SIAM J. NUMER. ANAL.
Vol. 41, No. 1, pp. 37–63

CONVERGENCE ANALYSIS OF A FINITE VOLUME METHOD FOR MAXWELL'S EQUATIONS IN NONHOMOGENEOUS MEDIA*

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Abstract. In this paper, we analyze a recently developed finite volume method for the time-dependent Maxwell's equations in a three-dimensional polyhedral domain composed of two dielectric materials with different parameter values for the electric permittivity and the magnetic permeability. Convergence and error estimates of the numerical scheme are established for general nonuniform tetrahedral triangulations of the physical domain. In the case of nonuniform rectangular grids, the scheme converges with second order accuracy in the discrete L^2-norm, despite the low regularity of the true solution over the entire domain. In particular, the finite volume method is shown to be superconvergent in the discrete $H(\mathrm{curl};\Omega)$-norm. In addition, the explicit dependence of the error estimates on the material parameters is given.

Key words. finite volume method, Maxwell's equations, inhomogeneous medium, stability, convergence

AMS subject classifications. 65M12, 65M15, 78-08

PII. S0036142901398453

1. Introduction.

Let Ω be a general polyhedral domain in $\mathbb{R}^3$, occupied by a material with electric permittivity ε and magnetic permeability μ. Maxwell's equations state that

$$\varepsilon \frac{\partial \mathbf{E}}{\partial t} - \mathbf{curl}\ \mathbf{H} = \mathbf{J} \quad \text{in} \quad \Omega \times (0,T), \tag{1.1}$$

$$\mu \frac{\partial \mathbf{H}}{\partial t} + \mathbf{curl}\ \mathbf{E} = \mathbf{0} \quad \text{in} \quad \Omega \times (0,T), \tag{1.2}$$

$$\mathrm{div}(\varepsilon \mathbf{E}) = \rho \quad \text{in} \quad \Omega \times (0,T), \tag{1.3}$$

$$\mathrm{div}(\mu \mathbf{H}) = 0 \quad \text{in} \quad \Omega \times (0,T), \tag{1.4}$$

where $\mathbf{E} = \mathbf{E}(x,t)$ and $\mathbf{H} = \mathbf{H}(x,t)$ denote the electric and magnetic fields, $\mathbf{J} = \mathbf{J}(x,t)$ denotes the applied current density, and $\rho = \rho(x,t)$ denotes the charge density. This paper is concerned with the case where the domain Ω is composed of two distinct dielectric materials. Let Ω_1 be a polyhedral subdomain strictly lying inside Ω, occupied by a material with electric permittivity ε_1 and magnetic permeability μ_1, and let $\Omega_2 = \Omega \backslash \bar{\Omega}_1$ be occupied by another material with electric permittivity ε_2 and magnetic permeability μ_2. For ease of exposition, we shall consider only the case where the parameters ε_i and μ_i are constant functions in Ω_i, $i = 1, 2$, but possibly with great differences in their values. We remark that our subsequent analyses can be

*Received by the editors November 16, 2001; accepted for publication (in revised form) June 4, 2002; published electronically February 12, 2003.
http://www.siam.org/journals/sinum/41-1/39845.html

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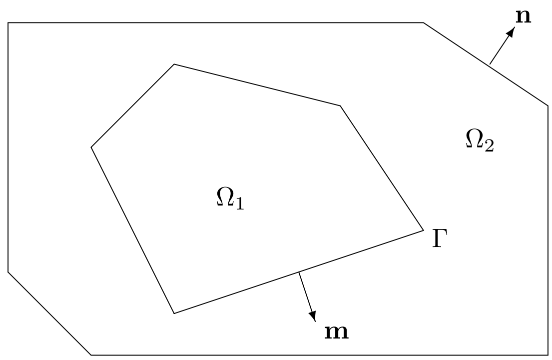

FIG. 1. *Two-dimensional cross-section of dielectric materials Ω_1, Ω_2 and their interface Γ.*

naturally extended to the case with piecewise smooth coefficients as well as multiple subdomains for which our methods have broad applications [3, 11].

Let $\Gamma = \partial\Omega_1$ be the boundary of Ω_1 with a unit outward normal vector $\mathbf{m}$, and let $\partial\Omega$ be the boundary of Ω with a unit outward normal vector $\mathbf{n}$; see Figure 1. We supplement the system (1.1)–(1.4) with the perfect conductor boundary condition and the initial condition given by

$$\mathbf{E} \times \mathbf{n} = \mathbf{0} \quad \text{on} \quad \partial\Omega \times (0, T)\ , \tag{1.5}$$

$$\mathbf{E}(x, 0) = \mathbf{E}_0(x) \quad \text{and} \quad \mathbf{H}(x, 0) = \mathbf{H}_0(x) \quad \forall x \in \Omega. \tag{1.6}$$

It is well known [3, 19] that the electric and magnetic fields $\mathbf{E}$ and $\mathbf{H}$ satisfy the following physical jump conditions across the interface Γ:

$$[\mathbf{E} \times \mathbf{m}] = \mathbf{0}, \quad [\varepsilon\mathbf{E} \cdot \mathbf{m}] = \rho_\Gamma, \tag{1.7}$$

$$[\mathbf{H} \times \mathbf{m}] = \mathbf{0}, \quad [\mu\mathbf{H} \cdot \mathbf{m}] = 0, \tag{1.8}$$

where $\rho_\Gamma = \rho_\Gamma(x)$ is the surface charge density and, throughout this paper, the jump of any function f across the interface Γ is defined by

$$[f] := f_2|_\Gamma - f_1|_\Gamma,$$

where $f_i = f|_{\Omega_i}$ for $i = 1, 2$.

In addition, we have the following constitutive relations:

$$\mathbf{D} = \varepsilon\mathbf{E}, \quad \mathbf{B} = \mu\mathbf{H}, \tag{1.9}$$

where $\mathbf{D}$ and $\mathbf{B}$ are the electric flux density and the magnetic flux density, respectively.

Over the past few decades, numerical methods for solving Maxwell's equations in homogeneous media have received much attention [11, 20]. The simple and popular Yee's scheme was proposed in 1966 [21], though its convergence analysis was not available until the work by Monk and Süli for nonuniform rectangular grids [14]. In order to handle domains with complicated geometry, both finite element and finite volume methods have been widely studied. For example, some fully discrete finite element methods were used to solve the decoupled time-dependent Maxwell's equations by Monk [13] and Raviart [18]. Second order convergence for the stationary case was established there, while a convergence analysis for the fully discrete time-dependent case was given by Ciarlet and Zou [7]. Chen and Yee proposed a finite volume method to solve Maxwell's equations in [4]. Convergence analyses for both semidiscrete and fully discrete schemes were given by Nicolaides and Wang [16].

For most real applications, however, one is often confronted with the solution of Maxwell's equations in nonhomogeneous media. Many of the aforementioned numerical methods either are not directly applicable or become inefficient (with lower order convergence) for these problems due to different physical characteristics reflected by the electric permittivities and magnetic permeabilities of different media, and due to the extra jump conditions the electric and magnetic fields need to satisfy on the interface; see (1.7)–(1.8). Several attempts have been made to handle the interface Maxwell's problems [4, 5, 20]. For example, Chen and Yee studied a hybrid FDTD/FVTD method for the interface problem [4], assuming that both the tangential components of the electric and magnetic fields are continuous across the interface and the electric field is tangentially piecewise constant on the interface. Chen, Du, and Zou [5] proposed an edge finite element method for solving Maxwell's system with general interface conditions and developed a general framework for its convergence analysis.

Recently, Chung and Zou presented a new finite volume method for Maxwell's equations in nonhomogeneous media [6], together with numerical experiments. In this paper, we will give the convergence analysis of the method for general tetrahedral triangulations. As in many interface problems, the regularity of the analytical solution of Maxwell's system in the entire physical domain is very low, which makes the convergence analysis very difficult. Regardless, we will show that, without making any extra regularity assumptions beyond those that are used for the case of a homogeneous medium [14, 16], the method under consideration is first order convergent for general tetrahedral triangulations and second order convergent for general nonuniform rectangular grids. Furthermore, it is shown that the proposed method has superconvergence in a discrete $H(\mathrm{curl};\Omega)$-norm, and the explicit dependence of the error estimates on the physical material parameters is given. To our knowledge, this seems to be the first rigorous work so far on the convergence of a finite volume method for Maxwell's equations with discontinuous coefficients.

We end this section with some notational conventions to be used in the subsequent analysis. For a nonnegative integer m and $1 \le p < \infty$, we use $W^{m,p}(\Omega)$ to denote the standard Sobolev space equipped with the norm [1]

$$\|u\|_{W^{m,p}(\Omega)} = \left(\sum_{0\le|\alpha|\le m} \|D^\alpha u\|^p_{L^p(\Omega)} \right)^{1/p}$$

and the seminorm

$$|u|_{W^{m,p}(\Omega)} = \left(\sum_{|\alpha|=m} \|D^\alpha u\|^p_{L^p(\Omega)} \right)^{1/p} .$$

Here $D^\alpha u$ denotes the αth order weak derivative of u. In addition, we define [10]

$$H(\mathrm{curl};\Omega) = \{\mathbf{u} \in L^2(\Omega)^3; \quad \mathbf{curl}\ \mathbf{u} \in L^2(\Omega)^3\},$$

with its seminorm and norm given by

$$|\mathbf{u}|_{H(\mathrm{curl};\Omega)} = \|\mathbf{curl}\ \mathbf{u}\|_{L^2(\Omega)^3}; \quad \|\mathbf{u}\|_{H(\mathrm{curl};\Omega)} = \{\|\mathbf{u}\|^2_{L^2(\Omega)^3} + \|\mathbf{curl}\ \mathbf{u}\|^2_{L^2(\Omega)^3}\}^{\frac{1}{2}} ,$$

respectively. Furthermore, for some $0 < \lambda < 1$, $C^{m,\lambda}(\Omega)$ denotes the standard Hölder spaces of functions whose mth order derivatives are Hölder continuous with exponent λ. The same definitions are adopted on Ω_1 and Ω_2.

We use $L^p(0,T;\mathbf{X})$ to denote the space of all L^p integrable functions $\mathbf{u}(t,\cdot)$ from $[0,T]$ into the Banach space $\mathbf{X}$, and we also define [12]

$$W^{m,p}(0,T;\mathbf{X}) = \left\{ \mathbf{u} \in L^p(0,T;\mathbf{X}); \quad \frac{\partial^\alpha \mathbf{u}}{\partial t^\alpha} \in L^p(0,T;\mathbf{X}) \quad \forall |\alpha| \leq m \right\},$$

with norm

$$\|\mathbf{u}\|_{W^{m,p}(0,T;\mathbf{X})} = \left\{ \sum_{0\leq|\alpha|\leq m} \left\| \frac{\partial^\alpha \mathbf{u}}{\partial t^\alpha} \right\|_{\mathbf{X}}^p \right\}^{1/p}.$$

When $p = 2$, we set $H^m(\Omega) = W^{m,2}(\Omega)$ and $H^m(0,T;\mathbf{X}) = W^{m,2}(0,T;\mathbf{X})$.

The rest of the paper is organized as follows. Some discrete vector fields and the finite volume method are introduced in sections 2 and 3, respectively. In section 4, we give a discussion of the discrete divergence constraints and stability. The convergence analysis for the general tetrahedral triangulation and the convergence analysis for the case of a nonuniform rectangular grid are given in section 5. Some concluding remarks are given in section 6.

2. Discrete vector fields. We now discuss the triangulation of the domain Ω. We use the Voronoi–Delaunay triangulation [9], which enjoys many elegant geometric properties that allow us to derive the numerical schemes in the subsequent sections. We adopt the notation developed by Nicolaides [15], Nicolaides and Wang [16], and Nicolaides and Wu [17], where a finite volume method was proposed for solving Maxwell's equations with smooth physical coefficients ε and μ.

We first triangulate Ω using the standard tetrahedral elements, which are called the *primal elements*. The triangulation is chosen so that the faces of the primal elements are aligned with the interface Γ. A primal element with at least one face lying on Γ is called an *interface primal element*, and a primal face (edge) lying on Γ is called an interface primal face (edge).

The *dual elements* are the Voronoi polyhedra formed by connecting the circumcenters of adjacent primal elements. Those dual elements (faces and edges) separated by the interface Γ into two parts lying in Ω_1 and Ω_2, respectively, are called the interface dual elements (faces and edges). The definitions and convergence analysis related to dual elements are much more complicated than those related to primal elements, due to the interface. From geometry, it is known that each primal edge is perpendicular to and is in one-to-one correspondence with a dual face, and each dual edge is perpendicular to and in one-to-one correspondence with a primal face.

