finite element space over $\mathcal{T}^{h_i}$, let M_{h_0} be the standard piecewise constant finite element space over Γ_{h_0}. Moreover, we define

$$T_{h_0}(\Gamma) = \{\mathbf{s}_h \subset T(\Gamma);\ \mathbf{s}_h = (\boldsymbol{\alpha}_\tau + \boldsymbol{\beta}_\tau \times \mathbf{x}) \times \mathbf{n} \text{ on any } \tau \in \Gamma_{h_0},\ \boldsymbol{\alpha}_\tau\,, \boldsymbol{\beta}_\tau \in R^3\}.$$

Note that on planar portions of the interface, the space $T_{h_0}(\Gamma)$ coincides with the lowest order Raviart–Thomas element (see [26]) which has been used often for electromagnetic integral equation calculations (cf. [4]).

Now set

$$X_h = X_{h_1} \times X_{h_2} \subset X, \qquad Q_h = Q_{h_1} \times Q_{h_2} \times T_{h_0}(\Gamma) \subset Q, \qquad M_h = M_{h_0} \subset M.$$

We will assume the following two inf-sup conditions:

(H2) There exists a constant $\beta^* > 0$ independent of h_0, h_1, h_2 such that

$$\sup_{\mathbf{B}_{h_i} \in X_{h_i}} \frac{\langle\!\langle \mathbf{s}_h, \mathbf{B}_{h_i} \rangle\!\rangle_{i,\Gamma}}{\|\mathbf{B}_{h_i}\|_{\mathbf{curl}\,,\Omega_i}} \geq \beta^* \, \|\mathbf{s}_h\|_{T(\Gamma)} \quad \forall\, \mathbf{s}_h \in T_{h_0}(\Gamma)\,, \qquad i = 1 \text{ or } 2.$$

(H3) There exists a constant $\gamma^* > 0$ independent of h_0, h_1, h_2 such that

$$\sup_{q_{h_i} \in Q_{h_i}} \frac{\langle q_{h_i}, \mu_h \rangle_\Gamma}{\|q_{h_i}\|_{1,\Omega_i}} \geq \gamma^* \, \|\mu_h\|_{-1/2,\Gamma} \quad \forall\, \mu_h \in M_{h_0}\,, \qquad i = 1 \text{ or } 2\,.$$

The assumptions (H2)–(H3) indicate that the mesh Γ_{h_0} should be coarse enough compared with the meshes $\mathcal{T}^{h_1}$ *or* $\mathcal{T}^{h_2}$ in order to stabilize the effect of the introduced Lagrangian multipliers. We will verify these two assumptions in the next subsection using a general compactness argument.

Now we are in a position to introduce the discrete version of Problem (P).

Problem (P_h). Find $(\mathbf{A}_h, (p_h, \mathbf{t}_h), \lambda_h) \in X_h \times Q_h \times M_h$ such that

$$\text{(4.28)}\; a(\mathbf{A}_h, \mathbf{B}_h) + b(\mathbf{B}_h, (p_h, \mathbf{t}_h)) = \sum_{i=1}^{2} \int_{\Omega_i} \mathbf{f} \cdot \mathbf{B}_{h_i} dx - \langle\!\langle \mathbf{f}_\Gamma, \mathbf{B}_{h_2} \rangle\!\rangle_{2,\Gamma} \;\; \forall\, \mathbf{B}_h \in X_h,$$

$$\text{(4.29)}\; b(\mathbf{A}_h, (q_h, \mathbf{s}_h)) + c((q_h, \mathbf{s}_h), \lambda_h) = \sum_{i=1}^{2} \int_{\Omega_i} g\, q_{h_i} dx + \langle g_\Gamma, q_{h_2} \rangle_\Gamma \;\; \forall\, (q_h, \mathbf{s}_h) \in Q_h,$$

$$\text{(4.30)}\; c((p_h, \mathbf{t}_h), \mu_h) = 0 \quad \forall\, \mu_h \in M_h.$$

The following theorem summarizes our main results of this section.

THEOREM 4.4. *Under the assumptions* (H1)–(H3) *the discrete Problem* (P_h) *has a unique solution* $(\mathbf{A}_h, (p_h, \mathbf{t}_h), \lambda_h) \in X_h \times Q_h \times M_h$. *It satisfies the error estimate, for some generic constant* $C > 0$,

$$\sum_{i=1}^{2}(\|\mathbf{A}_i - \mathbf{A}_{h_i}\|_{\mathbf{curl},\Omega_i} + \|p_i - p_{h_i}\|_{1,\Omega_i}) + \|\mathbf{t} - \mathbf{t}_h\|_{T(\Gamma)} + \|\lambda - \lambda_h\|_{-1/2,\Gamma}$$
$$\leq C\left\{ \inf_{\mathbf{B}_h \in X_h} \sum_{i=1}^{2} \|\mathbf{A}_i - \mathbf{B}_{h_i}\|_{\mathbf{curl},\Omega_i} + \inf_{(q_h,\mathbf{s}_h)\in Q_h} \left(\sum_{i=1}^{2} \|p_i - q_{h_i}\|_{1,\Omega_i} + \|\mathbf{t} - \mathbf{s}_h\|_{T(\Gamma)} \right) \right.$$
$$\left. + \inf_{\mu_h \in M_h} \|\lambda - \mu_h\|_{-1/2,\Gamma} \right\}.$$

Moreover, if $\mathbf{f} \in H^s(\Omega_1)^3 \times H^s(\Omega_2)^3$, $g \in H^s(\Omega_1) \times H^s(\Omega_2)$, $g_\Gamma \in H^{s-1/2}(\Gamma)$, $\mathbf{f}_\Gamma = \boldsymbol{\psi} \times \mathbf{n}$ *for some* $\boldsymbol{\psi} \in H^s(\mathbf{curl}\,;\Omega_1)$ *and the solution* $(\mathbf{A}, p)$ *of problem* (4.17)–(4.19) *has the regularity,*

$$\mathbf{A} \in H^s(\mathbf{curl}\,,\Omega_1) \times H^s(\mathbf{curl}\,,\Omega_2),\ \ p \in H^{1+s}(\Omega_1) \times H^{1+s}(\Omega_2),$$

where $s \in (1/2, 1]$; *then we have the following error estimate:*

$$\sum_{i=1}^{2}\{\|\mathbf{A}_i - \mathbf{A}_{h_i}\|_{\mathbf{curl},\Omega_i} + \|p_i - p_{h_i}\|_{1,\Omega_i}\} + \|\mathbf{t} - \mathbf{t}_h\|_{T(\Gamma)} + \|\lambda - \lambda_h\|_{-1/2,\Gamma}$$
$$\leq C\sum_{i=1}^{2} h_i^s(\|\mathbf{A}_i\|_{s,\mathbf{curl},\Omega_i} + \|p_i\|_{1+s,\Omega_i} + \|\mathbf{f}\|_{s,\Omega_i}) + Ch_0^s\|\lambda\|_{-1/2+s,\Gamma}. \tag{4.31}$$

We are going to apply the abstract framework in Lemma 4.2 to prove Theorem 4.4. We first note that

$$N_{1h} = \{(q_h, \mathbf{s}_h) \in Q_h;\ c((q_h, \mathbf{s}_h), \mu_h) = 0 \quad \forall\, \mu_h \in M_h\}$$
$$= \{(q_h, \mathbf{s}_h) \in Q_{h_1} \times Q_{h_2} \times T_{h_0}(\Gamma);\ \langle q_{h_1} - q_{h_2}, \mu_h\rangle_\Gamma = 0 \ \ \forall \mu_h \in M_{h_0}\}.$$

For those functions in N_{1h}, we have the following equivalence.

LEMMA 4.5. *For any* $q_h \in Q_{h_1} \times Q_{h_2}$, *the following two relations are equivalent:*

$$\int_\Gamma (q_{h_1} - q_{h_2})\, \mu_h d\sigma = 0 \quad \forall\, \mu_h \in \bar{M}_{h_0}, \tag{4.32}$$

$$\int_\Gamma \nabla(q_{h_1} - q_{h_2}) \cdot \mathbf{s}_h d\sigma = 0 \quad \forall\, \mathbf{s}_h \in T_{h_0}(\Gamma), \tag{4.33}$$

where $\bar{M}_{h_0} = \{\mu_h \in M_{h_0}\ ;\ \langle \mu_h,\, 1\rangle_\Gamma = 0\}$.

