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MINNAERT RESONANCES FOR BUBBLES IN SOFT ELASTIC MATERIALS*

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Abstract. Minnaert resonance is a widely known acoustic phenomenon, and it has many important applications, in particular in the effective realization of acoustic metamaterials using bubbly media in recent years. In this paper, motivated by the Minnaert resonance in acoustics, we consider the low-frequency resonance for acoustic bubbles embedded in soft elastic materials. This is a hybrid physical process that couples the acoustic and elastic wave propagations. By delicately and subtly balancing the acoustic and elastic parameters as well as the geometry of the bubble, we show that Minnaert resonance can occur (at least approximately) for rather general constructions. Our study highlights the great potential for the effective realization of negative elastic materials by using bubbly elastic media.

Key words. Minnaert resonance, bubbly elastic medium, hybrid Neumann–Poincaré operator, spectral, negative elastic materials

AMS subject classifications. 35R30, 35B30, 35Q60, 47G40

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1. Introduction.

The oscillation of bubbles in media is a classical problem. In particular, when bubbles are immersed in liquids, even a very small-volume fraction of bubbles can have a significant influence on the effective velocity of waves in liquids [10, 19]. This is due to the high oscillation of the bubbles caused by the high contrast in density between the bubbles and the surrounding liquid [2, 18]. In fact, at a particular low frequency known as the *Minnaert resonant frequency*, the bubbles can be treated as acoustic resonators [27]. The exceptional acoustic properties mentioned above can have many important applications and in particular can be used to design new materials, such as phononic crystals. In addition to much experimental progress, the bubbly acoustic materials have been systematically and comprehensively investigated recently in the mathematical literature. Furthermore, based on the mathematical theory developed, novel applications have also been proposed, especially for the effective realization of acoustic metamaterials. For the case that a single bubble is immersed in liquids, the authors in [2] provided a rigorous treatment of the Minnaert resonance and the monopole approximation. Later, they investigated the acoustic scattering by a large number of bubbles in liquids at frequencies near the Minnaert resonant frequency in [3]. Thus, by designing bubble metascreens, the superabsorption effect can be achieved [4]. Around the Minnaert resonant frequency, an effective medium theory was derived in [8].

Nevertheless, as pointed out in [20, 31], the practical constructions of acoustic

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bubbly designs are very challenging. The major difficulty arises from making bubbles have a uniform size and letting them remain inside the liquids. In order to overcome these challenges, substituting the host medium from liquids to soft elastic materials (the shear modulus is small) becomes a more practical scheme. In fact, the oscillation of a spherical cavity in an elastic material was investigated many years ago [28]. When a spherical bubble is immersed in a soft elastic material, it was shown in [11] that there also exists a certain low-frequency resonance. Using such resonant properties, the bubbly elastic structures have been used in the experimental design of new materials for strikingly new applications. For example, bubble phononic crystals were designed in [20], superabsorption of acoustic waves with bubble metascreens was achieved in [21], and reducing underwater sound transmission was shown in [12] by microfabricating cavities into silicone rubber (a soft elastic material).

Motivated by the aforementioned physical and mathematical studies, we consider in this paper the low-frequency resonance for the case where a bubble is embedded in a soft elastic medium. We aim to derive a systematic and comprehensive mathematical understanding of the resonance phenomena caused by the acoustic and elastic interactions. It turns out that the mathematical investigation on the resonance associated with the elastic bubbly media is more challenging than that for the acoustic bubbly media. Indeed, we note that, first, the wave scattering from an elastic bubbly medium is a hybrid physical process which couples the acoustic wave propagation inside the bubble and the elastic wave propagation outside the bubble. Second, since the shear modulus in the elastic material is nonzero, the resonance heavily depends on the geometry of the bubble [11]. This property is in sharp contrast to the case of bubbles in liquids which features weak shape dependence [2]. Therefore, one cannot expect an explicit expression of the resonant frequencies (unless the geometry of the bubble is simple—say, a radial one) as in the case for bubbles in liquids that was derived in [2]. Third, the bubble-liquid resonance only depends on the high contrast of the density between the bubble and the liquid. However, for the bubble–elastic material resonance, in addition to the high contrast of the density, the high contrast of the shear modulus and the compression modulus is required; see also [30] for a related discussion.

According to our discussion above, it is clear that the bubble-elastic resonance is of a different physical nature from the bubble-liquid resonance. Nevertheless, in order to reveal its origin of motivation, as well as for terminological convenience, we still call it the Minnaert resonance in the present paper. In order to derive the resonance results, in the spirit of the mathematical treatment in [2], we rely on the layer-potential techniques, which boil down our study to the asymptotic and spectral analysis of the layer-potential operators involved for the coupled PDE systems. By delicately and subtly balancing the acoustic and elastic parameters as well as the geometry of the bubble, we show that Minnaert resonance can (at least approximately) occur for a rather general construction in the three-dimensional case. In the two-dimensional case, due to technical constraints, we can only deal with the case that the bubble is in the radial geometry. Moreover, as mentioned earlier, we only consider the case with a single bubble embedded in a soft elastic material. We shall study the other case, e.g., the scattering from multiple bubbles, in our forthcoming work. It is emphasized that similar to the bubble-liquid case [5], our study highlights the great potential for the effective realization of negative elastic materials by using bubbly elastic media, which we shall also investigate in our near-future study.

The rest of the paper is organized as follows. In section 2, we present the general mathematical formulation of our study, especially the acoustic-elastic wave scattering from a bubble-elastic structure and its integral reformulation. In section 3, we discuss the general requirements for the medium configuration and also derive some auxiliary results for the subsequent use. Sections 4 and 5 are, respectively, devoted to the Minnaert resonances in three and two dimensions. Our study is concluded in section 6 with some related remarks.

2. Mathematical setup. In this section, we present the general mathematical formulation of our study. We consider an air bubble D in our study, and D is assumed to be a bounded domain in $\mathbb{R}^N$ ($N = 2, 3$), with a C^2-regular boundary ∂D. Let $\rho_b \in \mathbb{R}_+$ and $\kappa \in \mathbb{R}_+$ signify the density and the bulk modulus of the air inside the bubble, respectively. Assume that the background $\mathbb{R}^N \backslash \overline{D}$ is occupied by a regular and isotropic elastic material parameterized by the Lamé constants $(\tilde{\lambda}, \tilde{\mu})$ satisfying the following strong convexity conditions:

$$\text{(i)}\ \ \tilde{\mu} > 0 \qquad \text{and} \qquad \text{(ii)}\ \ N\tilde{\lambda} + 2\tilde{\mu} > 0. \tag{2.1}$$

The density of the background material is set to be $\rho_e \in \mathbb{R}_+$. Let $\mathbf{u}^i$ be an incident elastic wave, which is an entire solution to $\mathcal{L}_{\tilde{\lambda},\tilde{\mu}}\mathbf{u} + \omega^2 \rho_e \mathbf{u} = 0$ in $\mathbb{R}^N$. Here, $\omega \in \mathbb{R}_+$ denotes the frequency of the elastic wave. The acoustic-elastic wave interaction is described by the following coupled PDE system (cf. [29]):

$$\begin{cases} \mathcal{L}_{\tilde{\lambda},\tilde{\mu}}\mathbf{u}(\mathbf{x}) + \omega^2 \rho_e \mathbf{u}(\mathbf{x}) = 0, & \mathbf{x} \in \mathbb{R}^N \backslash \overline{D}, \\ \triangle u(\mathbf{x}) + \tilde{k}^2 u(\mathbf{x}) = 0, & \mathbf{x} \in D, \\ \mathbf{u}(\mathbf{x}) \cdot \boldsymbol{\nu} - \frac{1}{\rho_b \omega^2} \nabla u(\mathbf{x}) \cdot \boldsymbol{\nu} = 0, & \mathbf{x} \in \partial D, \\ \partial_{\tilde{\boldsymbol{\nu}}} \mathbf{u}(\mathbf{x}) + u(\mathbf{x})\boldsymbol{\nu} = 0, & \mathbf{x} \in \partial D, \\ \mathbf{u}(\mathbf{x}) - \mathbf{u}^i(\mathbf{x}) \qquad \text{satisfies the radiation condition,} & \end{cases} \tag{2.2}$$

where $\mathbf{u}$ is the total elastic wave field outside the domain D, u is the pressure inside the domain D, $\omega \in \mathbb{R}_+$ is the angular frequency, and $\tilde{k} = \omega / c_b$ with $c_b = \sqrt{\kappa / \rho_b}$ signifying the velocity of the wave in D. In (2.2), the Lamé operator $\mathcal{L}_{\tilde{\lambda},\tilde{\mu}}$ and the co-normal derivative $\partial_{\tilde{\boldsymbol{\nu}}}$, associated with the parameters $(\tilde{\lambda}, \tilde{\mu})$ are respectively defined by

$$\mathcal{L}_{\tilde{\lambda},\tilde{\mu}}\mathbf{w} := \tilde{\mu} \triangle \mathbf{w} + (\tilde{\lambda} + \tilde{\mu}) \nabla\nabla \cdot \mathbf{w} \tag{2.3}$$

and

$$\partial_{\tilde{\boldsymbol{\nu}}} \mathbf{w} = \tilde{\lambda} (\nabla \cdot \mathbf{w}) \boldsymbol{\nu} + 2\tilde{\mu} (\nabla^s \mathbf{w}) \boldsymbol{\nu}. \tag{2.4}$$