For the subsequent convergence analysis, we assume that all dihedral angles of each tetrahedron are uniformly acute and the triangulation restricted to each subdomain satisfies

$$K_r \leq \frac{h^r_{\max}}{h^r_{\min}} \leq \tilde{K}_r, \qquad r = 1, 2\ , \tag{2.1}$$

where $h^r_{\max}$ and $h^r_{\min}$ are, respectively, the local maximum and minimum side lengths of adjacent primal and dual elements in Ω_r, and K_r and $\tilde{K}_r$ are two positive constants.

Let N and L be the numbers of primal and dual elements, respectively, and let F be the number of primal faces (dual edges) and M the number of primal edges (dual faces). Assume that these quantities are numbered sequentially in some order. The individual elements, faces, edges, and nodes of the primal mesh are denoted by τ_i, κ_j, σ_k, and ν_l, respectively. Those quantities related to the dual mesh are denoted by the primed forms such as τ_i', κ_j', σ_k', and ν_l'. The area of κ_j is denoted by s_j, and the length of σ_k is given by h_k. A direction is assigned to each primal and dual edge by the rule that positive direction is from low to high node number. A direction is also assigned to each primal (dual) face so that it is the same as that of the corresponding dual (primal) edge. We denote by F_1 the number of interior primal faces (dual edges) and by M_1 the number of interior primal edges (dual faces). For each dual edge σ_j' of length h_j', we define a scaled length:

$$\bar{h}_j' = \begin{cases} \frac{1}{\mu_1} h_j' & \text{if} \quad \sigma_j' \in \Omega_1, \\ \frac{1}{\mu_2} h_j' & \text{if} \quad \sigma_j' \in \Omega_2, \\ (\frac{1}{\mu_1} a_j + \frac{1}{\mu_2}(1 - a_j)) h_j' & \text{otherwise,} \end{cases}$$

where $0 < a_j < 1$ is the ratio of the length of the portion of σ_j' that belongs to Ω_1 over the length of σ_j'. For any u and v in $\mathbb{R}^{F_1}$, we introduce a mesh and parameter dependent inner product defined by

$$(u, v)_W := \sum_{\kappa_j \subset \Omega} u_j v_j s_j \bar{h}_j' = (Su, D'v) = (D'u, Sv), \tag{2.2}$$

where $S := \mathrm{diag}(s_j)$ and $D' := \mathrm{diag}(\bar{h}_j')$ are $F_1 \times F_1$ diagonal matrices and $(\cdot, \cdot)$ denotes the standard Euclidean inner product. Similarly, for each dual face κ_j' with area s_j', we define a scaled area:

$$\bar{s}_j' = \begin{cases} \varepsilon_1 s_j' & \text{if} \quad \kappa_j' \in \Omega_1, \\ \varepsilon_2 s_j' & \text{if} \quad \kappa_j' \in \Omega_2, \\ (\varepsilon_1 b_j + \varepsilon_2 (1 - b_j)) s_j' & \text{otherwise,} \end{cases}$$

where $0 < b_j < 1$ is the ratio of the area of the portion of κ_j' that belongs to Ω_1 over the area of κ_j'. Also, we define a mesh and parameter dependent inner product in $\mathbb{R}^{M_1}$ by

$$(u, v)_{W'} := \sum_{\kappa_j' \subset \Omega} u_j v_j \bar{s}_j' h_j = (S'u, Dv) = (Du, S'v), \tag{2.3}$$

where $S' := \mathrm{diag}(\bar{s}_j')$ and $D := \mathrm{diag}(h_j)$ are $M_1 \times M_1$ diagonal matrices.

For any $\sigma_j \in \partial\kappa_i$, we say that σ_j is oriented positively along $\partial\kappa_i$ if the direction of σ_j agrees with the one of $\partial\kappa_i$ formed by the right-hand rule with the thumb pointing in the direction of σ_i'. Otherwise, we say that σ_j is oriented negatively along $\partial\kappa_i$. For each interior primal face κ_i, we define its discrete circulation by

$$(Cu)_{\kappa_i} := \sum_{\sigma_j \subset \partial\kappa_i} u_j \tilde{h}_j, \tag{2.4}$$

where

$$\tilde{h}_j = \begin{cases} h_j & \text{if } \sigma_j \text{ is oriented positively along } \partial\kappa_i, \\ -h_j & \text{if } \sigma_j \text{ is oriented negatively along } \partial\kappa_i. \end{cases}$$

Similarly, for each interior dual face κ_i' we define its discrete circulation by

$$(C'u)_{\kappa_i'} := \sum_{\sigma_j' \subset \partial \kappa_i'} u_j \tilde{h}_j', \tag{2.5}$$

where

$$\tilde{h}_j' = \begin{cases} \bar{h}_j' & \text{if } \sigma_j' \text{ is oriented positively along } \partial\kappa_i', \\ -\bar{h}_j' & \text{if } \sigma_j' \text{ is oriented negatively along } \partial\kappa_i'. \end{cases}$$

Clearly, C and C' are two linear mappings from $\mathbb{R}^M$ to $\mathbb{R}^{F_1}$ and $\mathbb{R}^{F_1}$ to $\mathbb{R}^{M_1}$, respectively. We remark that (2.4) and (2.5) are the discrete analogues of the integrals

$$\int_{\kappa_i'} \mathbf{E} \cdot \mathbf{n}_i \, d\sigma \quad \text{and} \quad \int_{\kappa_i} \mathbf{H} \cdot \mathbf{n}_i \, d\sigma$$

by Stokes' theorem, where in what follows $\mathbf{n}_i$ represents the unit normal vector for both primal and dual faces.

For each strictly interior dual edge σ_j' with both endpoints of σ_j' lying in Ω and the ith strictly interior dual face κ_i', we define the entries of a $F_1 \times M_1$ matrix G as

$$(G)_{ji} := \begin{cases} 1 & \text{if } \sigma_j' \text{ is oriented positively along } \partial\kappa_i', \\ -1 & \text{if } \sigma_j' \text{ is oriented negatively along } \partial\kappa_i', \\ 0 & \text{if } \sigma_j' \text{ does not meet } \partial\kappa_i'. \end{cases}$$

Let $w \in \mathbb{R}^M$ be a vector whose kth component is the value assigned to the kth primal edge. Let $w_1 \in \mathbb{R}^{M_1}$ be the restriction of w to the interior primal edges. Denote by $w|_{\partial\Omega}$ the components of w that are related to the boundary. Likewise, denote by $v \in \mathbb{R}^{F_1}$ the vector whose jth component represents a value on the jth interior dual edge. Similarly to [15, 16, 17], we have the following result.

LEMMA 2.1. *Let w, w_1, and v be defined as above, and $w|_{\partial\Omega} = 0$; then we have*

$$Cw = GDw_1 \, , \qquad C'v = G^T D' v \, . \tag{2.6}$$

Proof. To see the first relation in (2.6), we note that the ith component of both sides corresponds to the primal face κ_i. By the definition (2.4) and $w|_{\partial\Omega} = 0$, we have

$$(Cw)_{\kappa_i} = \sum_{\sigma_j \subset \partial\kappa_i} w_j \tilde{h}_j = \sum_{j=1}^{M_1} c_j w_j h_j, \qquad (GDw_1)_{\kappa_i} = \sum_{j=1}^{M_1} g_j h_j w_j,$$

where

$$c_j = \begin{cases} 1 & \text{if } \sigma_j \text{ is oriented postively along } \partial\kappa_i, \\ -1 & \text{if } \sigma_j \text{ is oriented negatively along } \partial\kappa_i, \\ 0 & \text{if } \sigma_j \text{ does not meet } \partial\kappa_i \end{cases}$$

for any interior primal edge σ_j, and $g_j = (G)_{ij}$. By the orthogonality between primal and dual meshes, we conclude that c_j and g_j are the same; the first relation in (2.6) is thus proved. The second relation can be proved by a similar technique. ∎

Using Lemma 2.1, we can show a discrete analogue of the following Green's formula:

$$\int_\Omega \mathbf{curl}\,\mathbf{E}\cdot\mathbf{B}\,dx = \int_\Omega \mathbf{curl}\,\mathbf{B}\cdot\mathbf{E}\,dx,$$

which holds when $\mathbf{E}\times\mathbf{n}=0$ on $\partial\Omega$.

LEMMA 2.2. *With the same definitions as in Lemma* 2.1, *we have*

$$(Cw, D'v) = (C'v, Dw_1). \tag{2.7}$$

Proof. Equation (2.7) follows directly from Lemma 2.1 and (2.6):

$$(C'v, Dw_1) = (G^T D'v, Dw_1) = (D'v, GDw_1) = (D'v, Cw). \qquad \square$$

With the definition of the discrete circulation operator C, we define the following inner product:

$$(u,v)_V := \sum_{\kappa_i\subset\Omega} (Cu)_i(Cv)_i s_i^{-1}\bar{h}'_i = (S^{-1}Cu, D'Cv) = (D'Cu, S^{-1}Cu) \tag{2.8}$$

for any vectors $u, v\in\mathbb{R}^M$, and the induced norm

$$|u|_V := (u,u)_V^{\frac{1}{2}}. \tag{2.9}$$

This norm is equivalent to the discrete seminorm of $H(\mathrm{curl};\Omega)$. We also define

$$\|u\|_V := (\|u\|_{W'}^2 + |u|_V^2)^{\frac{1}{2}}, \tag{2.10}$$

which is a discrete analogue of the norm in $H(\mathrm{curl};\Omega)$.

Let τ_i be a primal element and $\kappa_j\in\partial\tau_i$ be a primal face. We say κ_j is oriented positively along $\partial\tau_i$ if the dual edge σ'_j on κ_j is directed towards the outside of τ_i. Otherwise, we say κ_j is oriented negatively along $\partial\tau_i$. For each primal element τ_i we define a discrete flux by

$$(\mathcal{D}u)_i := \sum_{\kappa_j\subset\partial\tau_i} u_j\tilde{s}_j \qquad \forall u\in\mathbb{R}^{F_1}\ , \tag{2.11}$$

where no components of u on the boundary faces are involved, and $\tilde{s}_j$ is given by

$$\tilde{s}_j = \begin{cases} \ \ s_j & \text{if } \kappa_j \text{ is oriented positively along } \partial\tau_i,\\ -s_j & \text{if } \kappa_j \text{ is oriented negatively along } \partial\tau_i.\end{cases}$$

The mapping $\mathcal{D}$ is the discrete version of the divergence operator by noting that

$$\int_{\tau_i} \mathrm{div}\,\mathbf{u}\,dx = \int_{\partial\tau_i} \mathbf{u}\cdot\mathbf{n}\,ds.$$

Similarly, for each dual element τ'_i, we define a discrete flux by

$$(\mathcal{D}'u)_i := \sum_{\kappa'_j\subset\partial\tau'_i} u_j\tilde{s}'_j \quad \forall u\in\mathbb{R}^{M_1}\ , \tag{2.12}$$

where

$$\tilde{s}'_j = \begin{cases} \bar{s}'_j & \text{if } \kappa'_j \text{ is oriented positively along } \partial\tau'_i, \\ -\bar{s}'_j & \text{if } \kappa'_j \text{ is oriented negatively along } \partial\tau'_i. \end{cases}$$

Next we present a discrete analogue of the identity $\operatorname{div}(\mathbf{curl}\,\mathbf{u}) = 0$ for the discrete divergence operators $\mathcal{D}$ and $\mathcal{D}'$. To do so, we introduce two matrices B_1 and B'_1. B_1 is a $F_1 \times N$ matrix given by

$$(B_1)_{ji} := \begin{cases} 1 & \text{if } \kappa_j \text{ is oriented positively along } \partial\tau_i, \\ -1 & \text{if } \kappa_j \text{ is oriented negatively along } \partial\tau_i, \\ 0 & \text{if } \kappa_j \text{ does not meet } \partial\tau_i, \end{cases}$$

while B'_1 is a $M_1 \times L$ matrix given by

$$(B'_1)_{ji} := \begin{cases} 1 & \text{if } \kappa'_j \text{ is oriented positively along } \partial\tau'_i, \\ -1 & \text{if } \kappa'_j \text{ is oriented negatively along } \partial\tau'_i, \\ 0 & \text{if } \kappa'_j \text{ does not meet } \partial\tau'_i. \end{cases}$$

Then we have the following relations (cf. [6]).