Proof. We denote by $\mathcal{T}^{h_0}$ any shape regular triangulation of Ω_1 whose restriction on Γ coincides with Γ_{h_0}, and let X_{h_0} be the Nédélec $H(\mathbf{curl}\,,\Omega_1)$-conforming edge element space over $\mathcal{T}^{h_0}$. Then from the definition of $T_{h_0}(\Gamma)$ we know that $T_{h_0}(\Gamma) = \{\boldsymbol{\varphi}_h \times \mathbf{n}; \boldsymbol{\varphi}_h \in X_{h_0}\}$. Thus (4.33) is equivalent to

$$\int_\Gamma \nabla(q_{h_1} - q_{h_2}) \cdot (\boldsymbol{\varphi}_h \times \mathbf{n}) d\sigma = 0 \quad \forall\, \boldsymbol{\varphi}_h \in X_{h_0}.$$

Then, by integration by parts, we obtain (cf. [15, Lemma 2.1])

$$\int_\Gamma \nabla(q_{h_1} - q_{h_2}) \cdot (\boldsymbol{\varphi}_h \times \mathbf{n}) d\sigma = -\int_\Gamma (q_{h_1} - q_{h_2})(\mathbf{curl}\, \boldsymbol{\varphi}_h \cdot \mathbf{n}) d\sigma \quad \forall\, \boldsymbol{\varphi}_h \in X_{h_0}.$$

Now set $N_{h_0} = \{\mathbf{curl}\, \boldsymbol{\varphi}_h \cdot \mathbf{n}\,;\, \boldsymbol{\varphi}_h \in X_{h_0}\}$; then for our purpose it suffices to show $\bar{M}_{h_0} = N_{h_0}$. It is clear that $N_{h_0} \subset \bar{M}_{h_0}$. Let n_f, n_v, n_e stand for the number of faces, vertices, edges of the triangulation Γ_{h_0}, respectively. Using the equivalence between a two-dimensional tangential vector field (in this case the tangential components of the elements of X_{h_0} on the interface Γ), defined on the edges of the triangulation having zero circulation on each triangle, and that being induced by a scalar two-dimensional potential field defined on the vertices of the triangulation [23], we have $\dim N_{h_0} = n_e - (n_v - 1)$ which is equal to $n_f - 1$, the dimension of $\bar{M}_{h_0}$ due to the Euler formula. This completes the proof. ∎

Furthermore, we verify the following inf-sup condition.

LEMMA 4.6. *Under the assumptions* (H2)–(H3) *we have*

$$\sup_{\mathbf{B}_h \in X_h} \frac{b(\mathbf{B}_h, (q_h, \mathbf{s}_h))}{\|\mathbf{B}_h\|_X} \geq b^* \|(q_h, \mathbf{s}_h)\|_{Q_h} \quad \forall\, (q_h, \mathbf{s}_h) \in N_{1h}, \tag{4.34}$$

where $b^* > 0$ is a constant independent of h_0, h_1, h_2.

Proof. Without loss of generality we assume that (H2) is valid for $i = 1$. Thus for any $\mathbf{s}_h \in T_{h_0}(\Gamma)$, there exists a $\tilde{\mathbf{Q}}_{h_1} \in X_{h_1}$ such that

$$\beta^* \|\mathbf{s}_h\|_{T(\Gamma)} \leq \frac{\langle\!\langle \mathbf{s}_h, \tilde{\mathbf{Q}}_{h_1} \rangle\!\rangle_\Gamma}{\|\tilde{\mathbf{Q}}_{h_1}\|_{\mathbf{curl}\,,\Omega_1}}. \tag{4.35}$$

Then we define $\mathbf{Q}_{h_1} \in X_{h_1}$ to be the solution of the following problem:

$$\int_{\Omega_1} (\mathbf{curl}\, \mathbf{Q}_{h_1} \cdot \mathbf{curl}\, B_{h_1} + \mathbf{Q}_{h_1} \cdot B_{h_1}) dx = \langle\langle \mathbf{s}_h, \mathbf{B}_{h_1} \rangle\rangle_{1,\Gamma} \quad \forall\, \mathbf{B}_{h_1} \in X_{h_1}.$$

Taking $\mathbf{B}_h = \mathbf{Q}_{h_1}$ and $\mathbf{B}_h = \tilde{\mathbf{Q}}_{h_1}$ as test functions, respectively, and using (4.35), we obtain

$$\beta^* \|\mathbf{s}_h\|_{T(\Gamma)} \leq \|\mathbf{Q}_{h_1}\|_{\mathbf{curl}\,;\Omega_1} = |\!|\!|\mathbf{s}|\!|\!|_{1,\Gamma} \leq \|\mathbf{s}_h\|_{T(\Gamma)}. \tag{4.36}$$

Now we use $\mathbf{Q}_{h_1}$ to define $(u_h, \theta_h) \in (Q_{h_1} \times Q_{h_2}) \times M_h$ to be the solution of the following discrete elliptic interface problem:

$$\sum_{i=1}^{2} \int_{\Omega_i} \beta_i \nabla u_{h_i} \cdot \nabla v_{h_i} dx + \langle v_{h_1} - v_{h_2}, \theta_h \rangle_\Gamma = \int_{\Omega_1} \beta_1 \mathbf{Q}_{h_1} \cdot \nabla v_{h_1} dx \quad \forall\, v_h \in Q_{h_1} \times Q_{h_2}, \tag{4.37}$$

$$\langle u_{h_1} - u_{h_2}, \mu_h \rangle_\Gamma = 0 \qquad \forall\, \mu_h \in M_h. \tag{4.38}$$

Under the assumption (H3), this problem has a unique solution (u_h, θ_h). Choosing v_h to be u_h in (4.37) and using (4.38) we obtain

$$\sum_{i=1}^{2} \|\nabla u_{h_i}\|_{0,\Omega_i} \leq C\, \|\mathbf{Q}_{h_1}\|_{0,\Omega_1}. \tag{4.39}$$

Now set

$$\tilde{\mathbf{B}}_h = \begin{cases} -\nabla q_{h_1} + \nabla u_{h_1} - \mathbf{Q}_{h_1} & \text{in } \Omega_1, \\ -\nabla q_{h_2} + \nabla u_{h_2} & \text{in } \Omega_2. \end{cases}$$

It is clear that $\tilde{\mathbf{B}}_h \in X_h$, and we have from (4.39) that

$$\begin{aligned} \|\tilde{\mathbf{B}}_h\|_X &= \|\nabla u_{h_1} - \nabla q_{h_1} - \mathbf{Q}_{h_1}\|_{0,\Omega} + \|\mathbf{curl}\,\mathbf{Q}_{h_1}\|_{0,\Omega_1} + \|\nabla u_{h_2} - \nabla q_{h_2}\|_{0,\Omega_2} \\ &\le C\sum_{i=1}^{2} \|\nabla q_{h_i}\|_{0,\Omega} + \|\mathbf{Q}_{h_1}\|_{\mathbf{curl},\Omega_1}. \end{aligned} \tag{4.40}$$

But from Lemma 4.5 we know that

$$\langle\nabla(q_{h_1} - q_{h_2}), \mathbf{s}_h\rangle_\Gamma = 0, \qquad \langle\nabla(u_{h_1} - u_{h_2})\ ,\ \mathbf{s}_h\rangle_\Gamma = 0 \quad \forall\, \mathbf{s}_h \in T_{h_0}(\Gamma),$$

which yields, together with taking $v_h = q_h$ in (4.37) and using (4.36),

$$\begin{aligned} b(\tilde{\mathbf{B}}_h, (q_h, \mathbf{s}_h)) &= \sum_{i=1}^{2}\int_{\Omega_i} \beta_i|\nabla q_{h_i}|^2 dx + \langle\!\langle \mathbf{s}_h, \mathbf{Q}_{h_1}\rangle\!\rangle_{1,\Gamma} \\ &\ge\ C\big(\sum_{i=1}^{2}\|\nabla q_{h_i}\|_{0,\Omega_i}^2 + \|\,\mathbf{Q}_{h_1}\,\|_{\mathbf{curl},\Omega_1}^2\big) \quad \forall\,(q_h, \mathbf{s}_h) \in N_{1h}. \end{aligned}$$

Hence using (4.36) again we easily derive

$$\frac{b(\tilde{\mathbf{B}}_h, (q_h, \mathbf{s}_h))}{\|\tilde{\mathbf{B}}_h\|_X} \ge C\left\{\sum_{i=1}^{2}\|\nabla q_{h_i}\|_{0,\Omega_i} + \|\mathbf{s}_h\|_{T(\Gamma)}\right\} \quad \forall\,(q_h, \mathbf{s}_h) \in N_{1h}\ .$$