Here $\boldsymbol{\nu}$ represents the outward unit normal vector to ∂D, and the operator ∇^s is the symmetric gradient

$$\nabla^s \mathbf{w} := \frac{1}{2} \left(\nabla \mathbf{w} + \nabla \mathbf{w}^t \right), \tag{2.5}$$

with $\nabla \mathbf{w}$ denoting the matrix $(\partial_j w_i)_{i,j=1}^N$ and the superscript t signifying the matrix transpose. In (2.2), the third condition denotes the continuity of the normal component of the displacement on the boundary ∂D, and the fourth condition is the continuity of the stress across ∂D. Moreover, the radiation condition in (2.2) designates the following condition as $|\mathbf{x}| \to +\infty$ (cf. [22]):

$$\begin{aligned} (\nabla\times\nabla\times(\mathbf{u}-\mathbf{u}^i))(\mathbf{x})\times\frac{\mathbf{x}}{|\mathbf{x}|}-\mathrm{i}\tilde{k}_s\nabla\times(\mathbf{u}-\mathbf{u}^i)(\mathbf{x}) =&\mathcal{O}(|\mathbf{x}|^{-2}),\\ \frac{\mathbf{x}}{|\mathbf{x}|}\cdot\left(\nabla(\nabla\cdot(\mathbf{u}-\mathbf{u}^i))\right)(\mathbf{x})-\mathrm{i}\tilde{k}_p\nabla(\mathbf{u}-\mathbf{u}^i)(\mathbf{x}) =&\mathcal{O}(|\mathbf{x}|^{-2}), \end{aligned} \tag{2.6}$$

where $\mathrm{i}=\sqrt{-1}$,

$$\tilde{k}_s=\frac{\omega}{\tilde{c}_s}=\frac{\omega}{\sqrt{\tilde{\mu}/\rho_e}}, \quad \text{and} \quad \tilde{k}_p=\frac{\omega}{\tilde{c}_p}=\frac{\omega}{\sqrt{(\tilde{\lambda}+2\tilde{\mu})/\rho_e}}, \tag{2.7}$$

with $\tilde{\lambda}$ and $\tilde{\mu}$ as defined in (2.1).

Next we apply the potential theory to derive the integral representation of the solution to system (2.2) and give the definition of the resonance. First, we introduce the potential operators for the Helmholtz system and the Lamé system. Let $G^k(\mathbf{x})$ be the fundamental solution of the operator $\triangle + k^2$, namely

$$G^k(\mathbf{x})=\begin{cases} -\dfrac{\mathrm{i}}{4}H_0^{(1)}(k|\mathbf{x}|), & N=2,\\ -\dfrac{e^{\mathrm{i}k|\mathbf{x}|}}{4\pi|\mathbf{x}|}, & N=3, \end{cases} \tag{2.8}$$

where $H_0^{(1)}$ is the zeroth-order Hankel function of the first kind. The single layer potential associated with the Helmholtz system is defined for $\varphi(\mathbf{x})\in L^2(\partial D)$ by

$$S_{\partial D}^k[\varphi](\mathbf{x})=\int_{\partial D}G^k(\mathbf{x}-\mathbf{y})\varphi(\mathbf{y})ds(\mathbf{y}), \quad \mathbf{x}\in\mathbb{R}^N. \tag{2.9}$$

Then the conormal derivative of the single layer potential enjoys the jump formula

$$\nabla S_{\partial D}^k[\varphi]\cdot\boldsymbol{\nu}|_{\pm}(\mathbf{x})=\left(\pm\frac{1}{2}I+K_{\partial D}^{k,*}\right)[\varphi](\mathbf{x}), \quad \mathbf{x}\in\partial D, \tag{2.10}$$

where

$$K_{\partial D}^{k,*}[\varphi](\mathbf{x})=\text{p.v.}\int_{\partial D}\nabla_{\mathbf{x}}G^k(\mathbf{x}-\mathbf{y})\cdot\boldsymbol{\nu}_{\mathbf{x}}\varphi(\mathbf{y})ds(\mathbf{y}), \quad \mathbf{x}\in\partial D,$$

which is also known as the Neumann–Poincaré operator associated with the Helmholtz system. Here and in what follows, p.v. stands for the Cauchy principal value. Moreover, we introduce the following L^2-adjoint of the operator $K_{\partial D}^{k,*}$:

$$K_{\partial D}^{k}[\varphi](\mathbf{x})=\text{p.v.}\int_{\partial D}\nabla_{\mathbf{y}}G^k(\mathbf{x}-\mathbf{y})\cdot\boldsymbol{\nu}_{\mathbf{y}}\varphi(\mathbf{y})ds(\mathbf{y}), \quad \mathbf{x}\in\partial D.$$

In what follows, we denote $S_{\partial D}^k$, $K_{\partial D}^{k,*}$, and $K_{\partial D}^k$ by $S_{\partial D,0}$, $K_{\partial D,0}^*$, and $K_{\partial D,0}$ for $k=0$. We would like to point out that the operators $K_{\partial D,0}^*$ and $K_{\partial D,0}$ have the following expressions in three dimensions:

$$\begin{aligned} K_{\partial D,0}^*[\varphi](\mathbf{x})&=\int_{\partial D}\frac{\langle\mathbf{x}-\mathbf{y},\boldsymbol{\nu}_{\mathbf{x}}\rangle}{4\pi|\mathbf{x}-\mathbf{y}|^3}\varphi(\mathbf{y})ds(\mathbf{y}), \quad \mathbf{x}\in\partial D,\\ K_{\partial D,0}[\varphi](\mathbf{x})&=\int_{\partial D}\frac{\langle\mathbf{y}-\mathbf{x},\boldsymbol{\nu}_{\mathbf{y}}\rangle}{4\pi|\mathbf{x}-\mathbf{y}|^3}\varphi(\mathbf{y})ds(\mathbf{y}), \quad \mathbf{x}\in\partial D. \end{aligned} \tag{2.11}$$

We refer the reader to [9, 26] for the mapping properties of the operators introduced above.

Next we introduce the potential operators for the Lamé system. The fundamental solution $\mathbf{\Gamma}^\omega = (\Gamma^\omega_{i,j})^N_{i,j=1}$ for the operator $\mathcal{L}_{\tilde{\lambda},\tilde{\mu}} + \rho_e\omega^2$ can be decomposed into shear and pressure components (cf. [6]):

$$\mathbf{\Gamma}^\omega = \mathbf{\Gamma}^\omega_s + \mathbf{\Gamma}^\omega_p, \tag{2.12}$$

where

$$\mathbf{\Gamma}^\omega_p = -\frac{1}{\rho_e\omega^2}\nabla\nabla G^{\tilde{k}_p} \quad \text{and} \quad \mathbf{\Gamma}^\omega_s = \frac{1}{\rho_e\omega^2}(\tilde{k}_s^2\mathbf{I} + \nabla\nabla)G^{\tilde{k}_s},$$

with $\mathbf{I}$ denoting the $N \times N$ identity matrix, G^k given in (2.8) and $\tilde{k}_s$ as well as $\tilde{k}_p$ defined as in (2.7). The single layer potential operator associated with the fundamental solution $\mathbf{\Gamma}^\omega$ is defined by