LEMMA 2.3. *We have*

$$\mathcal{D} = B_1^T S, \qquad \mathcal{D}' = (B'_1)^T S', \tag{2.13}$$

$$B_1^T C = \mathbf{0}, \qquad (B'_1)^T C' = \mathbf{0}. \tag{2.14}$$

3. The finite volume method. The finite volume method proposed in Chung and Zou [6] for solving the interface Maxwell's equations (1.1)–(1.8) approximates the edge average of $\mathbf{E}$ on each primal edge and the face average of $\mathbf{B}$ on each primal face. The use of the magnetic flux density $\mathbf{B}$ in the approximation, instead of the magnetic field $\mathbf{H}$ as in most existing numerical methods, is crucial for maintaining accuracy in interface problems. This observation is supported by the numerical experiments presented in [6].

We now introduce some average quantities. For the magnetic flux density $\mathbf{B}$, we define its primal face average $B_f \in \mathbb{R}^{F_1}$ by

$$(B_f)_i := \frac{1}{s_i} \int_{\kappa_i} \mathbf{B} \cdot \mathbf{n}_i \, d\sigma$$

for each primal face κ_i and its dual edge average $B'_e \in \mathbb{R}^{F_1}$ by

$$(B'_e)_i := \frac{1}{h'_i} \int_{\sigma'_i} \mathbf{B} \cdot \mathbf{t}_i \, dl$$

for each noninterface dual edge σ'_i. Further, we let

$$\begin{aligned} (B'_e)_i &:= \alpha_i (B'_{e_1})_i + (1-\alpha_i)(B'_{e_2})_i \\ &:= \alpha_i \frac{1}{h_i^1} \int_{\sigma_i^1} \mathbf{B} \cdot \mathbf{t}_i \, dl + (1-\alpha_i) \frac{1}{h_i^2} \int_{\sigma_i^2} \mathbf{B} \cdot \mathbf{t}_i \, dl \end{aligned} \tag{3.1}$$

for each interface dual edge σ'_i. Here, for $r = 1, 2$, $\sigma_i^r = \sigma'_i \cap \Omega_r$ is the portion of σ'_i in Ω_r and $\alpha_i := \mu_r^{-1} h_i^r (\bar{h}'_i)^{-1}$ with h_i^r being the length of σ_i^r.

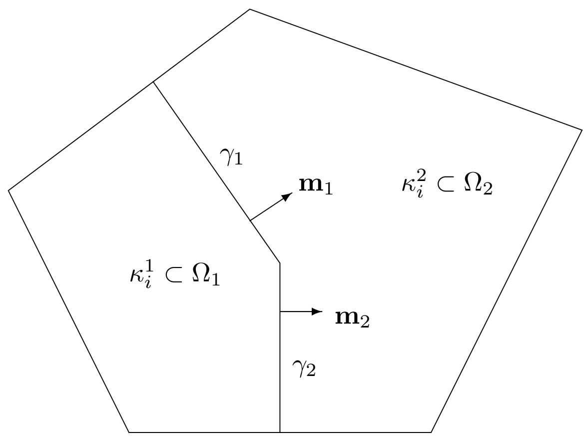

FIG. 2. *A dual face κ_i', divided by the interface into two parts κ_i^1, κ_i^2.*

For the electric field $\mathbf{E}$, we define its primal edge average $E_e \in \mathbb{R}^{M_1}$ by

$$(E_e)_i := \frac{1}{h_i} \int_{\sigma_i} \mathbf{E} \cdot \mathbf{n}_i \, dl$$

for each primal edge σ_i and its dual face average $E_f' \in \mathbb{R}^{M_1}$ by

$$(E_f')_i := \frac{1}{s_i'} \int_{\kappa_i'} \mathbf{E} \cdot \mathbf{n}_i \, d\sigma$$

for each non-interface dual face κ_i', and we let

$$\begin{aligned}(E_f')_i &:= \beta_i (E_{f_1}')_i + (1-\beta_i)(E_{f_2}')_i \\ &:= \beta_i \frac{1}{s_i^1} \int_{\kappa_i^1} \mathbf{E} \cdot \mathbf{n}_i \, d\sigma + (1-\beta_i) \frac{1}{s_i^2} \int_{\kappa_i^2} \mathbf{E} \cdot \mathbf{n}_i \, d\sigma \end{aligned} \tag{3.2}$$

for each interface dual face κ_i'; see Figure 2. Here, for $r = 1, 2$, $\kappa_i^r = \kappa_i' \cap \Omega_r$ is the portion of κ_i' in Ω_i with its area being s_i^r, and $\beta_i := \varepsilon_r s_i^r (\bar{s}_i')^{-1}$.

With the above notation, one can show that for each primal face κ_j and dual face κ_j' the true electric and magnetic fields $\mathbf{E}$ and $\mathbf{B}$ satisfy the equations [6]

$$s_j \frac{d}{dt}(B_f)_j + (CE_e)_{\kappa_j} = 0, \tag{3.3}$$

$$\bar{s}_j' \frac{d}{dt}(E_f')_j - (C' B_e')_{\kappa_j'} = \int_{\kappa_j'} \mathbf{J} \cdot \mathbf{n}_j \, d\sigma \, . \tag{3.4}$$

Let $E \in \mathbb{R}^{M_1}$ and $B \in \mathbb{R}^{F_1}$ be the approximations of the primal edge and face averages of the true solution $\mathbf{E}$ and $\mathbf{B}$ to (1.1)–(1.4), respectively. Note that each dual face (edge) average and the corresponding primal edge (face) average are approximately the same for sufficiently small h. Due to continuity of the tangential component of $\mathbf{E}$ and the normal component of $\mathbf{B}$ across the interface Γ, we naturally come to the following approximations based on (3.3) and (3.4):

Find $E \in \mathbb{R}^{M_1}$ and $B \in \mathbb{R}^{F_1}$ such that $E(0) = E_e(0)$, $B(0) = B_f(0)$, and

$$S' \frac{dE}{dt} - C'B = \tilde{J}, \tag{3.5}$$

$$S\frac{dB}{dt} + CE = \mathbf{0}, \tag{3.6}$$

where $\tilde{J} \in \mathbb{R}^{M_1}$ are defined by the right-hand sides of (3.4), while $E_e(0)$ and $B_f(0)$ are the primal edge average of $\mathbf{E}$ and primal face average of $\mathbf{B}$ at time $t = 0$.

Applying standard results concerning the well-posedness of systems of first order ordinary differential equations, we obtain the following theorem.

THEOREM 3.1. *The semi-discrete scheme* (3.5)–(3.6) *is well-posed.*

4. Discrete divergence constraints and stability. In this section, we show that the solutions E and B of the semidiscrete finite volume scheme (3.5)–(3.6) satisfy the divergence constraint conditions (1.3)–(1.4) at the discrete level.

THEOREM 4.1. *Let E and B be the solutions of* (3.5)–(3.6), *and let B_f, E_e, and E'_f be the average vectors of* $\mathbf{B}$ *or* $\mathbf{E}$ *as defined in section* 3. *Then*

$$\mathcal{D}B(t) = \mathbf{0}, \qquad \mathcal{D}B_f(t) = \mathbf{0}, \tag{4.1}$$

$$\mathcal{D}'E(t) = \tilde{\rho}(t) + \mathcal{D}'(E_e - E'_f)(0), \qquad \mathcal{D}'E_e(t) = \tilde{\rho}(t) + \mathcal{D}'(E_e - E'_f)(t) \tag{4.2}$$

for any $0 \le t \le T$, *where*

$$\tilde{\rho}_j(t) := \int_{\tau'_j} \rho(x,t)\, dx + \int_{\tau'_j \cap \Gamma} \rho_\Gamma(x,t)\, d\sigma. \tag{4.3}$$

Furthermore, we have the following discrete charge conservation law:

$$(B'_1)^T \tilde{J} = \frac{d\tilde{\rho}(t)}{dt}. \tag{4.4}$$

Proof. Multiplying (3.3) and (3.6) by the matrix B_1^T, and using (2.14), we have

$$\mathcal{D}\frac{dB_f}{dt} = \mathbf{0}, \quad \mathcal{D}\frac{dB}{dt} = \mathbf{0}.$$

So $\mathcal{D}B(t) = \mathcal{D}B_f(t) = \mathcal{D}B_f(0)$. Now (4.1) follows directly from the divergence-free condition (1.4).

To show (4.2) and (4.4), we multiply (3.4) by the matrix $(B'_1)^T$ and then use (2.13) to get

$$\mathcal{D}'\frac{dE'_f}{dt} = (B'_1)^T \tilde{J}. \tag{4.5}$$

Integrating the divergence condition (1.3) on each dual element, we obtain

$$\mathcal{D}'E'_f(t) = \tilde{\rho}(t) \tag{4.6}$$

for any $0 \le t \le T$, which is the second relation in (4.2). Also, the discrete charge conservation law (4.4) follows readily from the two equations above.

Now we multiply (3.5) by the matrix $(B'_1)^T$ and use (4.4) to get

$$\mathcal{D}'\frac{dE}{dt} = (B'_1)^T \tilde{J} = \frac{d\tilde{\rho}(t)}{dt}\ .$$

Integrating in time, we have

$$\mathcal{D}'E(t) = \tilde{\rho}(t) + \mathcal{D}'E(0) - \tilde{\rho}(0),$$

which is the first equation in (4.2) by applying (4.6) at $t = 0$. □

Next we state some stability results for the approximate solutions E and B.

THEOREM 4.2. *The solution* (E, B) *to the semidiscrete scheme* (3.5)–(3.6) *satisfies the following stability inequality:*

$$\max_{0\le t\le T}\{\|B(t)\|_W^2 + \|E(t)\|_{W'}^2\} \le 2\|B(0)\|_W^2 + 2\|E(0)\|_{W'}^2 + 4T\int_0^T \|S^{'-1}\tilde{J}(t)\|_{W'}^2\, dt.$$

Proof. Multiplying (3.6) by $D'B$ and (3.5) by DE, and adding up the resulting equations and using (2.7), we obtain

$$\left(S\frac{dB}{dt}, D'B\right) + \left(S'\frac{dE}{dt}, DE\right) = (\tilde{J}, DE),$$

and consequently

$$\frac{1}{2}\frac{d}{dt}\|B(t)\|_W^2 + \frac{1}{2}\frac{d}{dt}\|B(t)\|_{W'}^2 = (\tilde{J}, DE).$$

Integrating with respect to time, we get for any $0 \le s < t$

$$\|B(s)\|_W^2 + \|E(s)\|_{W'}^2 = \|B(0)\|_W^2 + \|E(0)\|_{W'}^2 + 2\int_0^s (\tilde{J}, DE)\, dt.$$

Using the above equation, the desired bound follows from the estimate

$$\begin{aligned}2\int_0^s (\tilde{J}, DE)\, dt &\le 2\int_0^s \|S^{'-1}\tilde{J}(t)\|_{W'}\|E(t)\|_{W'}\, dt\\ &\le 2T\int_0^s \|S^{'-1}\tilde{J}(t)\|_{W'}^2\, dt + \frac{1}{2T}\int_0^s \|E(t)\|_{W'}^2\, dt.\end{aligned}$$ □

5. Error estimates for the finite volume method. We devote this section to the error analysis of the finite volume scheme (3.5)–(3.6). We will present the discrete L^2-norm error estimates for both a tetrahedral grid and a rectangular grid, where the same convergence orders can be achieved as for noninterface Maxwell's equations. Also, we will show a discrete $H(\mathrm{curl};\Omega)$-norm error estimate, from which one can observe some superconvergence results for the finite volume method.

5.1. Discrete L^2-norm error estimate for tetrahedral grids. The purpose of this section is to develop the error analysis of the numerical scheme (3.5)–(3.6) in the discrete L^2-norms $\|\cdot\|_{W'}$ and $\|\cdot\|_W$. To do so, subtracting (3.3) from the jth component of (3.6), we obtain

$$S\frac{d}{dt}(B - B_f) + C(E - E_e) = 0\,; \tag{5.1}$$

then subtracting (3.4) from the jth component of (3.5) gives

$$S'\frac{d}{dt}(E - E_f') - C'(B - B_e') = \mathbf{0}\,. \tag{5.2}$$

Now multiplying (5.1) by $D'(B - B'_e)$ and (5.2) by $D(E - E_e)$, and then adding the resulting equalities, we have

$$
\begin{aligned}
&(S(\dot{B} - \dot{B}_f), D'(B - B_e)) + (S'(\dot{E} - \dot{E}'_f), D(E - E_e)) \\
&= (C'(B - B'_e), D(E - E_e)) - (C(E - E_e), D'(B - B'_e)),
\end{aligned}
\tag{5.3}
$$

where the dot represents the time derivative. By the boundary condition $\mathbf{E} \times \mathbf{n} = \mathbf{0}$ on $\partial\Omega$, we know that all the components of $E - E_e$ on the boundary vanish. So by Lemma 2.2 we see that

$$(C'(B - B'_e), D(E - E_e)) - (C(E - E_e), D'(B - B'_e)) = 0,$$

and consequently we obtain from (5.3) that

$$(\dot{B} - \dot{B}_f, B - B'_e)_W + (\dot{E} - \dot{E}'_f, E - E_e)_{W'} = 0. \tag{5.4}$$

Now we rewrite (5.4) as

$$
\begin{aligned}
&(\dot{B} - \dot{B}'_e, B - B'_e)_W + (\dot{E} - \dot{E}_e, E - E_e)_{W'} \\
&= (\dot{E}'_f - \dot{E}_e, E - E_e)_{W'} + (\dot{B}_f - \dot{B}'_e, B - B'_e)_W
\end{aligned}
$$

or, equivalently, as

$$\frac{1}{2}\frac{d}{dt}(\|B - B'_e\|_W^2 + \|E - E_e\|_{W'}^2) = (\dot{E}'_f - \dot{E}_e, E - E_e)_{W'} + (\dot{B}_f - \dot{B}'_e, B - B'_e)_W. \tag{5.5}$$

This enables us to show the following (optimal) first order convergence result for the finite volume scheme (3.5)–(3.6) for solving Maxwell's equations (1.1)–(1.4) on general tetrahedral grids.