Finally note the fact that $q_{h_2} = 0$ on $\partial\Omega$ and $\int_\Gamma q_{h_1}\, d\sigma = \int_\Gamma q_{h_2}\, d\sigma$ due to $(q_h, \mathbf{s}_h) \in N_{1h}$, and we have by means of the standard argument for Poincaré inequality (cf. [9]) that

$$\sum_{i=1}^{2}\|q_{h_i}\|_{1,\Omega_i} \le C\sum_{i=1}^{2}\|\nabla q_{h_i}\|_{0,\Omega_i}.$$

This completes the proof of Lemma 4.6. ∎

Proof of Theorem 4.4. We know from (H3) that $c : Q \times M \to R$ satisfies the inf-sup condition

$$\sup_{(q_h,\mathbf{s}_h)\in Q_h} \frac{c((q_h, \mathbf{s}_h), \mu_h)}{\|(q_h, \mathbf{s}_h)\|_Q} \ge \gamma^*\|\mu_h\|_M.$$

Thus, with this, (4.34), and the obvious coercivity of $a(\cdot,\cdot)$ we are able to apply Lemma 4.2 to conclude that

$$\begin{aligned} &\sum_{i=1}^{2}(\|\mathbf{A}_i - \mathbf{A}_{h_i}\|_{\mathbf{curl},\Omega_i} + \|p_i - p_{h_i}\|_{1,\Omega_i}) + \|\mathbf{t} - \mathbf{t}_h\|_{T(\Gamma)} + \|\lambda - \lambda_h\|_{1/2,\Gamma} \\ &\quad\le C\left\{\inf_{\mathbf{B}_h\in X_h}\|\mathbf{A}_i - \mathbf{B}_{h_i}\|_{\mathbf{curl},\Omega_i} + \sum_{i=1}^{2}\inf_{q_{h_i}\in Q_{h_i}}\|p_i - q_{h_i}\|_{1,\Omega_i}\right. \\ &\qquad\left. + \inf_{\mathbf{s}_h\in T_{h_0}(\Gamma)}\|t - \mathbf{s}_h\|_{T(\Gamma)} + \inf_{\mu_h\in M_h}\|\lambda - \mu_h\|_{-1/2,\Gamma}\right\}. \end{aligned} \tag{4.41}$$

Now using the standard interpolation estimates (cf. [9]) and Lemma 3.3, we get

$$\begin{aligned}\inf_{\mathbf{B}_{h_i}\in X_{h_i}} \|\mathbf{A}_i - \mathbf{B}_{h_i}\|_{\mathbf{curl},\Omega_i} + \inf_{q_{h_i}\in Q_{h_i}} \|p_i - q_{h_i}\|_{1,\Omega_i} \\ \le Ch_i^s\big(\|\mathbf{A}_i\|_{s,\mathbf{curl},\Omega_i} + \|p_i\|_{1+s,\Omega_i}\big)\,,\end{aligned} \tag{4.42}$$

$$\inf_{\mu_h\in M_h} \|\lambda-\mu_h\|_{-1/2,\Gamma} \le Ch_0^s\|\lambda\|_{-1/2+s,\Gamma}. \tag{4.43}$$

Next we introduce a triangulation $\mathcal{T}^{h_0}$ in Ω_1 whose restriction on Γ coincides with Γ_{h_0} and let X_{h_0} be the Nédélec $H(\mathbf{curl}\,,\Omega_1)$-conforming edge element over the mesh $\mathcal{T}^{h_0}$. Then from the definition of $T_{h_0}(\Gamma)$ we know that

$$T_{h_0}(\Gamma) = \{\boldsymbol{\varphi}_h\times\mathbf{n};\ \boldsymbol{\varphi}_h\in X_{h_0}\}\ .$$

Now using the fact that $\mathbf{t} = \alpha_1\mathbf{curl}\,\mathbf{A}_1\times\mathbf{n} := \mathbf{Q}\times\mathbf{n}$, we can easily show that, by (2.1), $\mathbf{Q}\in H^s(\mathbf{curl}\,,\Omega_1)$ and

$$\|\mathbf{Q}\|_{s,\mathbf{curl}\,,\Omega_1} \le C(\|\mathbf{A}_1\|_{s,\mathbf{curl}\,,\Omega_1} + \|p_1\|_{1+s,\Omega_1} + \|\mathbf{f}\|_{s,\Omega_1}). \tag{4.44}$$

Thus we obtain by Lemma 2.3 and Lemma 3.3 that

$$\inf_{\mathbf{s}_h\in T_{h_0}(\Gamma)} \|\mathbf{t}-\mathbf{s}_h\|_{T(\Gamma)} \le C\inf_{\boldsymbol{\varphi}_h\in X_{h_0}} \|\mathbf{Q}-\boldsymbol{\varphi}_h\|_{\mathbf{curl}\,,\Omega_1} \le Ch_0^s\|\mathbf{Q}\|_{s,\mathbf{curl}\,,\Omega_1}. \tag{4.45}$$

Now Theorem 4.4 follows from (4.41)–(4.45). $\square$

To conclude this subsection, we give some remarks on the technical difficulty encountered if the lower-order term $\gamma\beta\mathbf{A}$ in (2.1) is not included. First we show the following.

LEMMA 4.7. *There exists a constant $a^*(h)>0$ such that*

$$\sum_{i=1}^2 \|\mathbf{curl}\,\mathbf{B}_{h_i}\|_{0,\Omega_i}^2 \ge a^*(h)\|\mathbf{B}_h\|_{0,\Omega}^2 \quad \forall\,\mathbf{B}_h\in N_{2h}. \tag{4.46}$$

Proof. First we note that

$$N_{2h} = \Bigg\{\mathbf{B}_h\in X_h;\quad \langle\!\langle \mathbf{s}_h,\mathbf{B}_{h_1}\rangle\!\rangle_{1,\Gamma} - \langle\!\langle \mathbf{s}_h,\mathbf{B}_{h_2}\rangle\!\rangle_{2,\Gamma} = 0 \quad\text{and}\quad \sum_{i=1}^2\int_{\Omega_i}\beta_i\nabla q_{h_i}\cdot\mathbf{B}_{h_i}dx = 0 \quad \forall\,(q_h,\mathbf{s}_h)\in N_{1h}\Bigg\}.$$

To see (4.46) we need only to show that for any $\mathbf{B}_h\in N_{2h}$, $\mathbf{curl}\,\mathbf{B}_{h_i} = 0$ in Ω_i, $(i=1,2)$ implies $\mathbf{B}_h = 0$. Since $\mathbf{curl}\,\mathbf{B}_{h_i} = 0$ in Ω_i, there exists $\varphi_{h_i}\in Q_{h_i}$ such that $\mathbf{B}_{h_i} = \nabla\varphi_{h_i}$ in Ω_i. The choice of φ_{h_1} is unique up to a constant, and we thus may let

$$\langle\varphi_{h_1},1\rangle_\Gamma = \langle\varphi_{h_2},1\rangle_\Gamma\ . \tag{4.47}$$

Now since $\mathbf{B}_h\in N_{2h}$, we get

$$\langle\!\langle \mathbf{s}_h,\nabla\varphi_{h_2}\rangle\!\rangle_{2,\Gamma} - \langle\!\langle \mathbf{s}_h,\nabla\varphi_{h_1}\rangle\!\rangle_{1,\Gamma} = 0 \quad \forall\,\mathbf{s}_h\in T_{h_0}(\Gamma) \tag{4.48}$$

and

$$\sum_{i=1}^2\int_{\Omega_i}\beta_i\nabla q_{h_i}\cdot\nabla\varphi_{h_i}dx = 0 \tag{4.49}$$

for any $q_h \in Q_h$ such that

$$\int_\Gamma (q_{h_1} - q_{h_2})\, \mu_h \, ds = 0 \quad \forall\, \mu_h \in M_h. \tag{4.50}$$

Note that $\langle\!\langle \mathbf{s}_h, \nabla\varphi_{h_i} \rangle\!\rangle_{i,\Gamma} = \langle \mathbf{s}_h, \nabla\varphi_{h_i}\rangle_\Gamma$ since $\mathbf{s}_h, \nabla\varphi_{h_i} \in L^2(\Gamma)^3, i = 1,2$. Thus by Lemma 4.5 we know that the above $\varphi_h \in Q_{h_1} \times Q_{h_2}$ satisfies

$$\int_\Gamma (\varphi_{h_1} - \varphi_{h_2})\, \mu_h \, ds = 0 \quad \forall\, \mu_h \in \bar{M}_h.$$

Coupled with (4.47), we have that $\varphi_h \in Q_{h_1} \times Q_{h_2}$ satisfies (4.50). Thus we can take $q_{h_i} = \varphi_{h_i}$ in (4.49) and obtain $\nabla\varphi_{h_i} = 0$, or $\varphi_{h_i} = 0$ $(i = 1,2)$ since we have (4.47) and $\varphi_{h_2} = 0$ on $\partial\Omega$. This completes the proof. $\square$

The inequality (4.46) does not imply that the bilinear form corresponding to the term $\mathbf{curl}\,(\alpha \mathbf{curl}\,\mathbf{A})$ fulfills the N_{2h}-coercivity condition uniformly in h, thus the abstract results in section 4.1 cannot be applied without the lower-order term. Whether the dependence of $a^*(h)$ in (4.46) on h can be removed remains an interesting open question.