$$\mathbf{S}^\omega_{\partial D}[\boldsymbol{\varphi}](\mathbf{x}) = \int_{\partial D} \mathbf{\Gamma}^\omega(\mathbf{x}-\mathbf{y})\boldsymbol{\varphi}(\mathbf{y})ds(\mathbf{y}), \quad \mathbf{x} \in \mathbb{R}^N, \tag{2.13}$$

for $\boldsymbol{\varphi} \in L^2(\partial D)^N$. On the boundary ∂D, the conormal derivative of the single layer potential satisfies the following jump formula:

$$\frac{\partial \mathbf{S}^\omega_{\partial D}[\boldsymbol{\varphi}]}{\partial \boldsymbol{\nu}}\Big|_\pm(\mathbf{x}) = \left(\pm\frac{1}{2}\mathbf{I} + (\mathbf{K}^\omega_{\partial D})^*\right)[\boldsymbol{\varphi}](\mathbf{x}), \quad \mathbf{x} \in \partial D, \tag{2.14}$$

where

$$\mathbf{K}^{\omega,*}_{\partial D}[\boldsymbol{\varphi}](\mathbf{x}) = \text{p.v.}\int_{\partial D} \frac{\partial \mathbf{\Gamma}^\omega}{\partial \boldsymbol{\nu}(\mathbf{x})}(\mathbf{x}-\mathbf{y})\boldsymbol{\varphi}(\mathbf{y})ds(\mathbf{y}), \tag{2.15}$$

with the subscript $\pm$ indicating the limits from outside and inside D, respectively. The operator $\mathbf{K}^{\omega,*}_{\partial D}$ is called Neumann–Poincaré (N–P) operator of the Lamé system. In our subsequent analysis, we also need the following single layer potential operators associated with the p-wave (pressure wave) and s-wave (shear wave), respectively:

$$\mathbf{S}^{\omega,i}_{\partial D}[\boldsymbol{\varphi}](\mathbf{x}) = \int_{\partial D} \mathbf{\Gamma}^\omega_i(\mathbf{x}-\mathbf{y})\boldsymbol{\varphi}(\mathbf{y})ds(\mathbf{y}), \quad \mathbf{x} \in \mathbb{R}^N\backslash\overline{D}, \tag{2.16}$$

where $\boldsymbol{\varphi}(\mathbf{y}) \in L^2(\partial D)^N$ and the kernel functions $\mathbf{\Gamma}^\omega_i$, $i = p, s$, are defined as in (2.12). We refer the reader to [6] for the mapping properties of the operators introduced above.

With the help of the potential operators introduced above, the solution to (2.2) can be represented by the following integral ansatz:

$$\mathbf{u} = \begin{cases} S^{\tilde{k}}_{\partial D}[\varphi_b](\mathbf{x}), & \mathbf{x} \in D, \\ \mathbf{S}^\omega_{\partial D}[\boldsymbol{\varphi}_e](\mathbf{x}) + \mathbf{u}^i, & \mathbf{x} \in \mathbb{R}^N\backslash\overline{D}, \end{cases} \tag{2.17}$$

for some density functions $\varphi_b \in L^2(\partial D)$ and $\boldsymbol{\varphi}_e \in L^2(\partial D)^N$. By matching the transmission conditions on the boundary ∂D, along with the help of the jump formulas (2.10) and (2.14), it can be verified by some straightforward calculations that the density functions $\varphi_b, \boldsymbol{\varphi}_e$ satisfy the following system of boundary integral equations:

$$\tilde{\mathcal{A}}(\omega,\delta)[\Phi](\mathbf{x}) = F(\mathbf{x}), \quad \mathbf{x} \in \partial D, \tag{2.18}$$

where

$$\tilde{\mathcal{A}}(\omega,\delta)=\begin{pmatrix} \frac{1}{\rho_b\omega^2}\left(-\frac{I}{2}+K_{\partial D}^{\tilde{k},*}\right) & -\boldsymbol{\nu}\cdot\mathbf{S}_{\partial D}^{\omega} \\ \boldsymbol{\nu} S_{\partial D}^{\tilde{k}} & \frac{\mathbf{I}}{2}+\mathbf{K}_{\partial D}^{\omega,*} \end{pmatrix},\ \Phi=\begin{pmatrix}\varphi_b\\ \boldsymbol{\varphi}_e\end{pmatrix},\ \text{and } F=\begin{pmatrix}\boldsymbol{\nu}\cdot\mathbf{u}^i\\ -\partial_{\boldsymbol{\nu}}\mathbf{u}^i\end{pmatrix}.$$

Then the Minnaert resonance of the system (2.2) is defined for all $\omega\in\mathbb{C}$ such that the following equation holds:

$$\tilde{\mathcal{A}}(\omega,\delta)[\Phi](\mathbf{x})=0 \tag{2.19}$$

for a nontrivial solution $\Phi\in\mathcal{H}$. For notational convenience, we shall write $\mathcal{H}:=L^2(\partial D)\times L^2(\partial D)^N$. Moreover, in our subsequent study of the Minnaert resonance, we may weaken the condition (2.19) by finding a solution $\Phi\in\mathcal{H}$ with $\|\Phi\|_{\mathcal{H}}=1$ such that for $\omega\ll 1$,

$$\|\tilde{\mathcal{A}}(\omega,\delta)[\Phi](\mathbf{x})\|_{\mathcal{H}}\ll 1. \tag{2.20}$$

If the condition (2.20) is fulfilled, we say that the weak (Minnaert) resonance occurs for the system (2.2). In contrast to the weak Minnaert resonance, when the condition (2.19) is fulfilled, we say that the strong (Minnaert) resonance occurs.

We remark that the definition of the weak resonance in (2.20) is rather qualitative and heuristic at the moment, but it will become clearer and more substantiated in our subsequent study; see, e.g., (3.10), where all the parameters involved are dimensionless after appropriate normalization.

Finally, we make some remarks on the definition of the Minnaert resonance introduced above. In fact, the definition of the strong (Minnaert) resonance is similar to the one introduced in [2] for the bubble-liquid resonance. For the weak resonance, let us assume that

$$\tilde{\mathcal{A}}(\omega,\delta)[\Phi](\mathbf{x})=\tilde{\Psi}.$$

According to (2.20), one has that

$$\|\tilde{\Psi}\|_{\mathcal{H}}\ll 1.$$

Set $\Psi=\tilde{\Psi}/\|\tilde{\Psi}\|_{\mathcal{H}}$ such that $\|\Psi\|_{\mathcal{H}}=1$. If F in (2.18) is properly chosen, which has a component being Ψ, one can easily conclude from (2.17) that the scattering wave will blow up at the order $1/\|\tilde{\Psi}\|_{\mathcal{H}}$.

3. General requirements for the medium configuration and auxiliary results on the layer-potential operators.

In this section, we first introduce some general requirements for the medium configuration that are critical for the occurrence of the Minnaert resonances in our subsequent constructions of the bubble-elastic structures in sections 4 and 5. Then we derive some auxiliary results for subsequent use.

3.1. General requirements for the medium configuration.

We will consider

$$\delta=\rho_b/\rho_e=o(1), \tag{3.1}$$

which states that the contrast of the densities of the bubble and the elastic material is high. Moreover, we assume that the bulk modulus of the air κ and the compression modulus $\tilde{\lambda}$ as well as the shear modulus $\tilde{\mu}$ of the elastic material satisfy

$$\kappa/\tilde{\lambda}=\mathcal{O}(\delta)\quad\text{and}\quad\tilde{\mu}/\tilde{\lambda}=o(1). \tag{3.2}$$

Under these assumptions, we can easily derive

$$\tau = \frac{c_b}{\tilde{c}_p} = \frac{\sqrt{\kappa/\rho_b}}{\sqrt{\left(\tilde{\lambda} + 2\tilde{\mu}\right)/\rho_e}} = \mathcal{O}(1), \tag{3.3}$$

where c_b and $\tilde{c}_p$ are defined as in (2.2) and (2.7), respectively. Indeed, the assumptions in (3.1) and (3.2) are reasonable, and this is the case that air bubbles are embedded in the polydimethylsiloxane, a soft elastic material (cf. [11]).