THEOREM 5.1. *Assume that $\mathbf{E}, \mathbf{B} \in W^{1,1}(0,T; W^{1,p}(\Omega_i)^3)$, for $i = 1, 2$ and $p > 2$, are the solutions to Maxwell's system* (1.1)–(1.4)*, while E and B are the finite volume solution of* (3.5)–(3.6)*. Then the following error estimate holds for some constant K, independent of the mesh and the material parameters:*

$$
\begin{aligned}
&\max_{0\le t\le T}\{\|(E - E_e)(t)\|_{W'} + \|(B - B_f)(t)\|_W\} \\
&\le Kh\sum_{i=1}^{2}\{\|\varepsilon_i^{\frac{1}{2}}\mathbf{E}\|_{W^{1,1}(0,T;W^{1,p}(\Omega_i)^3)} + \|\mu_i^{-\frac{1}{2}}\mathbf{B}\|_{W^{1,1}(0,T;W^{1,p}(\Omega_i)^3)}\}.
\end{aligned}
\tag{5.6}
$$

Proof. We prove this theorem by using (5.5). For each noninterface interior primal edge σ_i, by definition we have

$$(\dot{E}'_f - \dot{E}_e)_i = \frac{1}{s'_i}\int_{\kappa'_i}\dot{\mathbf{E}}\cdot\mathbf{n}_i\,d\sigma - \frac{1}{h_i}\int_{\sigma_i}\dot{\mathbf{E}}\cdot\mathbf{t}_i\,dl,$$

where $\mathbf{n}_i$ is the unit normal vector to the dual face κ'_i. Let τ'_{i_1} and τ'_{i_2} be the two dual elements sharing the same dual face κ'_i; then by the Sobolev embedding theorem we have, for $p > 2$,

$$W^{1,p}(\tau'_{i_1}\cup\tau'_{i_2}) \hookrightarrow L^1(\kappa'_i), \qquad W^{1,p}(\tau'_{i_1}\cup\tau'_{i_2}) \hookrightarrow L^1(\sigma_i)\,.$$

Hence, $(\dot{E}'_f - \dot{E}_e)_i$ is a bounded linear functional on $W^{1,p}(\tau'_{i_1} \cup \tau'_{i_2})^3$ and vanishes for all constant functions. By the Bramble–Hilbert lemma and a standard scaling argument, we obtain

$$|(\dot{E}'_f - \dot{E}_e)_i| \le Kh^{1-\frac{3}{p}} |\dot{\mathbf{E}}|_{W^{1,p}(\tau'_{i_1} \cup \tau'_{i_2})^3} \tag{5.7}$$

for some generic constant K.

Next, for each interface primal edge σ_i corresponding to an interface dual face κ'_i, using (3.2) we get

$$\begin{aligned}(\dot{E}'_f - \dot{E}_e)_i &= (\beta_i \dot{E}'_{f_1} + (1-\beta_i)\dot{E}'_{f_2})_i - (\dot{E}_e)_i \\ &= \beta_i(\dot{E}'_{f_1} - \dot{E}_e)_i + (1-\beta_i)(\dot{E}'_{f_2} - \dot{E}_e)_i\,.\end{aligned}$$

Let $O_{i_1} = (\tau'_{i_2} \cup \tau'_{i_1}) \cap \Omega_1$ and $O_{i_2} = (\tau'_{i_2} \cup \tau'_{i_1}) \cap \Omega_2$; then the same reasoning as above shows that $(\dot{E}'_{f_1} - \dot{E}_e)_i$ and $(\dot{E}'_{f_2} - \dot{E}_e)_i$ are bounded linear functionals on $W^{1,p}(O_{i_1})^3$ and $W^{1,p}(O_{i_2})^3$, respectively, and vanish for all constant functions. Again, an application of the Bramble–Hilbert lemma and a scaling argument yield

$$|(\dot{E}'_{f_1} - \dot{E}_e)_i| \le Kh^{1-\frac{3}{p}} |\dot{\mathbf{E}}|_{W^{1,p}(O_{i_1})^3}, \tag{5.8}$$

$$|(\dot{E}'_{f_2} - \dot{E}_e)_i| \le Kh^{1-\frac{3}{p}} |\dot{\mathbf{E}}|_{W^{1,p}(O_{i_2})^3}\,. \tag{5.9}$$

By the definitions of $\bar{s}'_i$ and β_i, it is easy to see that $\bar{s}'_i\beta_i^2 \le \varepsilon_1 s_i^1$ and $\bar{s}'_i(1-\beta_i)^2 \le \varepsilon_2 s_i^2$. Thus we have

$$\begin{aligned}\bar{s}'_i h_i |(\dot{E}'_f - \dot{E}_e)_i|^2 &\le \bar{s}'_i h_i (2\beta_i^2 |(\dot{E}'_{f_1} - \dot{E}_e)_i|^2 + 2(1-\beta_i)^2 |(\dot{E}'_{f_2} - \dot{E}_e)_i|^2) \\ &\le 2\varepsilon_1 h_i s_i^1 |(\dot{E}'_{f_1} - \dot{E}_e)_i|^2 + 2\varepsilon_2 h_i s_i^2 |(\dot{E}'_{f_2} - \dot{E}_e)_i|^2\,.\end{aligned}$$

This, along with the estimates (5.7)–(5.9) and the Cauchy–Schwarz inequality, leads to

$$\begin{aligned}\|\dot{E}'_f - \dot{E}_e\|^2_{W'} &= \sum_{\kappa'_i \subset \Omega_1 \cup \Omega_2} \bar{s}'_i h_i |(\dot{E}'_f - \dot{E}_e)_i|^2 + \sum_{\kappa'_i \cap \Gamma \neq \phi} \bar{s}'_i h_i |(\dot{E}'_f - \dot{E}_e)_i|^2, \\ &\le Kh^{5-\frac{6}{p}} \sum_{i=1}^{M_1} \left\{ \varepsilon_1 |\dot{\mathbf{E}}|^2_{W^{1,p}(O_{i_1})^3} + \varepsilon_2 |\dot{\mathbf{E}}|^2_{W^{1,p}(O_{i_2})^3} \right\}, \\ &\le Kh^{5-\frac{6}{p}} \left\{ \sum_{i=1}^{M_1} \varepsilon_1^{p/2} |\dot{\mathbf{E}}|^p_{W^{1,p}(O_{i_1})^3} + \varepsilon_2^{p/2} |\dot{\mathbf{E}}|^p_{W^{1,p}(O_{i_2})^3} \right\}^{\frac{2}{p}} \left\{ \sum_{i=1}^{M_1} 1 \right\}^{1-\frac{2}{p}}.\end{aligned}$$

Noting the fact that $h^3 \sum_{i=1}^{M_1} 1 \le K$, we conclude that

$$\|\dot{E}'_f - \dot{E}_e\|_{W'} \le Kh \sum_{r=1}^{2} |\varepsilon_r^{\frac{1}{2}} \dot{\mathbf{E}}|_{W^{1,p}(\Omega_r)^3}. \tag{5.10}$$

Similarly, we have

$$\|\dot{B}_f - \dot{B}'_e\|_{W} \le Kh \sum_{r=1}^{2} |\mu_r^{-\frac{1}{2}} \dot{\mathbf{B}}|_{W^{1,p}(\Omega_r)^3}. \tag{5.11}$$

By integrating (5.5) over $(0,t)$ and applying the Cauchy–Schwarz inequality, we obtain

$$\begin{aligned}\|(B-B_e')(t)\|_W^2 + \|(E-E_e)(t)\|_{W'}^2 &\le 2\int_0^t (\|(B-B_e')(s)\|_W \|(\dot B_f - \dot B_e')(s)\|_W \\ &\qquad + \|(E-E_e)(s)\|_{W'}\|(\dot E_f' - \dot E_e)(s)\|_{W'})\, ds,\\ &\le 2\max_{0\le t\le T}(\|(B-B_e')(t)\|_W + \|(E-E_e)(t)\|_{W'})\\ &\qquad \times \int_0^T (\|(\dot B_f - \dot B_e')(s)\|_W + \|(\dot E_f' - \dot E_e)(s)\|_{W'})\, ds.\end{aligned}$$

Then, by (5.10) and (5.11), we have

$$\begin{aligned}&\max_{0\le t\le T}(\|(E-E_e)(t)\|_{W'} + \|(B-B_e')(t)\|_W)\\ &\le Kh\sum_{i=1}^2 (|\varepsilon_i^{\frac12}\mathbf{E}|_{W^{1,1}(0,T;W^{1,p}(\Omega_i))^3} + |\mu_i^{-\frac12}\mathbf{B}|_{W^{1,1}(0,T;W^{1,p}(\Omega_i))^3}).\end{aligned}$$

In order to complete the proof, we first observe that

$$\|(B-B_f)(t)\|_W \le \|(B-B_e')(t)\|_W + \|(B_e'-B_f)(t)\|_W.$$

So it remains to estimate $\|(B_e'-B_f)(t)\|_W$. Following the same argument as the one that led to (5.11), we have

$$\|B_f - B_e'\|_W \le Kh\sum_{r=1}^2 |\mu_r^{-\frac12}\mathbf{B}|_{W^{1,p}(\Omega_r)^3}.$$

Hence,

$$\begin{aligned}\max_{0\le t\le T}\|(B_f-B_e')(t)\|_W &\le Kh\sum_{r=1}^2 \max_{0\le t\le T}|\mu_r^{-\frac12}\mathbf{B}(t)|_{W^{1,p}(\Omega_r)^3}\\ &\le Kh\sum_{r=1}^2 \|\mu_r^{-\frac12}\mathbf{B}\|_{W^{1,1}(0,T;W^{1,p}(\Omega_r))^3}. \qquad \square\end{aligned}$$

Remark. There are very few studies in the literature concerning the regularity of the solution to the time-dependent Maxwell system (1.1)–(1.4) with discontinuous coefficients. However, for domains with smooth boundaries and interfaces, the regularity $\mathbf{B},\mathbf{E}\in L^2(0,T;W^{1,p}(\Omega_i))$ $(i=1,2)$ can be shown by slightly modifying the proof of Theorem 6.2 [8] in combination with the equivalence between the space $\{\mathbf{w}\in W^{1,p}(\Omega);\ \mathbf{w}\cdot\mathbf{n}=0 \text{ on } \partial\Omega\}$ and the space

$$\{\mathbf{w}\in L^p(\Omega)^3;\ \mathbf{curl}\,\mathbf{w}\in L^p(\Omega)^3, \mathbf{div}\,\mathbf{w}\in L^p(\Omega)^3,\ \mathbf{w}\cdot\mathbf{n}=0 \text{ on } \partial\Omega\}.$$

The additional time differentiability $\mathbf{B},\mathbf{E}\in W^{1,1}(0,T;W^{1,p}(\Omega_i))$ can be proved using standard arguments; see, e.g., [2].

5.2. Discrete L^2-norm error estimate for rectangular grids. The first order convergence of the finite volume scheme (3.5)–(3.6) given in the last subsection is generally optimal in terms of the regularities used. In this section, we intend to improve the convergence rate of the scheme (3.5)–(3.6) on rectangular grids by one order; namely, we establish second order convergence by making full use of the local regularities of the fields $\mathbf{E}$ and $\mathbf{B}$. Such a second order convergence result is invalid for general tetrahedral triangulations, even in the case of the noninterface Maxwell's equations [14, 16].