4.4. Verification of inf-sup conditions. In this subsection we show that the assumptions (H2)–(H3) are valid at least when the mesh size h_1 or h_2 is suitably small compared with h_0. Let us first introduce a projection operator R_{h_1} from $H(\mathbf{curl}\,, \Omega_1)$ to X_{h_1}. For any $\mathbf{Q} \in H(\mathbf{curl}\,, \Omega_1)$, $R_{h_1}\mathbf{Q} \in X_{h_1}$ is the unique solution of the equation

$$\int_{\Omega_1} \Big\{ \mathbf{curl}\,(R_{h_1}\mathbf{Q} - \mathbf{Q}) \cdot \mathbf{curl}\, B_{h_1} + (R_{h_1}\mathbf{Q} - \mathbf{Q}) \cdot \mathbf{B}_{h_1} \Big\} dx = 0 \;\; \forall\, \mathbf{B}_{h_1} \in X_{h_1}. \tag{4.51}$$

It is obvious that

$$\|R_{h_1}\mathbf{Q} - \mathbf{Q}\|_{\mathbf{curl}\,,\Omega_1} \le \inf_{\mathbf{B}_{h_1} \in X_{h_1}} \|\mathbf{Q} - \mathbf{B}_{h_1}\|_{\mathbf{curl}\,,\Omega_1},$$

which, together with Lemma 3.3 and the standard density argument, yields

$$\|R_{h_1}\mathbf{Q} - \mathbf{Q}\|_{\mathbf{curl}\,,\Omega_1} \to 0 \quad \text{as} \quad h_1 \to 0. \tag{4.52}$$

Similar to the proof of Lemma 2 in [27], we can show the following lemma which indicates that the convergence in (4.52) is uniform in any compact subset of $H(\mathbf{curl}\,, \Omega_1)$.

LEMMA 4.8. *Let W be a fixed compact subset of $H(\mathbf{curl}\,, \Omega_1)$. Then given any $\varepsilon > 0$, there exists a $\tilde{h}_1 = \tilde{h}_1(\varepsilon, W) > 0$ such that for any $\mathbf{Q} \in W$ and any $0 < h_1 < \tilde{h}_1$,*

$$\|R_{h_1}\mathbf{Q} - \mathbf{Q}\|_{\mathbf{curl}\,,\Omega_1} \le \varepsilon \quad \forall\, \mathbf{Q} \in W.$$

Now we can state the following theorem for (H2).

THEOREM 4.9. *For any given triangulation Γ_{h_0} on the interface Γ, there exists a constant $h_1^* = h_1^*(h_0)$ such that for any $h_1 < h_1^*$, we have*

$$\sup_{\mathbf{B}_{h_1} \in X_{h_1}} \frac{\langle\!\langle \mathbf{s}_h, \mathbf{B}_{h_1} \rangle\!\rangle_{1,\Gamma}}{\|\mathbf{B}_{h_1}\|_{\mathbf{curl}\,,\Omega_1}} \ge \beta^* \|\mathbf{s}_h\|_{T(\Gamma)} \quad \forall\, \mathbf{s}_h \in T_{h_0}(\Gamma), \tag{4.53}$$

where $\beta^* > 0$ is a constant independent of h_0, h_1, h_2.

Proof. Let $D_{h_0} = \{\mathbf{s}_h \in T_{h_0}(\Gamma)\ ; |||\mathbf{s}|||_{1,\Gamma} = 1\}$ be the unit sphere in $T_{h_0}(\Gamma)$. It is clear by Lemma 2.3 that D_{h_0} is compact in $T(\Gamma)$ since $T_{h_0}(\Gamma)$ is a finite dimensional subspace of $T(\Gamma)$. For any $\mathbf{s}_h \in T_{h_0}(\Gamma)$, let $\mathbf{Q} = \mathbf{Q}(\mathbf{s}_h) \in H(\mathbf{curl}\,, \Omega_1)$ be the unique solution of the problem

$$\int_{\Omega_1} (\mathbf{curl}\,\mathbf{Q} \cdot \mathbf{curl}\,\boldsymbol{\varphi} + \mathbf{Q} \cdot \boldsymbol{\varphi})dx = \langle\!\langle \mathbf{s}_h, \boldsymbol{\varphi} \rangle\!\rangle_{1,\Gamma} \quad \forall\, \boldsymbol{\varphi} \in H(\mathbf{curl}\,, \Omega_1). \tag{4.54}$$

From the definition (2.14) we have

$$\|\mathbf{Q}\|_{\mathbf{curl}\,,\Omega_1} \;=\; |||\mathbf{s}|||_{1,\Gamma}. \tag{4.55}$$

Let $W := \{\mathbf{Q} = \mathbf{Q}(\mathbf{s}_h);\ \mathbf{s}_h \in D_{h_0}\}$ be the subspace of $H(\mathbf{curl}\,, \Omega_1)$. It is easy to check that W is compact in $H(\mathbf{curl}\,, \Omega_1)$. Thus by Lemma 4.8, there exists an $h_1^* = h_1^*(h_0)$ such that for $h_1 < h_1^*$,

$$\|\mathbf{Q} - R_{h_1}\mathbf{Q}\|_{\mathbf{curl}\,,\Omega_1} \le 1/2 \quad \forall\, \mathbf{Q} \in W.$$

Thus, for any $\mathbf{s}_h \in T_{h_0}(\Gamma)$ satisfying $|||\mathbf{s}|||_{1,\Gamma} = 1$, we obtain

$$\frac{\langle\!\langle \mathbf{s}_h, R_{h_1}\mathbf{Q} \rangle\!\rangle_{1,\Gamma}}{\|R_{h_1}\mathbf{Q}\|_{\mathbf{curl}\,,\Omega_1}} = \|R_{h_1}\mathbf{Q}\|_{\mathbf{curl}\,,\Omega_1} \ge \|\mathbf{Q}\|_{\mathbf{curl}\,,\Omega_1} - \|\mathbf{Q} - R_{h_1}\mathbf{Q}\|_{\mathbf{curl}\,,\Omega_1} \ge \frac{1}{2},$$

where we have used (4.51), (4.54), and (4.55). This completes the proof of Theorem 4.9 by Lemma 2.3. □

We remark that similar results are valid for the inf-sup condition (H3) using the same arguments as that in the proof of Theorem 4.9. Moreover, by switching the definition of the projection to that on the domain Ω_2, we can easily restate the theorem with h_1 being replaced by h_2, with the corresponding spaces and norms defined on Ω_2.

The requirement on either h_1 or h_2 to be suitable small compared with h_0 in Theorem 4.9 has also been used in [14] to verify the discrete inf-sup conditions for the finite element approximations of an interface problem for the Poisson equations with nonmatching finite element meshes. In the case where we expect that the mesh in one of the subdomains is much finer than the other (say, $h_1 \ll h_2$), the condition on h_1 being suitably small compared with h_0 would appear only to be a very mild restriction on the meshes. How this restriction affects the practical performance of the numerical method and whether this condition could be lifted will be issues to be further examined, both theoretically and through numerical testings, in the future.