As a matter of fact, the low frequency is mainly caused by the fact that the size of the air bubble D is much smaller than the wavelength of the elastic wave. Since the elastic wave can be decomposed into the compressional wave (p-wave) and the shear wave (s-wave) [14], we are mainly concerned in this paper with the case that the wavelength of the p-wave is much larger than the size of the bubble D and the wavelength of the s-wave generically does not satisfy this requirement. That means that the Minnaert resonance is mainly caused by the p-wave. Thus by the coordinate transformation, we may assume that the size of the domain D is of order 1 and $\omega = o(1)$. Since c_b is fixed, we further have

$$\tilde{k} = o(1) \quad \text{and} \quad \tilde{k}_p = o(1),$$

where $\tilde{k}$ and $\tilde{k}_p$ are defined as in (2.2) and (2.7), respectively.

Let L be the typical length (average length) of the domain D. Then we introduce the following nondimensional parameters:

$$\begin{gathered} \mathbf{x}' = \mathbf{x}/L, \qquad k = \tilde{k}L, \qquad \mathbf{u}' = \mathbf{u}/L, \\ \mu = \tilde{\mu}/(\tilde{\lambda} + 2\tilde{\mu}), \quad \lambda = \tilde{\lambda}/(\tilde{\lambda} + 2\tilde{\mu}), \quad u' = u/(\rho_b c_b^2). \end{gathered} \tag{3.4}$$

Thus, from the previous assumptions, one has that

$$k = o(1), \quad \delta = \rho_b/\rho_e = o(1), \quad \tau = \mathcal{O}(1), \quad \mu = o(1), \quad \text{and} \quad \lambda = \mathcal{O}(1). \tag{3.5}$$

Substituting these parameters into (2.2) and dropping the primes, one can obtain the following coupled PDE system for our subsequent study:

$$\begin{cases} \mathcal{L}_{\lambda,\mu}\mathbf{u}(\mathbf{x}) + k^2\tau^2\mathbf{u}(\mathbf{x}) = 0, & \mathbf{x} \in \mathbb{R}^N \backslash \overline{D}, \\ \triangle u(\mathbf{x}) + k^2 u(\mathbf{x}) = 0, & \mathbf{x} \in D, \\ \mathbf{u}(\mathbf{x}) \cdot \boldsymbol{\nu} - \frac{1}{k^2}\nabla u(\mathbf{x}) \cdot \boldsymbol{\nu} = 0, & \mathbf{x} \in \partial D, \\ \partial_{\boldsymbol{\nu}}\mathbf{u}(\mathbf{x}) + \delta\tau^2 u(\mathbf{x})\boldsymbol{\nu} = 0, & \mathbf{x} \in \partial D, \\ \mathbf{u}(\mathbf{x}) - \mathbf{u}^i(\mathbf{x}) \qquad \text{satisfies the radiation condition,} \end{cases} \tag{3.6}$$

where τ is as defined in (3.3). Here we would like to point out that in (3.6), the p-wavenumber satisfies

$$k_p = \frac{k\tau}{c_p} = \frac{k\tau}{\sqrt{\lambda + 2\mu}} = o(1).$$

Following our earlier discussions in (2.17)–(2.19), the solution to the system (3.6) can be given by

$$\mathbf{u} = \begin{cases} S_{\partial D}^{k}[\varphi_b](\mathbf{x}), & \mathbf{x} \in D, \\ \mathbf{S}_{\partial D}^{k\tau}[\boldsymbol{\varphi}_e](\mathbf{x}) + \mathbf{u}^i, & \mathbf{x} \in \mathbb{R}^N \backslash \overline{D}, \end{cases} \tag{3.7}$$

for some surface densities $(\varphi_b, \boldsymbol{\varphi}_e) \in \mathcal{H}$ that satisfy

$$\mathcal{A}(k,\delta)[\Phi](\mathbf{x}) = F(\mathbf{x}), \quad \mathbf{x} \in \partial D, \tag{3.8}$$

where

$$\mathcal{A}(k,\delta) = \begin{pmatrix} \frac{1}{k^2}\left(-\frac{I}{2} + K_{\partial D}^{k,*}\right) & -\boldsymbol{\nu}\cdot\mathbf{S}_{\partial D}^{k\tau} \\ \delta\tau^2\boldsymbol{\nu}S_{\partial D}^{k} & \frac{\mathbf{I}}{2} + \mathbf{K}_{\partial D}^{k\tau,*} \end{pmatrix}, \ \Phi = \begin{pmatrix} \varphi_b \\ \boldsymbol{\varphi}_e \end{pmatrix}, \text{ and } F = \begin{pmatrix} \boldsymbol{\nu}\cdot\mathbf{u}^i \\ -\partial_{\boldsymbol{\nu}}\mathbf{u}^i \end{pmatrix}.$$

Based on our earlier definitions of the strong and weak Minnaert resonances, we shall establish the sufficient conditions for the occurrence of resonances associated with (3.7)–(3.8); that is, it holds for the strong resonance of (3.6) that

$$\mathcal{A}(k,\delta)[\Phi](\mathbf{x}) = 0, \tag{3.9}$$

while it holds for the weak resonance of (3.6) that

$$\|\mathcal{A}(k,\delta)[\Phi](\mathbf{x})\|_{\mathcal{H}} \ll 1 \tag{3.10}$$

for a nontrivial $\Phi \in \mathcal{H}$ with $\|\Phi\|_{\mathcal{H}} = 1$ and $k \ll 1$,

3.2. Some auxiliary results. We first introduce the following lemmas.

LEMMA 3.1. *If a vector field* $\mathbf{w} \in H^1(\mathbb{R}^3\backslash\overline{D})^3$ *satisfies the three equations*

$$\triangle\mathbf{w} + k^2\mathbf{w} = 0, \quad \nabla\times\mathbf{w} = 0, \quad \textit{and} \quad \nabla\cdot\mathbf{w} = 0,$$

with $k \neq 0$, *then* $\mathbf{w} \equiv 0$.

Proof. Direct calculations show that

$$\nabla\times\nabla\times\mathbf{w} = \nabla\nabla\cdot\mathbf{w} - \triangle\mathbf{w} = 0 + k^2\mathbf{w} = 0.$$

Thus one can obtain $\mathbf{w} \equiv 0$ since $k^2 \neq 0$. ∎

Recall that the operator $\mathbf{S}_{\partial D}^{\omega,s} : L^2(\partial D)^3 \to H^1(\mathbb{R}^3\backslash\overline{D})^3$ is defined in (2.16). In what follows, if $\boldsymbol{\varphi} \in L^2(\partial D)^3$ satisfies

$$\int_{\partial D} \frac{1}{\rho\omega^2}(k_s^2\mathbf{I} + \nabla\nabla)G^{k_s}(\mathbf{x}-\mathbf{y})\boldsymbol{\varphi}(\mathbf{y})ds(\mathbf{y}) = 0, \quad \mathbf{x} \in \mathbb{R}^3\backslash\overline{D},$$

then we say that $\boldsymbol{\varphi} \in \ker(\mathbf{S}_{\partial D}^{\omega,s})$.

LEMMA 3.2. *For* $\boldsymbol{\varphi} \in \ker(\mathbf{S}_{\partial D}^{\omega,s})$, *one has that*

$$\int_{\partial D} \nabla\nabla G^0(\mathbf{x}-\mathbf{y})\boldsymbol{\varphi}(\mathbf{y})ds(\mathbf{y}) = 0, \quad \mathbf{x} \in \mathbb{R}^3\backslash D,$$

where $G^k(\mathbf{x}-\mathbf{y})$ *is defined as in* (2.8) *with* $k = 0$.