Let Ω be a rectangular cuboid in R^3. Similarly to the case of a polyhedral domain in section 2, we generate the primal and dual triangulations of Ω by using smaller rectangular cuboids. Note that both the primal and dual meshes are now made up of rectangular cuboids. For simplicity, the directions of edges and faces are assigned as follows: a direction is assigned to each primal and dual edge by the rule that positive direction means that it points in the positive axis direction. The directions of primal and dual faces are the same as those of the corresponding dual and primal edges. Below, we adopt the same notations as in section 2.

Clearly, most of the arguments presented in the previous subsection remain valid for the case of rectangular domain Ω considered here. To begin, we rewrite (5.4) as

$$\begin{aligned}&(\dot{B}-\dot{B}_f, B-B_f)_W + (\dot{E}-\dot{E}_e, E-E_e)_{W'}\\ &= (\dot{B}-\dot{B}_f, B'_e-B_f)_W + (\dot{E}'_f-\dot{E}_e, E-E_e)_{W'}\end{aligned}$$

or, equivalently, as

$$\frac{1}{2}\frac{d}{dt}(\|B-B_f\|_W^2 + \|E-E_e\|_{W'}^2) \tag{5.12}$$

$$= (\dot{B}-\dot{B}_f, B'_e-B_f)_W + (\dot{E}'_f-\dot{E}_e, E-E_e)_{W'}. \tag{5.13}$$

Next we estimate the terms on the right-hand side of (5.13), and this needs the following two auxiliary lemmas.

LEMMA 5.2. *There exist functions $u(t)$ and $\xi(t) \in \mathbb{R}^{F_1}$ such that all the noninterface components of $\xi(t)$ vanish, all the components of u and ξ are bounded linear functionals of $\mathbf{B}$, and the following relation holds for all $\phi \in \mathbb{R}^M$ with $\phi|_{\partial\Omega} = 0$:*

$$(C\phi, D'(B_f - B'_e)) = (C\phi, D'u) + (C\phi, \xi)\,. \tag{5.14}$$

Furthermore, the following estimates hold for $u(t)$ and $\xi(t)$:

(5.15)

$$\|u\|_W \le Kh^2\sum_{i=1}^{2}\|\mu_i^{-\frac{1}{2}}\mathbf{B}\|_{H^3(\Omega_i)^3}\,, \qquad \|D'^{-1}\xi\|_W \le Kh^2\sum_{i=1}^{2}\|\mu_i^{-\frac{1}{2}}\mathbf{B}\|_{H^3(\Omega_i)^3}.$$

Proof. By definition, for any strictly interior primal face κ_j we have

$$(B_f - B'_e)_j = \frac{1}{s_j}\int_{\kappa_j}\mathbf{B}\cdot\mathbf{n}_j\,d\sigma - \frac{1}{h'_j}\int_{\sigma'_j}\mathbf{B}\cdot\mathbf{t}_j\,dl.$$

Assume that κ_j is parallel to the xy-plane, with P_1 as its center; see Figure 3. We know that the quadrature rule

$$\int_{\kappa_j}\mathbf{B}\cdot\mathbf{n}_j\,d\sigma = s_j\,(\mathbf{B}\cdot\mathbf{n}_j)(P_1)$$

is exact for linear functions.

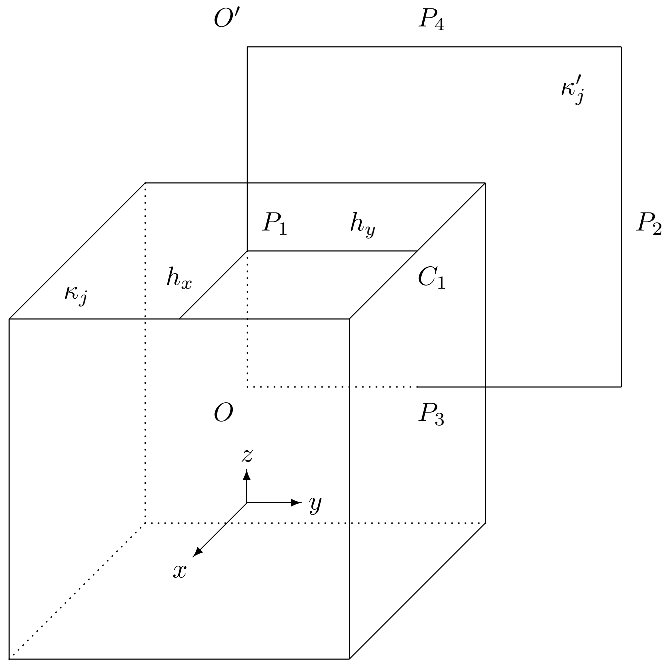

FIG. 3. *Noninterface element.*

Note that P_1 is not the center of the dual edge σ_j'. By adding a first order correction term, it is easy to see that the quadrature rule

$$\int_{\sigma_j'} \mathbf{B} \cdot \mathbf{t}_j \, dl = (\mathbf{B} \cdot \mathbf{t}_j)(P_1) h_j' + \frac{1}{2}(\overline{O'P_1}^2 \mathbf{B}_{3z}(O') - \overline{OP_1}^2 \mathbf{B}_{3z}(O))$$

is then exact for linear functions. Here $\overline{O'P_1}$ denotes the distance from O' to P_1 and $\mathbf{B}_{3z}$ denotes the derivative of the third component of $\mathbf{B}$ with respect to z. Similar notation will be used below. By the two relations above, we can rewrite $(B_f - B_e')_j$ as

$$(B_f - B_e')_j = \frac{1}{\bar{h}_j'} \tilde{u}_j + u_j, \tag{5.16}$$

where u_j vanishes for linear functions and the first order correction $\tilde{u}_j$ is given by

$$\begin{aligned} \tilde{u}_j &:= \frac{1}{2\mu_r}(\overline{OP_1}^2 \mathbf{B}_{3z} + h_x^2 \mathbf{B}_{1x} + h_y^2 \mathbf{B}_{2y})(O) \\ &\quad - \frac{1}{2\mu_r}(\overline{O'P_1}^2 \mathbf{B}_{3z} + h_x^2 \mathbf{B}_{1x} + h_y^2 \mathbf{B}_{2y})(O'). \end{aligned} \tag{5.17}$$

Here $r = 1$ or 2 is the index corresponding to the subdomain Ω_r in which κ_j lies. Moreover, notice the fact that $\mathbf{B}_{1x}(O) - \mathbf{B}_{1x}(O')$ and $\mathbf{B}_{2y}(O) - \mathbf{B}_{2y}(O')$ vanish for all linear functions, and the terms related to $\mathbf{B}_{1x}$ and $\mathbf{B}_{2y}$ are added to the above equation to make the relation more symmetric.

Next, by (3.1), for an interface primal face κ_i lying on Γ, we have

$$(B_f - B_e')_i = \alpha_i (B_f - B_{e_1}')_i + (1 - \alpha_i)(B_f - B_{e_2}')_i \, .$$

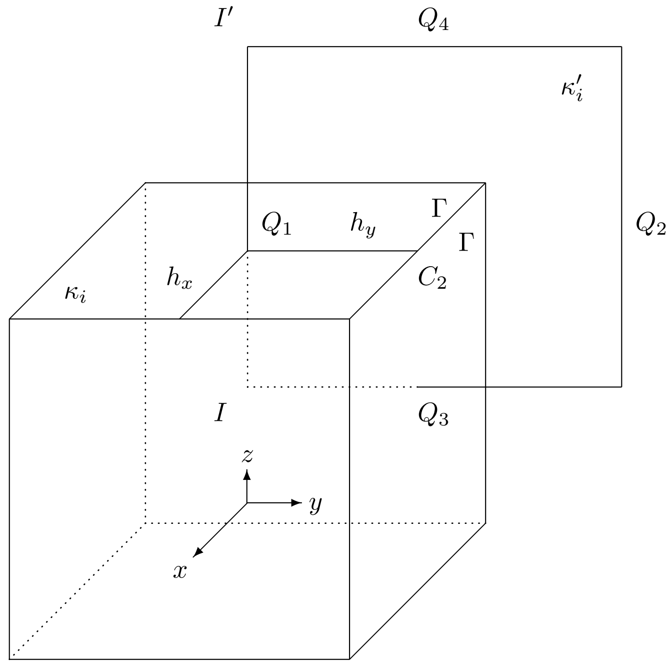

FIG. 4. *Interface element.*

Without loss of generality, we assume that κ_i is parallel to the xy-plane; see Figure 4. It is easy to verify that the quadrature rules

$$\int_{\kappa_i} \mathbf{B}\cdot\mathbf{n}_i\, d\sigma = s_i\,(\mathbf{B}\cdot\mathbf{n}_i)(Q_1)\ ,$$

$$(B'_{e_1})_i = \int_{\sigma_i^1} \mathbf{B}\cdot\mathbf{t}_i\, dl = (\mathbf{B}\cdot\mathbf{t}_i)(Q_1)h_i^1 - \frac{1}{2}\overline{IQ_1}^2\mathbf{B}_{3z}(I),$$

$$(B'_{e_2})_i = \int_{\sigma_i^2} \mathbf{B}\cdot\mathbf{t}_i\, dl = (\mathbf{B}\cdot\mathbf{t}_i)(Q_1)h_i^2 + \frac{1}{2}\overline{I'Q_1}^2\mathbf{B}_{3z}(I')$$

are all exact for linear functions. Using these relations, we can rewrite $(B_f - B'_e)_i$ as

$$(B_f - B'_e)_i = \frac{1}{h'_i}\tilde{u}_i + \frac{1}{h'_i}\xi_i + u_i, \tag{5.18}$$

where $u_i = \alpha_i u_i^1 + (1-\alpha_i)u_i^2$, u_i^1 and u_i^2 both vanish for linear functions, and the correction terms $\tilde{u}_i$ and ξ_i are given by

$$\begin{aligned}\tilde{u}_i := {}& \frac{1}{2\mu_1}(\overline{IQ_1}^2\mathbf{B}_{3z} + h_x^2\mathbf{B}_{1x} + h_y^2\mathbf{B}_{2y})(I)\\ & - \frac{1}{2\mu_2}(\overline{I'Q_1}^2\mathbf{B}_{3z} + h_x^2\mathbf{B}_{1x} + h_y^2\mathbf{B}_{2y})(I'),\end{aligned} \tag{5.19}$$

$$\xi_i := \frac{1}{2\mu_2}(h_x^2\mathbf{B}_{1x} + h_y^2\mathbf{B}_{2y})(I') - \frac{1}{2\mu_1}(h_x^2\mathbf{B}_{1x} + h_y^2\mathbf{B}_{2y})(I)\ . \tag{5.20}$$

For the same reason as earlier for the noninterface face κ_i, we have also added some extra terms related to $\mathbf{B}_{1x}$ and $\mathbf{B}_{2y}$ here. Note, however, that due to the jumps across the interface, ξ_i no longer vanishes for linear functions.

By (5.17), (5.19), and the definition of B_1, we can write $\tilde{u} = B_1\tilde{\phi}$ for some $\tilde{\phi} \in \mathbb{R}^N$. Hence for any $\phi \in \mathbb{R}^M$ with $\phi|_{\partial\Omega} = 0$, we get from (5.16) and (5.18) that

$$\begin{aligned}(C\phi, D'(B_f - B'_e)) &= (C\phi, \tilde{u}) + (C\phi, D'u) + (C\phi, \xi)\\ &= (C\phi, B_1\tilde{\phi}) + (C\phi, D'u) + (C\phi, \xi)\\ &= (B_1^T C\phi, \tilde{\phi}) + (C\phi, D'u) + (C\phi, \xi)\\ &= (C\phi, D'u) + (C\phi, \xi).\end{aligned}$$

This proves (5.14).