5. The time-dependent Maxwell equations. Finally we turn our attention to the main aim of the paper, i.e., to investigate finite element methods for the time-dependent electric field equations (1.7)–(1.10). Again we introduce a Lagrangian multiplier for the divergence constraint (1.8) and consider the problem,

$$\varepsilon\,\partial_{tt}\mathbf{E} + \mathbf{curl}\,(\mu^{-1}\mathbf{curl}\,\mathbf{E}) - \varepsilon\,\nabla p = \;\mathbf{f} \quad \text{in} \quad \Omega \times (0,T), \tag{5.1}$$

$$\operatorname{div}(\varepsilon\,\mathbf{E}) = \;\rho \quad \text{in} \quad \Omega \times (0,T), \tag{5.2}$$

$$[\mathbf{E} \times \mathbf{n}] = 0, \quad [\varepsilon\,\mathbf{E} \cdot \mathbf{n}] = \;\rho_\Gamma \quad \text{on}\ \Gamma \times (0,T), \tag{5.3}$$

$$[p] = 0, \quad [\mu^{-1}\mathbf{curl}\,\mathbf{E} \times \mathbf{n}] = \;\mathbf{f}_\Gamma \quad \text{on}\ \Gamma \times (0,T), \tag{5.4}$$

$$p = 0, \quad \mathbf{E} \times \mathbf{n} = \;0 \quad \text{on} \quad \partial\Omega \times (0,T), \tag{5.5}$$

together with the initial conditions

$$\mathbf{E}(x,0)=\mathbf{E}_0(x) \quad \text{and} \quad \mathbf{E}_t(x,0)=\mathbf{E}_1(x) \quad \text{in} \quad \Omega,$$

where $\mathbf{f} = \partial_t \mathbf{J}$ and $\mathbf{f}_\Gamma = \partial_t \mathbf{J}_\Gamma$, and $\mathbf{E}_1(x) = \varepsilon^{-1}(\mathbf{J}(x,0) + \mathbf{curl}\,\mathbf{H}_0(x))$, which is obtained from (1.1) with $t=0$.

We are going to approximate the system (5.1)–(5.5) using an implicit finite difference scheme in time and the edge element method in space. We shall study only the case with a nonmatching mesh below; the treatment of the case with a matching grid is similar and in fact much simpler. All the notations used in this section are carried over from those in section 4, unless otherwise specified. Let X, Q, and M be the Banach spaces defined in section 4.2; and then we introduce three bilinear forms $a : X \times X \to R$, $b : X \times Q \to R$, and $c : Q \times M \to R$ as follows:

$$a(\mathbf{A},\mathbf{B}) = \sum_{i=1}^{2} \int_{\Omega_i} \mu_i^{-1} \mathbf{curl}\,\mathbf{A}_i \cdot \mathbf{curl}\,\mathbf{B}_i \, dx \quad \forall\, \mathbf{A},\mathbf{B} \in X,$$

$$b(\mathbf{B},(q,\mathbf{s})) = \sum_{i=1}^{2} \int_{\Omega_i} -\varepsilon_i \nabla q_i \cdot \mathbf{B}_i \, dx + \langle\!\langle \mathbf{s},\mathbf{B}_2\rangle\!\rangle_{2,\Gamma} - \langle\!\langle \mathbf{s},\mathbf{B}_1\rangle\!\rangle_{1,\Gamma} \quad \forall\, \mathbf{B} \in X,\ (q,\mathbf{s}) \in Q,$$

$$c((q,\mathbf{s}),\mu) = \langle q_1 - q_2, \mu\rangle_\Gamma \quad \forall\, (q,\mathbf{s}) \in Q,\ \mu \in M.$$

Then the weak formulation of (5.1)–(5.5) reads as follows.

Find $(\mathbf{E},(p,\mathbf{t}),\lambda)$ in the following spaces

$$\mathbf{E} \in H^2(0,T;L^2(\Omega)^3) \cap L^2(0,T;X),\ (p,\mathbf{t}) \in L^2(0,T;Q),\ \lambda \in L^2(0,T;M)$$

such that it satisfies the initial conditions

$$\mathbf{E}(\mathbf{x},0) = \mathbf{E}_0(\mathbf{x}), \qquad \partial_t \mathbf{E}(\mathbf{x},0) = \mathbf{E}_1(\mathbf{x}), \qquad \mathbf{x} \in \Omega \tag{5.6}$$

and the equations

$$(\varepsilon\, \partial_{tt}\mathbf{E},\mathbf{B}) + a(\mathbf{E},\mathbf{B}) + b(\mathbf{B},(p,\mathbf{t})) = (\mathbf{f},\mathbf{B}) - \langle\!\langle \mathbf{f}_\Gamma,\mathbf{B}_2\rangle\!\rangle_{2,\Gamma} \quad \forall\, \mathbf{B} \in X, \tag{5.7}$$

$$b(\mathbf{E},(q,\mathbf{s})) + c((q,\mathbf{s}),\lambda) = (\rho,q) + \langle \rho_\Gamma, q_2\rangle_\Gamma \quad \forall\, (q,\mathbf{s}) \in Q, \tag{5.8}$$

$$c((p,\mathbf{t}),\mu) = 0 \quad \forall\, \mu \in M \tag{5.9}$$

for almost everywhere (a.e.) $t \in (0,T)$.

To see that the system (5.6)–(5.9) admits a unique solution $(\mathbf{E},(p,\mathbf{t}),\lambda)$, one can apply the standard argument of method of lines combining with the results in section 4.2; we omit the details here. We next introduce a fully discrete scheme for the system (5.6)–(5.9). Let us first divide the time interval $(0,T)$ into M equally spaced subintervals using the nodal points

$$0 = t^0 < t^1 < \cdots < t^M = T,$$

with $t^n = n\tau$ and $\tau = T/M$. For a given sequence $\{u^n\}_{n=0}^M$ in $L^2(\Omega)$ or $L^2(\Omega)^3$, we define the first and second order backward finite differences:

$$\partial_\tau u^n = \frac{u^n - u^{n-1}}{\tau}\ , \qquad \partial_\tau^2 u^n = \frac{\partial_\tau u^n - \partial_\tau u^{n-1}}{\tau}.$$

For a continuous mapping $u : [0,T] \to L^2(\Omega)$ or $L^2(\Omega)^3$, we define

$$u^n(\cdot) = u(\cdot, n\tau) \quad \text{and} \quad \bar{u}^n(\cdot) = \frac{1}{\tau}\int_{t^{n-1}}^{t^n} u(\cdot, s)ds$$

for $1 \le n \le M$.

Let $X_h \subset X$, $Q_h \subset Q$, and $M_h \subset M$ be the finite element spaces defined in section 4.3; then the fully discrete finite element approximation to (5.6)–(5.9) can be formulated as follows.

For $n = 0, 1, \ldots, M,$ find $\mathbf{E}_h^n \in X_h,$ $(p_h^n, \mathbf{t}_h^n) \in Q_h$, and $\lambda_h^n \in M_h$ such that

$$\mathbf{E}_h^0 = \pi_h \mathbf{E}_0, \qquad \mathbf{E}_h^0 - \mathbf{E}_h^{-1} = \tau\, \pi_h \mathbf{E}_1, \tag{5.10}$$

and for any $\mathbf{B}_h \in X_h,$ $(q_h, \mathbf{s}_h) \in Q_h$ and $\mu_h \in M_h$ the following equations hold:

$$(\varepsilon\partial_\tau^2 \mathbf{E}_h^n, \mathbf{B}_h) + a(\mathbf{E}_h^n, \mathbf{B}_h) + b(\mathbf{B}_h, (p_h^n, \mathbf{t}_h^n)) = (\bar{\mathbf{f}}^n, \mathbf{B}_h) - \langle\!\langle \bar{\mathbf{f}}_\Gamma^n, \mathbf{B}_{h_2} \rangle\!\rangle_{2,\Gamma}, \tag{5.11}$$

$$b(\mathbf{E}_h^n, (q_h, \mathbf{s}_h)) + c((q_h, \mathbf{s}_h), \lambda_h^n) = (\bar{\rho}^n, q_h) + \langle \bar{\rho}_\Gamma^n, q_{h_2} \rangle_\Gamma, \tag{5.12}$$

$$c((p_h^n, \mathbf{t}_h^n), \mu_h) = 0. \tag{5.13}$$

From Theorem 4.4 we know that under the hypotheses (H1)–(H3), the system (5.11)–(5.13) has a unique solution $\{\mathbf{E}_h^n, (p_h^n, \mathbf{t}_h^n), \lambda_h^n\}$ at each time step n, with $\gamma = \varepsilon/\tau^2$ here. Moreover, we have the following main results of this section.