Proof. From the definition of the fundamental solution in (2.12), if $\boldsymbol{\varphi} \in \ker(\mathbf{S}_{\partial D}^{\omega,s})$, one has that

$$\int_{\partial D} \frac{1}{\rho\omega^2}(k_s^2\mathbf{I} + \nabla\nabla)G^{k_s}(\mathbf{x}-\mathbf{y})\boldsymbol{\varphi}(\mathbf{y})ds(\mathbf{y}) = 0, \quad \mathbf{x} \in \mathbb{R}^3\backslash\overline{D}.$$

Thus one can further have that for $\mathbf{x} \in \mathbb{R}^3 \backslash \overline{D}$,

(3.11)
$$-\int_{\partial D} \nabla\nabla G^0(\mathbf{x}-\mathbf{y})\boldsymbol{\varphi}(\mathbf{y})ds(\mathbf{y}) = \int_{\partial D} \left(k_s^2 G^{k_s} + \nabla\nabla(G^{k_s} - G^0)\right)(\mathbf{x}-\mathbf{y})\boldsymbol{\varphi}(\mathbf{y})ds(\mathbf{y}).$$

From the expression
$$G^0(\mathbf{x}-\mathbf{y}) = -\frac{1}{4\pi|\mathbf{x}-\mathbf{y}|},$$
the integral possesses the following property:
$$\int_{\partial D} \nabla\nabla G^0(\mathbf{x}-\mathbf{y})\boldsymbol{\varphi}(\mathbf{y})ds(\mathbf{y}) \to 0 \quad \text{as} \quad |\mathbf{x}| \to \infty.$$

Therefore, from the expansion of the fundamental solution $G^{k_s}(\mathbf{x}-\mathbf{y})$ and (3.11), one can obtain that for $\mathbf{x} \in \mathbb{R}^3 \backslash \overline{D}$

(3.12)
$$-\int_{\partial D} \nabla\nabla G^0(\mathbf{x}-\mathbf{y})\boldsymbol{\varphi}(\mathbf{y})ds(\mathbf{y}) = -\frac{k_s^2}{4\pi}\int_{\partial D} \left(\frac{1}{|\mathbf{x}-\mathbf{y}|} + \nabla\nabla(|\mathbf{x}-\mathbf{y}|)\right)\boldsymbol{\varphi}(\mathbf{y})ds(\mathbf{y}).$$

Taking the Laplace operator $\triangle$ on both sides of the last equation gives that for $\mathbf{x} \in \mathbb{R}^3 \backslash \overline{D}$
$$0 = -\frac{k_s^2}{4\pi}\int_{\partial D} \nabla\nabla\left(\frac{1}{|\mathbf{x}-\mathbf{y}|}\right)\boldsymbol{\varphi}(\mathbf{y})ds(\mathbf{y}).$$

The proof is completed by noting that the function on the right side of the equation (3.12) is continuous from $\mathbb{R}^3 \backslash \overline{D}$ to $\mathbb{R}^3 \backslash D$. ∎

Lemma 3.3. *If $\boldsymbol{\varphi} \in \ker(\mathbf{S}_{\partial D}^{\omega,s})$ does not depend on k_s, then one has that*
$$\mathbf{S}_{\partial D}^{\omega,p}[\boldsymbol{\varphi}](\mathbf{x}) = \frac{1}{\lambda+2\mu}\int_{\partial D} G^{k_p}(\mathbf{x}-\mathbf{y})\boldsymbol{\varphi}(\mathbf{y})ds(\mathbf{y}), \quad \mathbf{x} \in \mathbb{R}^3 \backslash D,$$
where the operators $\mathbf{S}_{\partial D}^{\omega,i}$ $(i = p, s)$ are defined as in (2.16).

Proof. From the definition of the fundamental solution in (2.12), if $\boldsymbol{\varphi} \in \ker(\mathbf{S}_{\partial D}^{\omega,s})$, one has that
$$\int_{\partial D} \frac{1}{\rho\omega^2}(k_s^2\mathbf{I} + \nabla\nabla)G^{k_s}(\mathbf{x}-\mathbf{y})\boldsymbol{\varphi}(\mathbf{y})ds(\mathbf{y}) = 0, \quad \mathbf{x} \in \mathbb{R}^3 \backslash \overline{D}.$$

Replacing k_s with k_p in the last equation yields that for $\mathbf{x} \in \mathbb{R}^3 \backslash \overline{D}$,
$$\int_{\partial D} \frac{-1}{\rho\omega^2}(\nabla\nabla)G^{k_p}(\mathbf{x}-\mathbf{y})\boldsymbol{\varphi}(\mathbf{y})ds(\mathbf{y}) = \frac{1}{\lambda+2\mu}\int_{\partial D} G^{k_p}(\mathbf{x}-\mathbf{y})\boldsymbol{\varphi}(\mathbf{y})ds(\mathbf{y}).$$

The proof is readily completed by noting that the operator $S_{\partial D}^{k_p}$ is continuous from $\mathbb{R}^3 \backslash \overline{D}$ to $\mathbb{R}^3 \backslash D$. ∎

Remark 3.1. We can derive that $\ker(\mathbf{S}_{\partial D}^{\omega,s}) \neq \emptyset$ and $\boldsymbol{\nu} \in \ker(\mathbf{S}_{\partial D}^{\omega,s})$. Indeed, we set

$$\mathbf{w}(\mathbf{x}) = \int_{\partial D} \frac{1}{\rho\omega^2}(k_s^2\mathbf{I} + \nabla\nabla)G^{k_s}(\mathbf{x}-\mathbf{y})\boldsymbol{\nu}_{\mathbf{y}}ds(\mathbf{y}), \quad \mathbf{x} \in \mathbb{R}^3 \backslash \overline{D}. \tag{3.13}$$

It is directly verified that the function $\mathbf{w}$ defined in (3.13) satisfies the following two equations in $\mathbb{R}^3 \backslash \overline{D}$:
$$\triangle\mathbf{w} + k_s^2\mathbf{w} = 0 \quad \text{and} \quad \nabla \cdot \mathbf{w} = 0.$$

With the help of the identities

$$\nabla \times \nabla = 0 \quad \text{and} \quad \nabla_{\mathbf{x}} G^{k_s}(\mathbf{x}-\mathbf{y}) = -\nabla_{\mathbf{y}} G^{k_s}(\mathbf{x}-\mathbf{y}),$$

one furthermore has for $\mathbf{x} \in \mathbb{R}^3 \backslash \overline{D}$ that

$$\begin{aligned}
\nabla \times \mathbf{w} &= \int_{\partial D} \frac{1}{\rho\omega^2} \nabla_{\mathbf{x}} \times (k_s^2 \mathbf{I} + \nabla\nabla) G^{k_s}(\mathbf{x}-\mathbf{y}) \boldsymbol{\nu}_{\mathbf{y}} ds(\mathbf{y}) \\
&= \frac{1}{\rho\omega^2} \int_{\partial D} (k_s^2 \nabla_{\mathbf{x}} \times \mathbf{I} + \nabla_{\mathbf{x}} \times \nabla\nabla) G^{k_s}(\mathbf{x}-\mathbf{y}) \boldsymbol{\nu}_{\mathbf{y}} ds(\mathbf{y}) \\
&= \frac{-k_s^2}{\rho\omega^2} \int_{\partial D} \nabla_{\mathbf{y}} G^{k_s}(\mathbf{x}-\mathbf{y}) \times \boldsymbol{\nu}_{\mathbf{y}} ds(\mathbf{y}) \\
&= \frac{k_s^2}{\rho\omega^2} \int_{D} \nabla_{\mathbf{y}} \times \nabla_{\mathbf{y}} G^{k_s}(\mathbf{x}-\mathbf{y}) d\mathbf{y} = 0.
\end{aligned}$$

Finally, Lemma 3.1 shows that $\mathbf{w}$ defined in (3.13) vanishes in $\mathbb{R}^3 \backslash \overline{D}$, and one can conclude that $\boldsymbol{\nu} \in \ker(\mathbf{S}_{\partial D}^{\omega,s})$ and $\boldsymbol{\nu}$ does not depend on k_s.

LEMMA 3.4. *For the operators $S_{\partial D}^{k} : L^2(\partial D) \to H^1(\partial D)$ and $K_{\partial D}^{k,*} : L^2(\partial D) \to L^2(\partial D)$ defined in* (2.9) *and* (2.10)*, respectively, we have the following asymptotic expansions in three dimensions (cf.* [2]*):*

$$S_{\partial D}^{k} = \sum_{j=0}^{+\infty} k^j S_{\partial D,j}, \quad K_{\partial D}^{k,*} = \sum_{j=0}^{+\infty} k^j K_{\partial D,j}^{*}, \tag{3.14}$$

where

$$S_{\partial D,j}[\varphi](\mathbf{x}) = -\frac{\mathrm{i}}{4\pi} \int_{\partial D} \frac{(\mathrm{i}|\mathbf{x}-\mathbf{y}|)^{j-1}}{j!} \varphi(\mathbf{y}) ds(\mathbf{y}),$$

and

$$K_{\partial D,j}^{*}[\varphi](\mathbf{x}) = -\frac{\mathrm{i}^j (j-1)}{4\pi j!} \int_{\partial D} |\mathbf{x}-\mathbf{y}|^{j-3} (\mathbf{x}-\mathbf{y}) \cdot \boldsymbol{\nu}_{\mathbf{x}} \varphi(\mathbf{y}) ds(\mathbf{y}).$$