For the estimate (5.15), let u_j be a component of u corresponding to an interior primal face κ_j in Ω_r, $r = 1, 2$. We recall from (5.16) that

$$u_j = (B_f - B'_e)_j - \frac{1}{\bar{h}'_j}\tilde{u}_j.$$

By the Sobolev embedding theorem, we have

$$H^3(\tau_{j_1} \cup \tau_{j_2}) \hookrightarrow C^{1,\frac{1}{2}}(\tau_{j_1} \cup \tau_{j_2}),$$

where τ_{j_1} and τ_{j_2} are two elements in Ω_r and share the face κ_j. Hence, u_j is a bounded linear functional of $\mathbf{B}$ in $H^3(\tau_{j_1} \cup \tau_{j_2})^3$ and vanishes for linear fields $\mathbf{B}$. Then, by the Bramble–Hilbert lemma, we have

$$|u_j|^2 \le K(h)\left(|\mathbf{B}|^2_{H^2(\tau_{j_1}\cup\tau_{j_2})^3} + |\mathbf{B}|^2_{H^3(\tau_{j_1}\cup\tau_{j_2})^3}\right).$$

A standard scaling argument yields

$$|u_j|^2 \le Kh\left(|\mathbf{B}|^2_{H^2(\tau_{j_1}\cup\tau_{j_2})^3} + |\mathbf{B}|^2_{H^3(\tau_{j_1}\cup\tau_{j_2})^3}\right) \le Kh\|\mathbf{B}\|^2_{H^3(\tau_{j_1}\cup\tau_{j_2})^3}. \tag{5.21}$$

Now consider a component u_i of u corresponding to an interface face κ_i, which is shared by the element τ_{i_1} in Ω_1 and τ_{i_2} in Ω_2. Recall that

$$u_i = \alpha_i u_i^1 + (1 - \alpha_i)u_i^2,$$

where

$$\begin{aligned}h_i^1 u_i^1 &:= h_i^1(B'_{e_i^1} - (B_f)_i) - \frac{1}{2}\overline{IQ_1}^2\mathbf{B}_{3z}(I)\ ,\\ h_i^2 u_i^2 &:= h_i^2(B'_{e_i^2} - (B_f)_i) + \frac{1}{2}\overline{I'Q_1}^2\mathbf{B}_{3z}(I').\end{aligned}$$

By the Sobolev embedding theorem, u_i^r is a bounded linear functional of $\mathbf{B}$ in $H^3(\tau_{i_r})^3$ and vanishes for all linear fields for $r = 1$ or 2. Hence, again by the Bramble–Hilbert lemma and a scaling argument, we have

$$|u_i^1| \le Kh^{\frac{1}{2}}\|\mathbf{B}\|_{H^3(\tau_{i_1})^3}, \qquad |u_i^2| \le Kh^{\frac{1}{2}}\|\mathbf{B}\|_{H^3(\tau_{i_2})^3}.$$

Similarly to the proof of (5.10), using the above estimates and (5.21) we obtain

$$
\begin{aligned}
\|u\|_W^2 &= \sum_{\sigma_i' \subset \Omega_1 \cap \Omega_2} s_j \bar{h}_j' |u_j|^2 + \sum_{\sigma_i' \cap \Gamma \neq \phi} s_j \bar{h}_j' |u_j|^2 \\
&\le \sum_{\sigma_i' \subset \Omega_1 \cap \Omega_2} s_j \bar{h}_j' |u_j|^2 + \sum_{\sigma_j' \cap \Gamma \neq \phi} s_j \bar{h}_j' (2\alpha_j^2 |u_j^1|^2 + 2(1-\alpha_j)^2 |u_j^2|^2) \\
&\le K h^4 \left\{ \sum_{\tau_{i_1} \subset \Omega_1} \mu_1^{-1} \|\mathbf{B}\|_{H^3(\tau_{i_1})^3}^2 + \sum_{\tau_{i_2} \subset \Omega_2} \mu_2^{-1} \|\mathbf{B}\|_{H^3(\tau_{i_2})^3}^2 \right\} \\
&\le K h^4 \left\{ \sum_{r=1}^2 \|\mu_r^{-\frac{1}{2}} \mathbf{B}\|_{H^3(\Omega_r)^3}^2 \right\}^2 .
\end{aligned}
$$

We are now ready to estimate ξ. For each interface primal face κ_i shared by the element τ_{i_1} in Ω_1 and τ_{i_2} in Ω_2, we rewrite ξ_i using the interface condition (1.8) as

$$
\begin{aligned}
\xi_i := &\left\{ \frac{1}{2}(h_x^2 \mathbf{H}_{1x} + h_y^2 \mathbf{H}_{2y})(I') - \frac{1}{2}(h_x^2 \mathbf{H}_{1x} + h_y^2 \mathbf{H}_{2y})(Q_1) \right\} \\
&+ \left\{ \frac{1}{2}(h_x^2 \mathbf{H}_{1x} + h_y^2 \mathbf{H}_{2y})(Q_1) - \frac{1}{2}(h_x^2 \mathbf{H}_{1x} + h_y^2 \mathbf{H}_{2y})(I) \right\}.
\end{aligned} \tag{5.22}
$$

By the Hölder continuity of $\mathbf{H}_{1x}$, we have

$$
|\mathbf{H}_{1x}(I') - \mathbf{H}_{1x}(Q_1)| \le K h^{\frac{1}{2}} \|\mathbf{H}\|_{C^{1,\frac{1}{2}}(\tau_{i_2})^3}.
$$

Similar estimates hold for the other pairs in (5.22). This leads to

$$
|\xi_i| \le K h^{\frac{5}{2}} \left\{ \|\mathbf{H}\|_{C^{1,\frac{1}{2}}(\tau_{i_1})^3} + \|\mathbf{H}\|_{C^{1,\frac{1}{2}}(\tau_{i_2})^3} \right\}.
$$

Consequently, by the fact that $\xi_i = 0$ for any noninterface primal face, we get

$$
\begin{aligned}
\|D'^{-1}\xi\|_W^2 &= \sum_{i=1}^{F_1} s_i \bar{h}_i' |(\bar{h}_j')^{-1} \xi_i|^2 \\
&\le K h^6 \sum_{\kappa_i \subset \Gamma} \left\{ \mu_1 \|\mathbf{H}\|_{C^{1,\frac{1}{2}}(\tau_{i_1})^3}^2 + \mu_2 \|\mathbf{H}\|_{C^{1,\frac{1}{2}}(\tau_{i_2})^3}^2 \right\} \\
&\le K h^4 \sum_{r=1}^2 \|\mu_r^{\frac{1}{2}} \mathbf{H}\|_{C^{1,\frac{1}{2}}(\Omega_r)^3}^2 . \qquad \square
\end{aligned}
$$

LEMMA 5.3. *There exist functions $v(t)$, $\lambda(t) \in \mathbb{R}^{M_1}$, and $w(t) \in \mathbb{R}^{F_1}$, such that all the noninterface components of $\lambda(t)$ vanish and all the components of v, w, and λ are bounded linear functionals of $\mathbf{E}$, and the following relation holds for all $\phi \in \mathbb{R}^M$ with $\phi|_{\partial\Omega} = 0$:*

$$
(\dot{E}_f' - \dot{E}_e, \phi)_{W'} = (\dot{v}, \phi)_{W'} + (D'\dot{w}, C\phi) + (S'^{-1}\dot{\lambda}, \phi)_{W'} . \tag{5.23}
$$

Furthermore, we have the following estimates for $v(t)$, $\lambda(t)$, $w(t)$, and $p > 3$:

$$
\|\dot{v}\|_{W'} \le K h^2 \sum_{i=1}^2 \|\epsilon_i^{\frac{1}{2}} \dot{\mathbf{E}}\|_{H^3(\Omega_i)^3} , \qquad \|\dot{w}\|_W \le K h^2 \sum_{i=1}^2 \|\epsilon_i^{\frac{1}{2}} \dot{\mathbf{E}}\|_{W^{2,p}(\Omega_i)^3}, \tag{5.24}
$$

$$
\|S'^{-1}\dot{\lambda}\|_{W'} \ \le \ K h^2 \sum_{i=1}^2 \|\epsilon_i^{\frac{1}{2}} \dot{\mathbf{E}}\|_{H^3(\Omega_i)^3} . \tag{5.25}
$$

Proof. The proof is similar to that of Lemma 5.2. First, we consider a noninterface dual face κ'_j lying in Ω_r $(r = 1, 2)$. Recall that

$$(E'_f - E_e)_j = \frac{1}{s'_j} \int_{\kappa'_j} \mathbf{E} \cdot \mathbf{n}_j \; d\sigma - \frac{1}{h_j} \int_{\sigma_j} \mathbf{E} \cdot \mathbf{t}_j \; dl.$$

We see from Figure 3 that C_1 is the center of the primal edge σ_j, so the quadrature rule

$$\int_{\sigma_j} \mathbf{E} \cdot \mathbf{t}_j \; dl = \mathbf{E}_1(C_1) h_j$$

is exact for all linear functions. However, C_1 is not the center of the dual face κ'_j. By adding a first order correction term $\tilde{w}_j$, the quadrature rule

$$\int_{\kappa'_j} \mathbf{E} \cdot \mathbf{n}_j \; d\sigma = \mathbf{E}_1(C_1) s'_j + \tilde{w}_j$$

is then exact for all linear functions, where $\tilde{w}_j$ is given by

$$2\tilde{w}_j = [\mathbf{E}_{1y}(P_2)\overline{P_2C_1}^2 - \mathbf{E}_{1y}(P_1)\overline{P_1C_1}^2]\overline{P_3P_4} + [\mathbf{E}_{1z}(P_4)\overline{P_4C_1}^2 - \mathbf{E}_{1z}(P_3)\overline{P_3C_1}^2]\overline{P_1P_2}.$$

By direct computations, $\tilde{w}_j$ can be represented by the discrete circulation as follows:

$$\tilde{w}_j := \frac{1}{\varepsilon_r}(C'w)_j \;, \tag{5.26}$$

where the components of w corresponding to the four edges of κ'_j containing the points P_1, P_2, P_3, and P_4 are assigned, respectively, the following values:

$$\begin{aligned}
w(P_1) &:= \frac{1}{2}\varepsilon_r\mu_r(h_y^2\mathbf{E}_{1y}(P_1) - h_x^2\mathbf{E}_{2x}(P_1)),\\
w(P_2) &:= \frac{1}{2}\varepsilon_r\mu_r((\overline{P_2C_1}^2\mathbf{E}_{1y}(P_2) - h_x^2\mathbf{E}_{2x}(P_2)),\\
w(P_3) &:= \frac{1}{2}\varepsilon_r\mu_r(h_x^2\mathbf{E}_{3x}(P_3) - \overline{P_3C_1}^2\mathbf{E}_{1z}(P_3)),\\
w(P_4) &:= \frac{1}{2}\varepsilon_r\mu_r(h_x^2\mathbf{E}_{3x}(P_4) - \overline{P_4P_1}^2\mathbf{E}_{1z}(P_4)).
\end{aligned}$$

We remark that for the verification of (5.26) we have used the simple fact that $\mathbf{E}_{2x}(P_1)$ and $\mathbf{E}_{2x}(P_2)$, as well as $\mathbf{E}_{3x}(P_1)$ and $\mathbf{E}_{3x}(P_2)$, are equal, respectively, for all linear functions. Using (5.26), we can rewrite $\dot{E}'_f - \dot{E}_e$ as

$$(\dot{E}'_f - \dot{E}_e)_j = \frac{1}{\bar{s}'_j}(C'\dot{w})_j + \dot{v}_j, \tag{5.27}$$

where $\dot{v}_j$ is a functional which vanishes for all linear functions.

Now consider an interface dual face κ'_i. By the definition (3.2) we have

$$\bar{s}'_i(\dot{E}'_f - \dot{E}_e)_i = \frac{d}{dt}\{\varepsilon_1 s_i^1(E'_{f_1} - E_e)_i + \varepsilon_2 s_i^2(E'_{f_2} - E_e)_i\}.$$

Without loss of generality, we assume that κ_i' is parallel to the zy-plane and perpendicular to the interface primal face κ_i; see Figure 4. It is straightforward to verify that the three quadrature rules

$$\begin{aligned}
(E_e)_i &= \frac{1}{h_i}\int_{\sigma_i} \mathbf{E}\cdot\mathbf{t}_i\, dl = \mathbf{E}_1(C_2),\\
(E'_{f_1})_i &= \frac{1}{s_i^1}\int_{\kappa_i^1} \mathbf{E}\cdot\mathbf{n}_i\, d\sigma = \frac{1}{s_i^1}\{\mathbf{E}_1(C_2)s_i^1 + \tilde{w}_i^1\},\\
(E'_{f_2})_i &= \frac{1}{s_i^2}\int_{\kappa_i^2} \mathbf{E}\cdot\mathbf{n}_i\, d\sigma = \frac{1}{s_i^2}\{\mathbf{E}_1(C_2)s_i^2 + \tilde{w}_i^2\}
\end{aligned}$$

are all exact for linear functions, where

$$\tilde{w}_i^1 := \frac{1}{2}[-\mathbf{E}_{1y}(Q_1)\overline{Q_1C_2}^2]\overline{Q_3C_2} + \frac{1}{2}[-\mathbf{E}_{1z}(Q_3)\overline{Q_3C_2}^2]\overline{Q_1C_2}$$

and

$$\begin{aligned}
\tilde{w}_i^2 :=& \frac{1}{2}[\mathbf{E}_{1y}(Q_2)\overline{Q_2C_2}^2]\overline{Q_3Q_4} + \frac{1}{2}[-\mathbf{E}_{1y}(Q_1)\overline{Q_1C_2}^2]\overline{Q_4C_2}\\
&+ \frac{1}{2}[\mathbf{E}_{1z}(Q_4)\overline{Q_4C_2}^2]\overline{Q_1Q_2} + \frac{1}{2}[-\mathbf{E}_{1z}(Q_3)\overline{Q_3C_2}^2]\overline{Q_2C_2}.
\end{aligned}$$