THEOREM 5.1. *Assume that for some $1/2 < s \le 1$, the solution of the continuous problem* (5.7)–(5.9) *has the regularity*

$$\mathbf{E} \in H^2(0,T; H^s(\mathbf{curl}\,, \Omega_1) \times H^s(\mathbf{curl}\,, \Omega_2)), \qquad \mathbf{E} \in H^3(0,T; L^2(\Omega)),$$
$$p \in H^1(0,T; H^{1+s}(\Omega_1) \times H^{1+s}(\Omega_2)).$$

Moreover, let

$$\mathbf{f} \in H^1(0,T; H^s(\Omega_1)^3 \times H^s(\Omega_2)^3), \qquad \rho \in H^1(0,T; H^s(\Omega_1) \times H^s(\Omega_2)),$$
$$\rho_\Gamma \in H^1(0,T; H^{s-1/2}(\Gamma)) \;\; \textit{and} \;\; \mathbf{f}_\Gamma = \boldsymbol{\varphi} \times \mathbf{n} \;\; \textit{for some} \;\; \boldsymbol{\varphi} \in H^1(0,T; H^s(\mathbf{curl}\,, \Omega_1)).$$

Then we have the following error estimates:

$$\max_{1\le n\le M}\left(\|\partial_\tau \mathbf{E}_h^n - \mathbf{E}_t^n\|_{0,\Omega} + \sum_{i=1}^2 \|\mathbf{E}_h^n - \mathbf{E}^n\|_{\mathbf{curl}\,,\Omega_i} \right) \le C\,\tau + \sum_{i=1}^2 C_i\, h_i^s + C_0\, h_0^s\,.$$

Proof. The proof is standard in the sense that we first estimate the errors between the discrete time-dependent solution and the so-called *elliptic* projection of the exact solution. The desired error estimates will then follow from the triangle inequality and the error estimates obtained for the elliptic projection (namely, Theorem 4.4). We define the projection operator $P_h : X \times Q \times M \to X_h \times Q_h \times M_h$ to be $(\mathbf{A}_h, (p_h, \mathbf{t}_h), \lambda_h) = P_h(\mathbf{A}, (p, \mathbf{t}), \lambda)$, for any $(\mathbf{A}, (p, t), \lambda)$ in $X \times Q \times M$, with $(\mathbf{A}_h, (p_h, \mathbf{t}_h), \lambda_h)$ in $X_h \times Q_h \times M_h$ satisfying

$$\begin{aligned}
(\mathbf{A}_h - \mathbf{A}, \mathbf{B}_h) + a(\mathbf{A}_h - \mathbf{A}, \mathbf{B}_h) + b(\mathbf{B}_h, (p_h - p, \mathbf{t}_h - \mathbf{t})) &= 0 \quad \forall\, \mathbf{B}_h \in X_h,\\
b(\mathbf{A}_h - \mathbf{A}, (q_h, \mathbf{s}_h)) + c((q_h, \mathbf{s}_h), \lambda_h - \lambda) &= 0 \quad \forall\, (q_h, \mathbf{s}_h) \in Q_h,\\
c((p_h - p, \mathbf{t}_h - \mathbf{t}), \mu_h) &= 0 \quad \forall\, \mu_h \in M_h.
\end{aligned}$$

We remark that the term $(\mathbf{A}_h - \mathbf{A}, \mathbf{B}_h)$ above is not necessary for the case with matching finite element grids; see section 3. Now, if we let $\mathbf{B} = \tau^{-1}\mathbf{B}_h \in X_h$, $(q, \mathbf{s}) = \tau^{-1}(q_h, \mathbf{s}_h) \in Q_h$, and $\mu = \tau^{-1}\mu_h \in M_h$ in (5.7)–(5.9) and integrate over (t^{n-1}, t^n), we obtain

$$(\varepsilon\, \partial_\tau \mathbf{E}_t^n, \mathbf{B}_h) + a(\bar{\mathbf{E}}^n, \mathbf{B}_h) + b(\mathbf{B}_h, (\bar{p}^n, \bar{\mathbf{t}}^n)) = (\bar{\mathbf{f}}^n, \mathbf{B}_h) - \langle\!\langle \bar{\mathbf{f}}_\Gamma^n, \mathbf{B}_{h_2} \rangle\!\rangle_{2,\Gamma}\,, \tag{5.14}$$

$$b(\bar{\mathbf{E}}^n, (q_h, \mathbf{s}_h)) + c((q_h, \mathbf{s}_h), \bar{\lambda}^n) = (\bar{\rho}^n, q_h) + \langle \bar{\rho}_\Gamma^n, q_{h_2} \rangle_\Gamma\,, \tag{5.15}$$

$$c((\bar{p}^n, \bar{\mathbf{t}}^n), \mu_h) = 0\,. \tag{5.16}$$

Letting

$$(\eta_h^n, \zeta_h^n, \delta_h^n) = (\mathbf{E}_h^n, (p_h^n, \mathbf{t}_h^n), \lambda_h^n) - P_h(\bar{\mathbf{E}}^n, (\bar{p}^n, \bar{\mathbf{t}}^n), \bar{\lambda}^n)$$

and subtracting equations (5.14)–(5.16) from the equations (5.11)–(5.13), together with the definition of the projection operator P_h, we derive

$$\begin{aligned}(\varepsilon\, \partial_\tau^2 \eta_h^n, \mathbf{B}_h) + a(\eta_h^n, \mathbf{B}_h) + b(\mathbf{B}_h, \zeta_h^n) &= \varepsilon\, (\partial_\tau \mathbf{E}_t^n - \partial_\tau^2 P_h \bar{\mathbf{E}}^n, \mathbf{B}_h)\\ &\quad + (P_h \bar{\mathbf{E}}^n - \bar{\mathbf{E}}^n, \mathbf{B}_h) \quad \forall \mathbf{B}_h \in X_h,\end{aligned} \tag{5.17}$$

$$b(\eta_h^n, (q_h, \mathbf{s}_h)) + c((q_h, \mathbf{s}_h), \delta_h^n) = 0 \quad \forall (q_h, \mathbf{s}_h) \in Q_h\,, \tag{5.18}$$

$$c(\zeta_h^n, \mu_h) = 0 \quad \forall \mu_h \in M_h. \tag{5.19}$$

Taking $\mathbf{B}_h = 2\tau\partial_\tau \eta_h^n$ in (5.17) and using (5.18)–(5.19), we then have

$$\begin{aligned}&\varepsilon \|\partial_\tau \eta_h^n\|^2 - \varepsilon \|\partial_\tau \eta_h^{n-1}\|^2 + a(\eta_h^n, \eta_h^n) - a(\eta_h^{n-1}, \eta_h^{n-1})\\ &\qquad \le 2\tau\, \varepsilon\, (\partial_\tau \mathbf{E}_t^n - \partial_\tau^2 P_h \bar{\mathbf{E}}^n, \partial_\tau \eta_h^n) + 2\tau\, (P_h \bar{\mathbf{E}}^n - \bar{\mathbf{E}}^n, \partial_\tau \eta_h^n).\end{aligned}$$

Now, using the discrete Gronwall's inequality and the estimates we have obtained for the elliptic projections P_h (cf. Theorem 4.4), we get the estimate on $\|\partial_\tau \eta_h^n\|_{0,\Omega}$ and $\|\mathbf{curl}\, \eta_h^n\|_{0,\Omega_i}$. The remaining L^2-norm $\|\eta_h^n\|_{0,\Omega}$ follows from the identity

$$\|\eta_h^n\|_{0,\Omega}^2 = \sum_{k=0}^{n} \tau\, (\partial_\tau \eta_h^n, \eta_h^n) + (\eta_h^0, \eta_h^n).$$

This implies the final estimate given in the theorem with the help of the triangle inequality. We omit the details. ∎

6. Conclusion. In this paper, we have studied the finite element approximations to the stationary and time-dependent Maxwell equations in a polyhedral domain using matching and nonmatching grids. We focus on the particular case where the coefficients are allowed to display discontinuous behavior as they vary from subdomain to subdomain. In many practical electromagnetic applications, such scenarios arise frequently due to the spatial inhomogeneities. Aside from the technical results we have proved in the paper, it is worthwhile to point out that the freedom in choosing nonmatching meshes for different subdomains will be a nice feature when developing effective numerical methods to simulate the complicated spatial structures. The abstract framework for the nonmatching grids outlined here will also be useful to the development of domain decomposition methods for the resulting linear (or even nonlinear) algebraic systems. We will pursue this and other issues as well as actual numerical testings in the future.

Appendix. The proofs of some technical results quoted earlier in the paper are provided in this appendix.

A1. Proof of Lemma 2.2. The argument is similar to the proof for extensions of $W^{1,p}$ functions ($1 \le p < \infty$); see [16], for example. But our construction here contains essential differences.

Step 1. Since ∂U is Lipschitz, for any point x on ∂U, there exist, upon rotating and relabeling the coordinate axes if necessary, a system of orthogonal coordinates (y_1, y_2, y_3), a cube C_x containing x, $C_x = \Pi_{i=1}^3(-a_i, a_i)$, and a Lipschitz continuous function $\Phi : (-a_1, a_1) \times (-a_2, a_2) \to (-a_3, a_3)$ such that

$$U \cap C_x = \{y \in C_x;\ y_3 > \Phi(y_1, y_2)\},$$
$$\partial U \cap C_x = \{y \in C_x;\ y_3 = \Phi(y_1, y_2)\}.$$

Let C'_x be the reduced cube $C'_x = (-a_1, a_1) \times (-a_2, a_2) \times (-a_3/2, a_3/2)$. We write in what follows

$$U^+ = U \cap C'_x, \quad U^- = C'_x - \bar{U}.$$

It is clear that if we define

$$\mathbf{n}(y) = \left(\frac{\partial \Phi}{\partial y_1}(y_1, y_2), \frac{\partial \Phi}{\partial y_2}(y_1, y_2), -1\right) \quad \forall y = (y_1, y_2, y_3) \in C_x,$$

then $\mathbf{n}$ is normal to ∂U for $y \in \partial U \cap C_x$.