Moreover, $S_{\partial D,j}$ and $K_{\partial D,j}^{*}$ are uniformly bounded with respect to j, and the two series in (3.14) *are convergent in $\mathcal{L}(L^2(\partial D), H^1(\partial D))$ and $\mathcal{L}(L^2(\partial D))$, respectively.*

As discussed earlier, only the wavelength of the p-wave is required to be asymptotically larger than the size of the domain D, and the wavelength of the s-wave is not required to satisfy such a requirement; thus the low-frequency resonance is mainly caused by the p-wave, and the s-wave generically makes no contribution. Therefore, in the following analysis for the low-frequency resonance, we choose to mainly consider the density function $\boldsymbol{\varphi} \in \ker(\mathbf{S}_{\partial D}^{\omega,s})$, which can be proved not depending on k_s later. From Lemma 3.3, we can focus our attention on the operator $\mathbf{S}_{\partial D}^{\omega,p}$ with the kernel $\delta_{ij} G^{k_p}/(\lambda + 2\mu)$ and the operator $\mathbf{K}_{\partial D}^{\omega,p,*} : L^2(\partial D)^3 \to L^2(\partial D)^3$ defined in (2.15) with the kernel function $\boldsymbol{\Gamma}^{\omega}$ replaced by $\delta_{ij} G^{k_p}/(\lambda + 2\mu)$. By straightforward calculations, we have the following asymptotic expansions for the operators $\mathbf{S}_{\partial D}^{\omega,p}$ and $\mathbf{K}_{\partial D}^{\omega,p,*}$ for the density function $\boldsymbol{\varphi} \in \ker(\mathbf{S}_{\partial D}^{\omega,s})$ that does not depend on k_s.

LEMMA 3.5. *For the density function $\boldsymbol{\varphi} \in \ker(\mathbf{S}_{\partial D}^{\omega,s})$ not depending on k_s, the operators $\mathbf{S}_{\partial D}^{\omega,p}$ from $L^2(\partial D)^3$ to $H^1(\partial D)^3$ and $\mathbf{K}_{\partial D}^{\omega,p,*}$ from $L^2(\partial D)^3$ to $L^2(\partial D)^3$ enjoy the following asymptotic expansions in three dimensions:*

$$\mathbf{S}_{\partial D}^{\omega,p} = \sum_{j=0}^{+\infty} k_p^j \mathbf{S}_{\partial D,j}^{p}, \quad \mathbf{K}_{\partial D}^{\omega,p,*} = \sum_{j=0}^{+\infty} k_p^j \mathbf{K}_{\partial D,j}^{p,*}, \tag{3.15}$$

where

$$\mathbf{S}^p_{\partial D,j}[\boldsymbol{\varphi}](\mathbf{x}) = -\frac{\mathrm{i}}{4\pi(\lambda+2\mu)}\int_{\partial D}\frac{(\mathrm{i}|\mathbf{x}-\mathbf{y}|)^{j-1}}{j!}\boldsymbol{\varphi}(\mathbf{y})ds(\mathbf{y}),$$

and

$$\mathbf{K}^{p,*}_{\partial D,j}[\boldsymbol{\varphi}](\mathbf{x}) = \frac{\lambda}{\lambda+2\mu}\mathbf{R}_{1,j}[\boldsymbol{\varphi}](\mathbf{x}) + \frac{\mu}{\lambda+2\mu}\mathbf{R}_{2,j}[\boldsymbol{\varphi}](\mathbf{x}), \tag{3.16}$$

with $\mathbf{R}_{1,j}$ *and* $\mathbf{R}_{2,j}$ *given by*

$$\mathbf{R}_{1,j}[\boldsymbol{\varphi}](\mathbf{x}) = -\frac{\mathrm{i}^j(j-1)\boldsymbol{\nu}_{\mathbf{x}}}{4\pi j!}\int_{\partial D}|\mathbf{x}-\mathbf{y}|^{j-3}\langle\mathbf{x}-\mathbf{y},\boldsymbol{\varphi}(\mathbf{y})\rangle ds(\mathbf{y}),$$

and

$$\begin{aligned}\mathbf{R}_{2,j}[\boldsymbol{\varphi}](\mathbf{x}) = &-\frac{\mathrm{i}^j(j-1)}{4\pi j!}\left(\int_{\partial D}|\mathbf{x}-\mathbf{y}|^{j-3}\langle\mathbf{x}-\mathbf{y},\boldsymbol{\nu}_{\mathbf{x}}\rangle\boldsymbol{\varphi}(\mathbf{y})ds(\mathbf{y})\right.\\ &\left.+\int_{\partial D}|\mathbf{x}-\mathbf{y}|^{j-3}(\mathbf{x}-\mathbf{y})\langle\boldsymbol{\nu}_{\mathbf{x}},\boldsymbol{\varphi}(\mathbf{y})\rangle ds(\mathbf{y})\right).\end{aligned}$$

Moreover, $\mathbf{S}^p_{\partial D,j}$ *and* $\mathbf{K}^{p,*}_{\partial D,j}$ *are uniformly bounded with respect to* j*, and the two series in* (3.15) *are convergent in* $\mathcal{L}(L^2(\partial D)^3, H^1(\partial D)^3)$ *and* $\mathcal{L}(L^2(\partial D)^3)$*, respectively.*

LEMMA 3.6. *If* $\boldsymbol{\varphi}\in\ker(\mathbf{S}^{\omega,s}_{\partial D})$*,* $k_p\ll 1$*,* $\mu\ll 1$*, and* $\lambda=\mathcal{O}(1)$*, then one has that for* $\mathbf{x}\in\partial D$*,*

$$\mathbf{K}^{\omega,*}_{\partial D}[\boldsymbol{\varphi}](\mathbf{x}) = \mathbf{R}_{1,0}[\boldsymbol{\varphi}] + \mathcal{O}(\mu) + \mathcal{O}(k_p^2),$$

where $\mathbf{R}_{1,0}$ *is defined as in* (3.16).

Proof. From the definition of the fundamental solution in (2.12) and the fact $\boldsymbol{\varphi}\in\ker(\mathbf{S}^{\omega,s}_{\partial D})$, we only need to deal with the kernel function $-\frac{1}{\omega^2}\nabla\nabla G^{k_p}$. Moreover, the traction operator $\partial_{\boldsymbol{\nu}}$ defined in (2.4) can also be written as

$$\partial_{\boldsymbol{\nu}}\mathbf{w} = 2\mu\nabla\mathbf{w}\cdot\boldsymbol{\nu} + \lambda(\nabla\cdot\mathbf{w})\boldsymbol{\nu} + \mu\boldsymbol{\nu}\times(\nabla\times\mathbf{w}). \tag{3.17}$$

From Lemma 3.2, we have that

$$\begin{aligned}&\int_{\partial D}2\mu\nabla\left(-\frac{1}{\omega^2}\nabla\nabla G^{k_p}(\mathbf{x}-\mathbf{y})\boldsymbol{\varphi}(\mathbf{y})\right)\cdot\boldsymbol{\nu}ds(\mathbf{y})\\ =&\int_{\partial D}2\mu\nabla\left(-\frac{1}{\omega^2}\nabla\nabla\left(G^{k_p}-G^0\right)(\mathbf{x}-\mathbf{y})\boldsymbol{\varphi}(\mathbf{y})\right)\cdot\boldsymbol{\nu}ds(\mathbf{y})\\ =&\mathcal{O}(\mu),\end{aligned} \tag{3.18}$$

where the last identity follows from the Lemma 3.5. It also holds that

$$\begin{aligned}&\lambda\boldsymbol{\nu}\int_{\partial D}\nabla\cdot\left(-\frac{1}{\omega^2}\nabla\nabla G^{k_p}(\mathbf{x}-\mathbf{y})\boldsymbol{\varphi}(\mathbf{y})\right)ds(\mathbf{y})\\ =&\lambda\boldsymbol{\nu}\int_{\partial D}-\frac{1}{\omega^2}\triangle\nabla G^{k_p}(\mathbf{x}-\mathbf{y})\cdot\boldsymbol{\varphi}(\mathbf{y})ds(\mathbf{y})\\ =&\boldsymbol{\nu}\int_{\partial D}\nabla G^{k_p}(\mathbf{x}-\mathbf{y})\cdot\boldsymbol{\varphi}(\mathbf{y})ds(\mathbf{y})\\ =&\mathbf{R}_{1,0}[\boldsymbol{\varphi}(\mathbf{y})]+\mathcal{O}(k_p^2),\end{aligned} \tag{3.19}$$

where $\mathbf{R}_{1,0}$ is defined as in (3.16). Moreover, since $\nabla \times \nabla = 0$, one finally concludes that from (3.17), (3.18), and (3.19)

$$\mathbf{K}_{\partial D}^{\omega,*}[\boldsymbol{\varphi}](\mathbf{x}) = \mathbf{R}_{1,0}[\boldsymbol{\varphi}] + \mathcal{O}(\mu) + \mathcal{O}(k_p^2).$$

The proof is completed. ∎

Remark 3.2. Lemma 3.6 holds for any $\boldsymbol{\varphi} \in \ker(\mathbf{S}_{\partial D}^{\omega,s})$, which could depend on k_s.