Then we have

$$\bar{s}_i'(\dot{E}_f' - \dot{E}_e)_i = \frac{d}{dt}(\varepsilon_1\tilde{w}_i^1 + \varepsilon_2\tilde{w}_i^2) + \frac{d}{dt}(\varepsilon_1 s_i^1 v_i^1 + \varepsilon_2 s_i^2 v_i^2),$$

where v_i^1 and v_i^2 are linear functionals which vanish for all linear functions. We further write

$$\frac{d}{dt}(\varepsilon_1\tilde{w}_i^1 + \varepsilon_2\tilde{w}_i^2) = (C'\dot{w})_i + \dot{\lambda}_i, \tag{5.28}$$

where the components of w on the four edges of κ_i' containing the points Q_1, Q_2, Q_3, and Q_4 are assigned, respectively, the following values:

$$\begin{aligned}
w(Q_1) :=& \frac{1}{2\bar{h}'_{i_1}}(\varepsilon_1\overline{Q_3C_2}h_y^2\mathbf{E}_{1y}(Q_1) + \varepsilon_2\overline{Q_4C_2}h_y^2\mathbf{E}_{1y}(Q_1))\\
&+ \frac{1}{2\bar{h}'_{i_1}}(-\varepsilon_1\overline{Q_3C_2}h_x^2\mathbf{E}_{2x}(Q_1) - \varepsilon_2\overline{C_2Q_4}h_x^2\mathbf{E}_{2x}(Q_1)),\\
w(Q_2) :=& \frac{1}{2\bar{h}'_{i_2}}(\varepsilon_2\overline{Q_3Q_4}\,\overline{C_2Q_2}^2\mathbf{E}_{1y}(Q_2) - \varepsilon_2\overline{Q_3Q_4}h_x^2\mathbf{E}_{2x}(Q_2)),\\
w(Q_3) :=& \frac{1}{2\bar{h}'_{i_3}}(-\varepsilon_1\overline{Q_1C_2}\,\overline{Q_3C_2}^2\mathbf{E}_{1z}(Q_3) - \varepsilon_2\overline{C_2Q_2}\,\overline{Q_3C_2}^2\mathbf{E}_{1z}(Q_3))\\
&+ \frac{1}{2\bar{h}'_{i_3}}(\varepsilon_1\overline{Q_1C_2}h_x^2\mathbf{E}_{3x}(Q_3) + \varepsilon_2\overline{C_2Q_2}h_x^2\mathbf{E}_{3x}(Q_3)),\\
w(Q_4) :=& \frac{1}{2\bar{h}'_{i_4}}(-\varepsilon_2\overline{Q_1Q_2}\,\overline{C_2Q_4}^2\mathbf{E}_{1z}(Q_4) + \varepsilon_2\overline{Q_1Q_2}h_x^2\mathbf{E}_{3x}(Q_4)),
\end{aligned}$$

and λ_i is a term due to the jump in the coefficients across the interface:

$$\begin{aligned}\lambda_i =& \frac{1}{2}\{-\varepsilon_1\overline{Q_3C_2}h_x^2\mathbf{E}_{2x}(Q_1) - \varepsilon_2\overline{C_2Q_4}h_x^2\mathbf{E}_{2x}(Q_1) + \varepsilon_2\overline{Q_3Q_4}h_x^2\mathbf{E}_{2x}(Q_2)\}\\ &+ \frac{1}{2}\{-\varepsilon_1\overline{Q_1C_2}h_x^2\mathbf{E}_{3x}(Q_3) - \varepsilon_2\overline{C_2Q_2}h_x^2\mathbf{E}_{3x}(Q_3) + \varepsilon_2\overline{Q_1Q_2}h_x^2\mathbf{E}_{3x}(Q_4)\}\\ :\equiv& \frac{1}{2}\,\mathrm{I} + \frac{1}{2}\,\mathrm{II}\,.\end{aligned}$$

As above, we can write

$$(\dot{E}'_f - \dot{E}_e)_i = \frac{1}{\bar{s}'_i}(C'\dot{w})_i + \frac{1}{\bar{s}'_i}\dot{\lambda}_i + \dot{v}_i, \tag{5.29}$$

where $\dot{v}_i = (\varepsilon_1 s_i^2\dot{v}_i^1 + \varepsilon_2 s_i^2\dot{v}_i^2)/\bar{s}'_i$. It is easy to see by using (5.27) and (5.29) that

$$\begin{aligned}(\dot{E}'_f - \dot{E}_e, \phi)_{W'} &= (C'\dot{w}, D\phi) + (\dot{v}, \phi)_{W'} + (S'^{-1}\dot{\lambda}, \phi)_{W'}\\ &= (D'\dot{w}, C\phi) + (\dot{v}, \phi)_{W'} + (S'^{-1}\dot{\lambda}, \phi)_{W'}.\end{aligned}$$

The estimates in (5.24) can be proved similarly to those in Lemma 5.2. We show only (5.25).

First, we rewrite $\dot{\mathrm{I}}$ as $\dot{\mathrm{I}} = \dot{\delta}_1 + \dot{\delta}_2$ with

$$\begin{aligned}\dot{\delta}_1 &= -\varepsilon_2\overline{C_2Q_4}h_x^2\dot{\mathbf{E}}_{2x}(Q_1) + \varepsilon_2\overline{C_2Q_4}h_x^2\dot{\mathbf{E}}_{2x}(Q_2),\\ \dot{\delta}_2 &= -\varepsilon_1\overline{Q_3C_2}h_x^2\dot{\mathbf{E}}_{2x}(Q_1) + \varepsilon_2\overline{Q_3C_2}h_x^2\dot{\mathbf{E}}_{2x}(Q_2).\end{aligned}$$

Note that the term $\dot{\delta}_1$ clearly vanishes for any linear field $\mathbf{E}$, so it can be absorbed into the term $\dot{v}_i$. The remaining term $\dot{\delta}_2$ can be written as

$$\dot{\delta}_2 = \varepsilon_1\overline{Q_3C_2}h_x^2\{-\dot{\mathbf{E}}_{2x}(Q_1) + \dot{\mathbf{E}}_{2x}(C_2)\} - \varepsilon_2\overline{Q_3C_2}h_x^2\{\dot{\mathbf{E}}_{2x}(C_2) - \dot{\mathbf{E}}_{2x}(Q_2)\}$$

by using the interface condition (1.7) and the fact that the function ρ_Γ depends only on the spatial variables. Then, by the Hölder continuity of $\dot{\mathbf{E}}_{2x}$, we have

$$\begin{aligned}|\dot{\mathbf{E}}_{2x}(Q_1) - \dot{\mathbf{E}}_{2x}(C_2)| &\le Kh^{\frac{1}{2}}\|\dot{\mathbf{E}}\|_{C^{1,\frac{1}{2}}(\tau'_{i_1})},\\ |\dot{\mathbf{E}}_{2x}(Q_2) - \dot{\mathbf{E}}_{2x}(C_2)| &\le Kh^{\frac{1}{2}}\|\dot{\mathbf{E}}\|_{C^{1,\frac{1}{2}}(\tau'_{i_2})},\end{aligned}$$

where τ'_{i_r} is the intersection of Ω_r with the union of all dual elements sharing the dual face κ'_i $(r = 1, 2)$. Hence,

$$|\delta_2| \le Kh^{\frac{7}{2}}\left\{\epsilon_1^{\frac{1}{2}}\|\dot{\mathbf{E}}\|_{C^{1,\frac{1}{2}}(\tau'_{i_1})} + \epsilon_2^{\frac{1}{2}}\|\dot{\mathbf{E}}\|_{C^{1,\frac{1}{2}}(\tau'_{i_2})}\right\}.$$

The term II can be estimated in the same manner. The rest of the proof is the same as the proof for ξ in (5.15). □

We are now ready to give the main result of this section.

THEOREM 5.4. *Assume that the following regularity hypotheses hold for the solution of the interface Maxwell system* (1.1)–(1.8):

$$\mathbf{E} \in W^{1,1}(0,T;H^3(\Omega_i)^3) \cap W^{2,1}(0,T;W^{2,p}(\Omega_i)^3), \qquad \mathbf{B} \in W^{1,1}(0,T;H^3(\Omega_i)^3)$$

for $i = 1, 2$ and $p > 3$, and $(E,\ B)$ is the solution of (3.5)–(3.6) *on a nonuniform rectangular grid of size h. Then we have*

$$\begin{aligned}
&\max_{0\le t\le T}\{\|(E-E_e)(t)\|_{W'} + \|(B-B_f)(t)\|_W\} \\
&\quad \le Kh^2 \sum_{i=1}^{2}\{\|\epsilon_i^{\frac{1}{2}}\mathbf{E}\|_{W^{1,1}(0,T;H^3(\Omega_i)^3)} \\
&\quad + \|\epsilon_i^{\frac{1}{2}}\mathbf{E}\|_{W^{2,1}(0,T;W^{2,p}(\Omega_i)^3)} + \|\mu_i^{-\frac{1}{2}}\mathbf{B}\|_{W^{1,1}(0,T;H^3(\Omega_i)^3)}\}.
\end{aligned} \tag{5.30}$$

Proof. It follows from (5.1), (5.13), (5.14), and (5.23) that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(\|B-B_f\|_W^2 + \|E-E_e\|_{W'}^2) \\
&\quad = (C(E-E_e), D'(B_f - B_e')) + (\dot v, E-E_e)_{W'} \\
&\qquad + (D'\dot w, C(E-E_e)) + (S'^{-1}\dot\lambda, E-E_e)_{W'} \\
&\quad = (C(E-E_e), D'u) + (C(E-E_e), \xi) + (\dot v, E-E_e)_{W'} \\
&\qquad - (\dot w, \dot B - \dot B_f)_W + (S'^{-1}\dot\lambda, E-E_e)_{W'} \\
&\quad = -\,(\dot B - \dot B_f, u)_W - (\dot B - \dot B_f, D'^{-1}\xi)_W + (\dot v, E-E_e)_{W'} \\
&\qquad - (\dot w, \dot B - \dot B_f)_W + (S'^{-1}\dot\lambda, E-E_e)_{W'}\,.
\end{aligned}$$

Integrating over $(0, t_1)$, we have

$$\begin{aligned}
&\frac{1}{2}(\|B-B_f\|_W^2 + \|E-E_e\|_{W'}^2)(t_1) \\
&\quad = \int_0^{t_1}[-(\dot B - \dot B_f, u)_W - (\dot B - \dot B_f, D'^{-1}\xi)_W + (\dot v, E-E_e)_{W'} \\
&\qquad -(\dot w, \dot B - \dot B_f)_W + (S'^{-1}\dot\lambda, E-E_e)_{W'}]dt\,.
\end{aligned}$$

Then, by integration by parts,

$$\begin{aligned}
&\frac{1}{2}(\|B-B_f\|_W^2 + \|E-E_e\|_{W'}^2)(t_1) \\
&\quad = \int_0^{t_1}[(\dot v, E-E_e)_{W'} + (S'^{-1}\dot\lambda, E-E_e)_{W'}]\,dt \\
&\qquad + \int_0^{t_1}(B-B_f, \dot u + \ddot w)_W\,dt + \int_0^{t_1}(B-B_f, D'^{-1}\dot\xi)_W\,dt \\
&\qquad - (B-B_f, \dot w + u)_W(t_1) - (B-B_f, D'^{-1}\xi)_W(t_1).
\end{aligned}$$

Now the desired estimate follows from the Cauchy–Schwarz inequality and the estimates in Lemmas 5.2 and 5.3. ∎