Step 2. Let $\mathbf{v} = (v_1, v_2, v_3) \in C^1(\bar{U})^3$ and suppose for the moment that $\mathrm{supp}(\mathbf{v}) \subset C'_x \cap \bar{U}$. Note that if $y = Qx + b$ is the coordinate transformation with $Q \in R^{3\times 3}$ being orthogonal matrix, then by the same technique as used by Nédélec [21], we know that

$$\tilde{\mathbf{v}}(y) = Q\mathbf{v}(Q^T y - Q^T b)$$

is in $H(\mathbf{curl}, U^+)$ in the coordinates y if and only if $\mathbf{v} \in H(\mathbf{curl}; U^+)$ in the coordinates x. Thus, without lost of generality, we can take Q as the identity matrix and $b = 0$ in the following. Let $e_3 = (0, 0, 1)$ and $z = y + 2(\Phi(y_1, y_2) - y_3)e_3$. Notice that we have $z \in \bar{U}^+$ for $y \in \bar{U}^-$, and we set

$$\mathbf{v}^+(y) = \mathbf{v}(y) \quad \text{if } y \in \bar{U}^+$$
$$\mathbf{v}^-(y) = \mathbf{v}(z) + 2v_3(z)\mathbf{n}(y) \quad \text{if } y \in \bar{U}^-.$$

Note that, on $\partial U \cap C'_x$, we obviously have $z = y$ and

$$\mathbf{v}^+(y) \times \mathbf{n} = \mathbf{v}^-(y) \times \mathbf{n}\ .$$

Step 3. We have

$$\| \mathbf{v}^- \|_{\mathbf{curl}; U^-} \le C \| \mathbf{v} \|_{\mathbf{curl}; U}\ . \tag{6.1}$$

To prove this, let $\{\Phi_k\}$ be a sequence of C^∞ functions such that (cf. [16, p. 136])

$$\Phi_k \ge \Phi, \quad \sup_k \| D\Phi_k \|_{L^\infty} < +\infty, \quad \Phi_k \to \Phi, D\Phi_k \to D\Phi \text{ uniformly a.e.},$$

and let $z^k = y + 2(\Phi_k(y_1, y_2) - y_3)e_3$ with

$$\mathbf{n}_k(y) = \left(\frac{\partial \Phi_k}{\partial y_1}(y_1, y_2), \frac{\partial \Phi_k}{\partial y_2}(y_1, y_2), -1\right) \quad \forall y = (y_1, y_2, y_3) \in C_x\ .$$

For $y \in \bar{U}^-$, denote

$$\mathbf{v}^k(y) = \mathbf{v}(z^k) + 2v_3(z^k)\mathbf{n}_k(y).$$

Then, for $y \in \bar{U}^-$, simple calculations yield

$$\begin{aligned}
\nabla_y \times \mathbf{v}(z^k) &= \nabla_z \times \mathbf{v}(z^k) + 2\mathbf{n}_k(y) \times \frac{\partial}{\partial z_3}\mathbf{v}(z^k)\ , \\
\nabla_y \times \big(v_3(z^k)\mathbf{n}_k(y)\big) &= \left(\nabla_z v_3(z^k) + 2\frac{\partial v_3}{\partial z_3}(z^k)\mathbf{n}_k(y)\right) \times \mathbf{n}_k(y) \\
&\quad + v_3(z^k)\nabla_y \times \mathbf{n}_k(y) = \nabla_z v_3(z^k) \times \mathbf{n}_k(y)\ .
\end{aligned}$$

Thus we have, as $k \to \infty$,

$$\mathbf{curl}\,\mathbf{v}^k(y) \to \mathbf{curl}_{\,z}\mathbf{v}(z) + 2\mathbf{n}(y) \times \left(\frac{\partial v_1}{\partial z_3} - \frac{\partial v_3}{\partial z_1}, \frac{\partial v_2}{\partial z_3} - \frac{\partial v_3}{\partial z_2}, 0\right)(z)$$

a.e. for y in U^-. Now we obtain, for any $\boldsymbol{\varphi} \in C_0^\infty(U^-)^3$,

$$\begin{aligned}
&\int_{U^-} \mathbf{v}^- \cdot \mathbf{curl}\,\boldsymbol{\varphi}\, dy = \lim_{k\to\infty} \int_{U^-} \mathbf{v}^k \cdot \mathbf{curl}\,\boldsymbol{\varphi}\, dy \\
&= \lim_{k\to\infty} \int_{U^-} \mathbf{curl}\,\mathbf{v}^k \cdot \boldsymbol{\varphi}\, dy \\
&= \int_{U^-} \left[\mathbf{curl}_{\,z}\mathbf{v}(z) + 2\mathbf{n}(y) \times \left(\frac{\partial v_1}{\partial z_3} - \frac{\partial v_3}{\partial z_1}, \frac{\partial v_2}{\partial z_3} - \frac{\partial v_3}{\partial z_2}, 0\right)(z)\right] \cdot \boldsymbol{\varphi}\, dy\ .
\end{aligned}$$

Recall that $\| D\Phi \|_{L^\infty} < +\infty$, we get $\| \mathbf{n} \|_{L^\infty} < +\infty$, and thus

$$\| \mathbf{v}^- \|_{\mathbf{curl}\,;U^-} \le C \| \mathbf{v} \|_{\mathbf{curl}\,;U}$$

by change of variable formula.

Step 4. Define

$$E\mathbf{v} = \begin{cases} \mathbf{v}^+ & \text{in} \quad \bar{U}^+, \\ \mathbf{v}^- & \text{in} \quad U^-, \\ 0 & \text{in} \quad R^3 - (\bar{U}^+ \cup U^-). \end{cases}$$

Note that $E\mathbf{v} \times \mathbf{n}$ is continuous on $\partial U \cap C'_x$ and $\operatorname{supp}(E\mathbf{v}) \subset C'_x \subset D$. Now it is easy to see by using (6.1) that $E\mathbf{v} \in H(\mathbf{curl}\,; R^3)$ and

$$\| E\mathbf{v} \|_{\mathbf{curl}\,;R^3} \le C \| \mathbf{v} \|_{\mathbf{curl}\,;U}.$$

This completes the proof in the case that $\mathbf{v}$ is C^1, with support in $C'_x \cap \bar{U}$.

Step 5. The rest of the argument is standard. We first assume $\mathbf{v} \in C^1(\bar{U})$ but then drop the restriction on its support. By using the compactness of ∂U and the partition of unity, we can show that there exists a $E\mathbf{v} \in H(\mathbf{curl}\,; R^3)$ with support in D such that

$$\| E\mathbf{v} \|_{\mathbf{curl}\,;R^3} \le C \| \mathbf{v} \|_{\mathbf{curl}\,;U}.$$

Finally, if $\mathbf{v} \in H(\mathbf{curl}\,; U)$, we approximate $\mathbf{v}$ by functions $\mathbf{v}_k \in C^1(\bar{U})$ due to the density of $C^1(\bar{U})$ in $H(\mathbf{curl}\,; U)$ (see, for example, [17]), and set

$$E\mathbf{v} = \lim_{k\to\infty} E\mathbf{v}_k.$$

This concludes the proof of the extension theorem. $\square$

A2. Proofs of the abstract results in section 4.1. Here, the proofs for the abstract results stated in section 4.1 are provided.