For later convenience, we introduce an important subspace of $L^2(\partial D)$:

$$L_0^2(\partial D) = \left\{\varphi \in L^2(\partial D) : \int_{\partial D} \varphi ds = 0\right\} \tag{3.20}$$

and the following results, which can be found in [7].

LEMMA 3.7. *Let ξ be a real number. The operator $\xi - K_{\partial D,0}^*$ is invertible on $L_0^2(\partial D)$ if $|\xi| \geq 1/2$, where $K_{\partial D,0}^*$ is as given in* (2.11). *Furthermore, the kernel of the operator $(-\frac{I}{2} + K_{\partial D,0}^*)$, restricted in the space $L^2(\partial D)$, is one-dimensional, and*

$$\ker\left(-\frac{I}{2} + K_{\partial D,0}^*\right) = \operatorname{span}\{S_{\partial D,0}^{-1}[1]\},$$

where the operator $S_{\partial D,0}^{-1}$ is the inverse of the operator $S_{\partial D,0}$ defined in (2.9).

LEMMA 3.8. *All $f \in L^2(\partial D)$ satisfying $(-\frac{I}{2} + K_{\partial D,0})f = 0$, with $K_{\partial D,0}$ defined as in* (2.11)*, are constant.*

4. Minnaert resonances in three dimensions. In this section, we show the Minnaert resonances for the system (3.6) in $\mathbb{R}^3$. We first prove that the weak resonance always occurs provided that the parameters are properly chosen. Then with a proper choice of the geometry of the domain D, we further show that enhanced or even strong resonances can occur.

THEOREM 4.1. *Consider the system* (3.6) *in three dimensions. If the parameters are chosen according to* (3.1)–(3.2) *(or equivalently* (3.5)*), then weak Minnaert resonance occurs.*

Proof. The proof proceeds by construction. By the definition of the weak resonance in (3.10) for the system (3.6), we construct in what follows a density function $\Phi \in \mathcal{H}$ with $\|\Phi\|_{\mathcal{H}} = 1$ such that condition (3.10) is fulfilled.

Since we consider the low-frequency resonance, i.e., $k \ll 1$, the density function $\Phi \in \mathcal{H}$ should satisfy the following asymptotic expansion:

$$\Phi = \Phi_0 + k\Phi_1 + k^2\Phi_2 + \cdots,$$

where

$$\Phi_j = \begin{pmatrix} \varphi_j \\ \boldsymbol{\varphi}_j \end{pmatrix}, \quad j = 0, 1, 2, \ldots.$$

As we discussed earlier, the resonance is mainly caused by the p-wave in our study. Therefore, we choose

$$\boldsymbol{\varphi}_j \in \ker(\mathbf{S}_{\partial D}^{\omega,s}), \quad j = 0, 1, 2, \ldots.$$

From the assumption that $\delta \ll 1$ and the operator $S_{\partial D}^k$ is bounded, we can derive from the definition of the operator $\mathcal{A}(k,\delta)$ in (3.8) and Lemmas 3.4–3.6 that

(4.1)
$$\begin{aligned}\mathcal{A}(k,\delta)[\Phi] =&\frac{1}{k^2}\begin{pmatrix} \left(-\frac{I}{2}+K^*_{\partial D,0}\right)[\varphi_0] \\ 0\end{pmatrix} + \frac{1}{k}\begin{pmatrix} \left(-\frac{I}{2}+K^*_{\partial D,0}\right)[\varphi_1] \\ 0\end{pmatrix} \\ &+ \begin{pmatrix} K^*_{\partial D,2}[\varphi_0] - \boldsymbol{\nu}\cdot\mathbf{S}^{k\tau}_{\partial D}[\boldsymbol{\varphi}_0] + \left(-\frac{I}{2}+K^*_{\partial D,0}\right)[\varphi_2] \\ \left(\frac{\mathbf{I}}{2}+\mathbf{K}^{k\tau,*}_{\partial D}\right)[\boldsymbol{\varphi}_0]\end{pmatrix} + \mathcal{O}(k)+\mathcal{O}(\delta).\end{aligned}$$

Since $k \ll 1$, the first two terms in (4.1) should vanish. Thus one can conclude from Lemma 3.7 that

$$\varphi_0, \varphi_1 \in \ker\left(-\frac{I}{2}+K^*_{\partial D,0}\right) = \mathrm{span}\{S^{-1}_{\partial D,0}[1]\}. \tag{4.2}$$

Next we deal with the third term in (4.1). Since $\boldsymbol{\varphi}_0 \in \ker(\mathbf{S}^{\omega,s}_{\partial D})$, one has that from Lemma 3.6

$$\left(\frac{\mathbf{I}}{2}+\mathbf{K}^{k\tau,*}_{\partial D}\right)[\boldsymbol{\varphi}_0] = \frac{1}{2}\boldsymbol{\varphi}_0 + \mathbf{R}_{1,0}[\boldsymbol{\varphi}_0] + \mathcal{O}(\mu)+\mathcal{O}(k_p^2),$$

where

$$\mathbf{R}_{1,0}[\boldsymbol{\varphi}_0] = \boldsymbol{\nu}_{\mathbf{x}}\int_{\partial D}\frac{\langle \mathbf{x}-\mathbf{y},\boldsymbol{\varphi}_0\rangle}{4\pi|\mathbf{x}-\mathbf{y}|^3}ds(\mathbf{y}) \tag{4.3}$$

and is bounded from $L^2(\partial D)^3$ to $L^2(\partial D)^3$. Therefore the leading term $\frac{1}{2}\boldsymbol{\varphi}_0+\mathbf{R}_{1,0}[\boldsymbol{\varphi}_0]$ should vanish. From (4.3), $\mathbf{R}_{1,0}[\boldsymbol{\varphi}_0]$ contains only the normal component; therefore the function $\boldsymbol{\varphi}_0$ should also contain only the normal component, namely

$$\boldsymbol{\varphi}_0 = \varphi\boldsymbol{\nu}$$

for some $\varphi \in L^2(\partial D)$. Thus

$$\frac{1}{2}\boldsymbol{\varphi}_0 + \mathbf{R}_{1,0}[\boldsymbol{\varphi}_0] = \boldsymbol{\nu}\left(\frac{1}{2}\varphi - K_{\partial D,0}[\varphi]\right),$$

and φ should be a constant thanks to Lemma 3.8. Hence one finally obtains that

$$\boldsymbol{\varphi}_0 = c_0\boldsymbol{\nu}$$

for some constant c_0, which will be further determined later. Since $\boldsymbol{\varphi}_0$ derived in the last equation does not depend on k_s, one has the following expansion from Lemma 3.5:

$$\mathbf{S}^{k\tau}_{\partial D}[\boldsymbol{\varphi}_0] = \mathbf{S}^{\omega,p}_{\partial D}[\boldsymbol{\varphi}_0] = \sum_{j=0}^{+\infty} k_p^j\mathbf{S}^p_{\partial D,j}[\boldsymbol{\varphi}_0],$$

and

$$\mathbf{K}^{k\tau,*}_{\partial D}[\boldsymbol{\varphi}_0] = \mathbf{K}^{\omega,p,*}_{\partial D}[\boldsymbol{\varphi}_0] = \sum_{j=0}^{+\infty} k_p^j\mathbf{K}^{p,*}_{\partial D,j}[\boldsymbol{\varphi}_0].$$

We proceed to deal with the first component of the third term in (4.1) by solving the following equation:

$$\left(-\frac{I}{2}+K^*_{\partial D,0}\right)[\varphi_2] = c_0\boldsymbol{\nu}\cdot\mathbf{S}^p_{\partial D,0}[\boldsymbol{\nu}] - K^*_{\partial D,2}[\varphi_0]. \tag{4.4}$$