5.3. Superconvergence in the discrete $\boldsymbol{H}(\mathbf{curl};\boldsymbol{\Omega})$-norm. We now show that the finite volume scheme (3.5)–(3.6) has certain superconvergence property; namely, the errors $E - E_e$ and $B - B_f$ are also second order convergent in a discrete $H(\mathrm{curl};\Omega)$-norm. To do so, we first differentiate (3.5) with respect to t to obtain

$$S'\frac{d^2E}{dt^2} - C'\frac{dB}{dt} = \frac{d\tilde J}{dt},$$

and then by (3.6) we obtain

$$S'\frac{d^2E}{dt^2}+C'S^{-1}CE=\frac{d\tilde{J}}{dt}. \tag{5.31}$$

We supplement (5.31) with the following initial conditions:

$$E(0)=E_e(0), \qquad \dot{E}(0)=\dot{E}_e(0). \tag{5.32}$$

Upon rewriting (5.31) as

$$S'\frac{d^2}{dt^2}(E-E_e)+C'S^{-1}C(E-E_e)=\frac{d\tilde{J}}{dt}-S'\frac{d^2E_e}{dt^2}-C'S^{-1}CE_e,$$

and by (3.3), we then have

$$S'\frac{d^2}{dt^2}(E-E_e)+C'S^{-1}C(E-E_e)=S'\frac{d^2}{dt^2}(E'_f-E_e)+\frac{d}{dt}(C'(B_f-B'_e)). \tag{5.33}$$

This indicates that $E-E_e$ satisfies the ordinary differential equation (5.33) with the homogeneous initial conditions

$$(E-E_e)(0)=0, \qquad (\dot{E}-\dot{E}_e)(0)=0. \tag{5.34}$$

Multiplying (5.33) by $D(\dot{E}-\dot{E}_e)$, we obtain

$$\begin{aligned}&(S'(\ddot{E}-\ddot{E}_e),D(\dot{E}-\dot{E}_e))+(C'S^{-1}C(E-E_e),D(\dot{E}-\dot{E}_e))\\ &=(S'(\ddot{E}'_f-\ddot{E}_e),D(\dot{E}-\dot{E}_e))+(C'(\dot{B}_f-\dot{B}'_e),D(\dot{E}-\dot{E}_e)).\end{aligned}$$

Then, using (2.7), we get

$$\begin{aligned}&(S'(\ddot{E}-\ddot{E}_e),D(\dot{E}-\dot{E}_e))+(D'S^{-1}C(E-E_e),C(\dot{E}-\dot{E}_e))\\ &=(S'(\ddot{E}'_f-\ddot{E}_e),D(\dot{E}-\dot{E}_e))+(D'(\dot{B}_f-\dot{B}'_e),C(\dot{E}-\dot{E}_e)),\end{aligned}$$

which can be written as

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|\dot{E}-\dot{E}_e\|^2_{W'}+\frac{1}{2}\frac{d}{dt}\|E-E_e\|^2_V\\ &=(\ddot{E}'_f-\ddot{E}_e,\dot{E}-\dot{E}_e)_{W'}+(D'(\dot{B}_f-\dot{B}'_e),C(\dot{E}-\dot{E}_e)).\end{aligned} \tag{5.35}$$

The following theorem gives a superconvergence result for $E-E_e$.

THEOREM 5.5. *Assume that*

$$\mathbf{E}\in W^{2,1}(0,T;H^3(\Omega_i)^3)\cap W^{3,1}(0,T;W^{2,p}(\Omega_i)^3), \qquad \mathbf{B}\in W^{2,1}(0,T;H^3(\Omega_i)^3)$$

satisfy the interface Maxwell system (1.1)–(1.8) *for* $i=1,2$ *and* $p>3$, *and* $(E,\,B)$ *is the solution of* (3.5)–(3.6) *on a nonuniform rectangular grid of size* h. *Then we have*

$$\begin{aligned}&\max_{0\le t\le T}\{\|(\dot{E}-\dot{E}_e)(t)\|_{W'}+\|(E-E_e)(t)\|_V\}\\ &\le Kh^2\sum_{i=1}^{2}\{\|\epsilon_i^{\frac{1}{2}}\mathbf{E}\|_{W^{2,1}(0,T;H^3(\Omega_i)^3)}\\ &+\|\epsilon_i^{\frac{1}{2}}\mathbf{E}\|_{W^{3,1}(0,T;W^{2,p}(\Omega_i)^3)}+\|\mu_i^{-\frac{1}{2}}\mathbf{B}\|_{W^{2,1}(0,T;H^3(\Omega_i)^3)}\}.\end{aligned} \tag{5.36}$$

Proof. By Lemma 5.3 we have

$$(\dot{E}'_f - \dot{E}_e, E - E_e)_{W'} = (\dot{v}, E - E_e)_{W'} + (D'\dot{w}, C(E - E_e)) + (S'^{-1}\dot{\lambda}, E - E_e)_{W'}.$$

A proof similar to the one for (5.23) leads to the following relations:

$$(\ddot{E}'_f - \ddot{E}_e, E - E_e)_{W'} = (\ddot{v}, E - E_e)_{W'} + (D'\ddot{w}, C(E - E_e)) + (S'^{-1}\ddot{\lambda}, E - E_e)_{W'},$$
$$(\dddot{E}'_f - \dddot{E}_e, E - E_e)_{W'} = (\dddot{v}, E - E_e)_{W'} + (D'\dddot{w}, C(E - E_e)) + (S'^{-1}\dddot{\lambda}, E - E_e)_{W'},$$

with $\ddot{v}, \dddot{v}, \ddot{w}, \dddot{w}, \ddot{\lambda}, \dddot{\lambda}$ obeying the same estimates as those stated in Lemma 5.3. In addition, by Lemma 5.2 and (5.1), we have

$$(C(E - E_e), D'(B_f - B'_e)) = (C(E - E_e), u) + (C(E - E_e), \xi).$$

Again, by a proof similar to the one of Lemma 5.2 we deduce that

$$(C(E - E_e), D'(\dot{B}_f - \dot{B}'_e)) = (C(E - E_e), \dot{u}) + (C(E - E_e), \dot{\xi}),$$
$$(C(E - E_e), D'(\ddot{B}_f - \ddot{B}'_e)) = (C(E - E_e), \ddot{u}) + (C(E - E_e), \ddot{\xi}),$$

with the corresponding estimates for $\dot{u}, \ddot{u}, \dot{\xi}$, and $\ddot{\xi}$ as those stated in Lemma 5.2. Now, integrating (5.35) over $[0, t_1]$, and by (5.34), we obtain

$$\begin{aligned}&\|(\dot{E} - \dot{E}_e)(t_1)\|^2_{W'} + \|(E - E_e)(t_1)\|^2_V \\ &= 2\int_0^{t_1} (\ddot{E}'_f - \ddot{E}_e, \dot{E} - \dot{E}_e)_{W'}\, ds + 2\int_0^{t_1} (D'(\dot{B}_f - \dot{B}'_e), C(\dot{E} - \dot{E}_e))\, ds.\end{aligned}$$

An application of integration by parts yields

$$\begin{aligned}&\|(\dot{E} - \dot{E}_e)(t_1)\|^2_{W'} + \|(E - E_e)(t_1)\|^2_V \\ &= 2\int_0^{t_1} (\ddot{E}'_f - \ddot{E}_e, \dot{E} - \dot{E}_e)_{W'}\, ds \\ &\quad + 2(D'(\dot{B}_f - \dot{B}'_e), C(E - E_e))(t_1) - 2\int_0^{t_1} (D'(\ddot{B}_f - \ddot{B}'_e), C(E - E_e))\, ds.\end{aligned}$$

Substituting the relations given in the beginning of the proof into the above equation, and using the Cauchy–Schwarz inequality together with the estimates in Lemmas 5.2 and 5.3, we obtain the desired estimate. ∎

The following theorem gives a superconvergence result for $B - B_f$.

THEOREM 5.6. *Under the same assumptions as in Theorem* 5.5, *we have*

$$\begin{aligned}&\max_{0\le t\le T}\left\{\|(\dot{B} - \dot{B}_f)(t)\|_W + \sup_{\phi\in\mathbb{R}^{M_1}} \frac{|(C'(B - B_f), D\phi)|}{\|\phi\|_V}\right\} \\ &\quad\le Kh^2 \sum_{i=1}^{2}\{\|\epsilon_i^{\frac{1}{2}}\mathbf{E}\|_{W^{2,1}(0,T;H^3(\Omega_i)^3)} + \|\epsilon_i^{\frac{1}{2}}\mathbf{E}\|_{W^{3,1}(0,T;W^{2,p}(\Omega_i)^3)} \\ &\qquad + \|\mu_i^{-\frac{1}{2}}\mathbf{B}\|_{W^{2,1}(0,T;H^3(\Omega_i)^3)}\}.\end{aligned}$$

Proof. By (5.1) and (5.36), we obtain

$$\begin{aligned}&\max_{0\le t\le T}\|(\dot{B} - \dot{B}_f)(t)\|_W \\ &\quad\le Kh^2 \sum_{i=1}^{2}\{\|\epsilon_i^{\frac{1}{2}}\mathbf{E}\|_{W^{2,1}(0,T;H^3(\Omega_i)^3)} + \|\epsilon_i^{\frac{1}{2}}\mathbf{E}\|_{W^{3,1}(0,T;W^{2,p}(\Omega_i)^3)} \\ &\qquad + \|\mu_i^{-\frac{1}{2}}\mathbf{B}\|_{W^{2,1}(0,T;H^3(\Omega_i)^3)}\}.\end{aligned}$$

By (5.2), we have

$$C'(B - B_f) = S' \frac{d}{dt}(E - E_e) - S' \frac{d}{dt}(E'_f - E_e) - C'(B_f - B'_e). \tag{5.37}$$

For any $\phi \in \mathbb{R}^{M_1}$, multiplying (5.37) by $D\phi$ and using (2.7), we obtain

$$(C'(B - B_f), D\phi) = (\dot{E} - \dot{E}_e, \phi)_{W'} - (\dot{E}'_f - \dot{E}_e, \phi)_{W'} - (D'(B_f - B'_e), C\phi).$$

First, by (5.36) we have

$$\begin{aligned} &|(\dot{E} - \dot{E}_e, \phi)_{W'}| \\ &\quad \le\ Kh^2 \|\phi\|_{W'} \sum_{i=1}^{2} \{ \|\epsilon_i^{\frac{1}{2}} \mathbf{E}\|_{W^{2,1}(0,T;H^3(\Omega_i)^3)} +\ \|\epsilon_i^{\frac{1}{2}} \mathbf{E}\|_{W^{3,1}(0,T;W^{2,p}(\Omega_i)^3)} \\ &\qquad\qquad + \|\mu_i^{-\frac{1}{2}} \mathbf{B}\|_{W^{2,1}(0,T;H^3(\Omega_i)^3)} \}. \end{aligned}$$

Then, using (5.23) and (5.25), we easily derive

$$|(\dot{E}'_f - \dot{E}_e, \phi)_{W'}| \le Kh^2 \|\phi\|_V \sum_{i=1}^{2} \{ \|\epsilon_i^{\frac{1}{2}} \mathbf{E}\|_{W^{2,1}(0,T;H^3(\Omega_i)^3)} + \|\epsilon_i^{\frac{1}{2}} \mathbf{E}\|_{W^{3,1}(0,T;W^{2,p}(\Omega_i)^3)} \},$$

while using (5.14) and (5.15) we have

$$|(D'(B_f - B'_e), C\phi)| \le Kh^2 \|\phi\|_V \sum_{i=1}^{2} \|\mu_i^{-\frac{1}{2}} \mathbf{B}\|_{W^{2,1}(0,T;H^3(\Omega_i)^3)}.$$

Collecting the above results leads to

$$\begin{aligned} \frac{|(C'(B - B_f), C\phi)|}{\|\phi\|_V} &\le K_1 h^2 \sum_{i=1}^{2} (\|\epsilon_i^{\frac{1}{2}} \mathbf{E}\|_{W^{2,1}(0,T;H^3(\Omega_i)^3)} + \|\epsilon_i^{\frac{1}{2}} \mathbf{E}\|_{W^{3,1}(0,T;W^{2,p}(\Omega_i)^3)}) \\ &\quad + K_2 h^2 \sum_{i=1}^{2} \|\mu_i^{-\frac{1}{2}} \mathbf{B}\|_{W^{2,1}(0,T;H^3(\Omega_i)^3)} \end{aligned}$$

for any $\phi \in \mathbb{R}^{M_1}$. ∎

6. Conclusion. Through a detailed analysis, we have established some rigorous convergence results for a finite volume method for the time-dependent Maxwell's equations in a three-dimensional polyhedral domain. Different materials are allowed to occupy portions of the domain, and interface conditions are imposed. Our analysis does not require extra regularity assumptions on the solutions of the interface problem beyond those for the analogous convergence results for noninterface Maxwell's equations, and our estimates also exhibit the detailed dependence on the material parameters. For brevity, we have chosen the case of two subdomains in our derivations, though much of our theory can be generalized to cases involving multiple subdomains. Implementations and applications of the methods discussed here are currently underway, and the results will be reported elsewhere.

Acknowledgments. The authors would like to thank the editor and the referees whose constructive suggestions and comments significantly improved the presentation of the paper.

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