Let X, Q, and M be three real Hilbert spaces with norms $\|\cdot\|_X$, $\|\cdot\|_Q$, and $\|\cdot\|_M$, respectively, and let X', Q', and M' be their corresponding dual spaces with the norms $\|\cdot\|_{X'}$, $\|\cdot\|_{Q'}$, and $\|\cdot\|_{M'}$. We will use the same notation $\langle\cdot,\cdot\rangle$ to denote the duality pairings between X and X', Q and Q', M and M'. Suppose there are three given continuous bilinear forms

$$a : X \times X \to R, \qquad b : X \times Q \to R, \qquad c : Q \times M \to R,$$

and their operator norms are denoted as $\|a\|$, $\|b\|$, and $\|c\|$. With the bilinear forms $b(\cdot,\cdot)$ and $c(\cdot,\cdot)$, we associate two linear operators $B \in \mathcal{L}(X, Q')$ and $C \in \mathcal{L}(Q, M')$ with their dual operators $B' \in \mathcal{L}(Q, X')$ and $C' \in \mathcal{L}(M, Q')$ defined as follows:

$$\begin{aligned}\langle B'q, v\rangle = \langle q, Bv\rangle = b(v, q) \quad \forall\, v \in X,\ q \in Q,\\ \langle C'\mu, q\rangle = \langle \mu, Cq\rangle = c(q, \mu) \quad \forall\, q \in Q,\ \mu \in M.\end{aligned}$$

Later we will use the notation $V = \ker(B)$ and $V^0 = \{\tilde{f} \in X';\ \langle \tilde{f}, v\rangle = 0, \quad v \in V\}$. To study the existence and uniqueness of the problem (4.1)–(4.3) we first recall the classical results of the Babuska–Brezzi theory (cf., for example, [6], [17]).

LEMMA A.1. *The following three properties are equivalent:*

(i) *There exists a constant $\beta > 0$ such that*

$$\inf_{q\in Q} \sup_{v\in X} \frac{b(v, q)}{\|v\|_X \|q\|_Q} \geq \beta. \tag{6.2}$$

(ii) *The operator B' is an isomorphism from Q onto V^0 and*

$$\|B'q\|_{X'} \geq \beta\|q\|_Q \quad \forall\, q \in Q.$$

(iii) *The operator B is an isomorphism from $V^\perp$ onto Q' and*

$$\|Bv\|_{Q'} \geq \beta\|v\|_X \quad \forall\, v \in V^\perp.$$

LEMMA A.2. *Assume that the bilinear form $a(\cdot,\cdot)$ is V-elliptic, i.e., there exists a constant $\alpha > 0$ such that*

$$a(v, v) \geq \alpha\|v\|_X^2 \quad \forall\, v \in V,$$

and the bilinear form $b(\cdot,\cdot)$ satisfies the inf-sup *condition* (6.2). *Then there exists a unique solution $(u, p) \in X \times Q$ to the following problem:*

$$\begin{aligned}a(u, v) + b(v, p) = \langle f, v\rangle \quad \forall\, v \in X,\\ b(u, q) = \langle g, q\rangle \quad \forall\, q \in Q.\end{aligned}$$

We now come to the proofs of Lemmas 4.1–4.2 in section 4.1.

Proof of Lemma 4.1. For any given $\chi \in M'$, by Lemma A.1 and the inf-sup condition (4.8) there exists a unique $p_\perp \in N_1^\perp$ such that $C\, p_\perp = \chi$. Now by Lemma A.2 and (4.6)–(4.7), we know that there is a unique $(u, p_0) \in N_2 \times N_1$ such that

$$Au + B'p_0 = f - B'p_\perp \text{ in } X', \tag{6.3}$$

$$Bu = g \text{ in } N_1'. \tag{6.4}$$

From (6.4) we know that $g - Bu \in N_1^0$, where

$$N_1^0 = \{\ell \in N_1';\ \langle l, q\rangle = 0 \quad \forall\, q \in N_1\}.$$

Thus, again by (4.8) and Lemma A.1, there exists a unique $\lambda \in M$ such that

$$C'\lambda = g - Bu \quad \text{in} \quad Q'.$$

This proves Lemma 4.1 by letting $p = p_0 + p_\perp \in Q$. ∎

For the proof of Lemma 4.2 we need the following lemma, for which we introduce

$$\begin{aligned} Q_h(\chi) &= \{q_h \in X_h;\ c(q_h, \mu_h) = \langle \chi, \mu_h\rangle\ \forall\, \mu_h \in M_h\},\\ X_h(g) &= \{v_h \in X_h;\ b(v_h, q_h) = \langle g, q_h\rangle\ \forall\, q_h \in N_{1h}\}. \end{aligned}$$

Clearly we have $Q_h(0) = N_{1h}$ and $X_h(0) = N_{2h}$.

LEMMA A.3. *With the* inf-sup *conditions* (4.15)–(4.16), *we have the following estimates:*

$$\begin{aligned} \inf_{q_h \in Q_h(\chi)} \|p - q_h\|_Q &\le \left(1 + \frac{\|c\|}{c^*}\right) \inf_{q_h \in Q_h} \|p - q_h\|_Q,\\ \inf_{w_h \in X_h(g)} \|u - w_h\|_X &\le \left(1 + \frac{\|b\|}{b^*}\right) \inf_{v_h \in X_h} \|u - v_h\|_X + \frac{\|c\|}{b^*} \inf_{\mu_h \in M_h} \|\lambda - \mu_h\|_M. \end{aligned}$$

The proof of Lemma A.3 follows as a minor modification of the proof for the classical Babuska–Brezzi theory (cf., for example, [17, p. 114]).

Proof of Lemma 4.2. We argue the proof in the following three steps.

Step 1. Since $u_h \in X_h(g)$, then for any $w_h \in X_h(g)$, we have $v_h = u_h - w_h \in N_{2h}$. Thus, using (4.2) and (4.10), we obtain

$$\begin{aligned} a(v_h, v_h) = a(u_h - w_h, v_h) &= a(u_h - u, v_h) + a(u - w_h, v_h)\\ &= b(v_h, p - p_h) + a(u - w_h, v_h). \end{aligned}$$

Now for any $q_h \in Q_h(\chi)$ we have $p_h - q_h \in Q_h(0) = N_{1h}$, which yields

$$b(v_h, p_h - q_h) = 0 \quad \forall\, q_h \in Q_h(\chi).$$

Note that $v_h \in N_{2h}$; hence, for any $q_h \in Q_h(\chi)$ and $w_h \in X_h(g)$,

$$a^* \|v_h\|_X^2 \le a(v_h, v_h) = b(v_h, p - q_h) + a(u - w_h, v_h) \quad \forall\, w_h \in X_h(g), \quad q_h \in Q_h(\chi),$$

which, together with the triangle inequality and Lemma A.3, implies

$$\|u - u_h\|_X \le C \left\{ \inf_{v_h \in X_h} \|u - v_h\|_X + \inf_{q_h \in Q_h} \|p - q_h\|_Q + \inf_{\mu_h \in M_h} \|\lambda - \mu_h\|_M \right\}.$$

Step 2. For any $\mu_h \in M_h$, it follows from (4.16), (4.2), and (4.10) that

$$\begin{aligned} \|\lambda_h - \mu_h\|_M &\le \frac{1}{c^*} \sup_{q_h \in Q_h} \frac{c(q_h, \lambda_h - \mu_h)}{\|q_h\|_Q}\\ &= \frac{1}{c^*} \sup_{q_h \in Q_h} \frac{b(u - u_h, q_h) + c(q_h, \lambda - \mu_h)}{\|q_h\|_Q}\\ &\le \frac{1}{c^*} (\|b\|\ \|u - u_h\|_X + \|c\|\ \|\lambda - \mu_h\|_M), \end{aligned}$$

which, along with the result from Step 1, yields

$$\|\lambda - \lambda_h\|_M \le C \left\{ \inf_{v_h \in X_h} \|u - v_h\|_X + \inf_{q_h \in Q_h} \|p - q_h\|_Q + \inf_{\mu_h \in M_h} \|\lambda - \mu_h\|_M \right\}.$$

Step 3. For any $q_h \in Q_h(\chi)$, since $p_h - q_h \in N_{1h}$, by means of (4.15), (4.1), and (4.9) we have

$$\begin{aligned}
\|p_h - q_h\|_Q &\le \frac{1}{b^*} \sup_{v_h \in X_h} \frac{b(v_h, p_h - q_h)}{\|v_h\|_X} \\
&= \frac{1}{b^*} \sup_{v_h \in X_h} \frac{a(u - u_h, v_h) + b(v_h, p - q_h)}{\|v_h\|_X} \\
&\le \frac{1}{b^*} \big(\|a\| \, \|u - u_h\|_X + \|b\| \, \|p - q_h\|_Q \big) \quad \forall\, q_h \in Q_h(\chi),
\end{aligned}$$

which, combined with Lemma A.3 and the result from Step 1, implies

$$\|p - p_h\|_Q \le C \left\{ \inf_{v_h \in X_h} \|u - v_h\|_X + \inf_{q_h \in Q_h} \|p - q_h\|_Q + \inf_{\mu_h \in M_h} \|\lambda - \mu_h\|_M \right\}.$$

This completes the proof of Lemma 4.2. $\square$

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