Lemma 3.7 shows that the operator $\left(-\frac{I}{2}+K^*_{\partial D}\right)$ is invertible on $L^2_0(\partial D)$; thus c_0 should be chosen as

$$c_0=\frac{\int_{\partial D}K^*_{\partial D,2}[\varphi_0](\mathbf{x})ds(\mathbf{x})}{\int_{\partial D}\boldsymbol{\nu}\cdot\mathbf{S}^p_{\partial D,0}[\boldsymbol{\nu}](\mathbf{x})ds(\mathbf{x})}\tag{4.5}$$

such that (4.4) is solvable. Thus

$$\varphi_2=\left(-\frac{I}{2}+K^*_{\partial D,0}\right)^{-1}\left[c_0\boldsymbol{\nu}\cdot\mathbf{S}^p_{\partial D,0}[\boldsymbol{\nu}]-K^*_{\partial D,2}[\varphi_0]\right].$$

Following the process above, one can construct

$$\Phi=\begin{pmatrix}\varphi_0\\ c_0\boldsymbol{\nu}\end{pmatrix}+k\begin{pmatrix}\varphi_1\\ c_1\boldsymbol{\nu}\end{pmatrix}+k^2\begin{pmatrix}\varphi_2\\ c_2\boldsymbol{\nu}\end{pmatrix}+k^3\begin{pmatrix}\varphi_3\\ 0\end{pmatrix}+k^4\begin{pmatrix}\varphi_4\\ 0\end{pmatrix},\tag{4.6}$$

where φ_0,φ_1 are as given in (4.2), c_0 is as given in (4.5), and

$$c_1=\frac{\int_{\partial D}K^*_{\partial D,2}[\varphi_1](\mathbf{x})+K^*_{\partial D,3}[\varphi_0](\mathbf{x})-(\tau/c_p)\boldsymbol{\nu}\cdot\mathbf{S}^p_{\partial D,1}[c_0\boldsymbol{\nu}](\mathbf{x})d\mathbf{x}}{\int_{\partial D}\boldsymbol{\nu}\cdot\mathbf{S}^p_{\partial D,0}[\boldsymbol{\nu}](\mathbf{x})d\mathbf{x}},$$

$$\varphi_3=\left(-\frac{I}{2}+K^*_{\partial D,0}\right)^{-1}\left[c_1\boldsymbol{\nu}\cdot\mathbf{S}^p_{\partial D,0}[\boldsymbol{\nu}]-K^*_{\partial D,2}[\varphi_1]+(\tau/c_p)c_0\boldsymbol{\nu}\cdot\mathbf{S}^p_{\partial D,1}[\boldsymbol{\nu}]-K^*_{\partial D,3}[\varphi_0]\right],$$

$$\begin{aligned}c_2=&\frac{\int_{\partial D}K^*_{\partial D,2}[\varphi_2](\mathbf{x})+K^*_{\partial D,3}[\varphi_1](\mathbf{x})+K^*_{\partial D,4}[\varphi_0](\mathbf{x})d\mathbf{x}}{\int_{\partial D}\boldsymbol{\nu}\cdot\mathbf{S}^p_{\partial D,0}[\boldsymbol{\nu}](\mathbf{x})d\mathbf{x}}\\&-\frac{\int_{\partial D}(\tau/c_p)\boldsymbol{\nu}\cdot\mathbf{S}^p_{\partial D,1}[c_1\boldsymbol{\nu}](\mathbf{x})+(\tau/c_p)^2\boldsymbol{\nu}\cdot\mathbf{S}^p_{\partial D,2}[c_0\boldsymbol{\nu}](\mathbf{x})d\mathbf{x}}{\int_{\partial D}\boldsymbol{\nu}\cdot\mathbf{S}^p_{\partial D,0}[\boldsymbol{\nu}](\mathbf{x})d\mathbf{x}},\end{aligned}$$

$$\begin{aligned}\varphi_4=\left(-\frac{I}{2}+K^*_{\partial D,0}\right)^{-1}&\left[c_2\boldsymbol{\nu}\cdot\mathbf{S}^p_{\partial D,0}[\boldsymbol{\nu}]-K^*_{\partial D,2}[\varphi_2](\mathbf{x})-K^*_{\partial D,3}[\varphi_1](\mathbf{x})\right.\\&\left.+(\tau/c_p)\boldsymbol{\nu}\cdot\mathbf{S}^p_{\partial D,1}[c_1\boldsymbol{\nu}](\mathbf{x})-K^*_{\partial D,4}[\varphi_0](\mathbf{x})+(\tau/c_p)^2\boldsymbol{\nu}\cdot\mathbf{S}^p_{\partial D,2}[c_0\boldsymbol{\nu}](\mathbf{x})\right].\end{aligned}$$

Then one can have that

$$\begin{aligned}\mathcal{A}(\omega,\delta)[\Phi]=&\begin{pmatrix}0\\ \delta\tau^2\boldsymbol{\nu}+\mu\left(\frac{1}{\lambda+2\mu}(\mathbf{R}_{2,0}-2\mathbf{R}_{1,0})[c_0\boldsymbol{\nu}]\right)+k^2(\tau/c_p)^2\mathbf{R}_{1,2}[c_0\boldsymbol{\nu}]\end{pmatrix}\\&+\begin{pmatrix}\mathcal{O}(k^3)\\ \mathcal{O}(k^3)+\mathcal{O}(\delta k)+\mathcal{O}(\mu k)\end{pmatrix},\end{aligned}\tag{4.7}$$

with Φ defined as in (4.6) and the operator $\mathbf{R}_{i,j}$ defined as in Lemma 3.5. Finally one can conclude that

$$\|\mathcal{A}(\omega,\delta)[\Phi](\mathbf{x})\|_{\mathcal{H}}=\mathcal{O}(\delta)+\mathcal{O}(\mu)+\mathcal{O}(k^2)\ll 1,$$

which clearly shows that the weak resonance occurs. ∎

From the proof of Theorem 4.1, one readily sees that we cannot enhance the resonance by diminishing the parameter k only. The parameter k should be chosen in an appropriate way that is correlated to the parameters δ and μ in order to achieve enhanced resonance effects. In fact, we have the following results.

PROPOSITION 4.1. *Consider the same setup as that in Theorem 4.1. If the equation*

$$\delta\tau^2\boldsymbol{\nu} + \mu\left(\frac{1}{\lambda+2\mu}(\mathbf{R}_{2,0} - 2\mathbf{R}_{1,0})[c_0\boldsymbol{\nu}]\right) + k^2(\tau/c_p)^2\mathbf{R}_{1,2}[c_0\boldsymbol{\nu}] = 0 \tag{4.8}$$

is solvable, where $\mathbf{R}_{2,0}$, $\mathbf{R}_{1,0}$, and $\mathbf{R}_{1,2}$ are defined as in Lemma 3.5, then one has that

$$\|\mathcal{A}(\omega,\delta)[\Phi](\mathbf{x})\|_{\mathcal{H}} = \mathcal{O}(\delta k) + \mathcal{O}(\mu k) + \mathcal{O}(k^3), \tag{4.9}$$

which indicates that the enhanced resonance can be achieved. If (4.8) is solvable, the parameter k should fulfil

$$k = \sqrt{\mathcal{O}(\delta) + \mathcal{O}(\mu)}. \tag{4.10}$$

Proof. If we take the density function Φ as in (4.6), and if (4.8) is solvable, then from (4.7) one has that

$$\mathcal{A}(\omega)[\Phi] = \begin{pmatrix} \mathcal{O}(k^3) \\ \mathcal{O}(k^3) + \mathcal{O}(\delta k) + \mathcal{O}(\mu k) \end{pmatrix}. \tag{4.11}$$

Thus the estimate in (4.9) is proved. Moreover, by noting that the functions

$$(\mathbf{R}_{2,0} - 2\mathbf{R}_{1,0})[c_0\boldsymbol{\nu}] \quad \text{and} \quad \mathbf{R}_{1,2}[c_0\boldsymbol{\nu}]$$

are bounded in $L^2(\partial D)^3$ and

$$\tau = \mathcal{O}(1) \quad \text{and} \quad \lambda = \mathcal{O}(1),$$

one can show by direct computations that if (4.8) is solvable, then the parameter k fulfils

$$k = \sqrt{\mathcal{O}(\delta) + \mathcal{O}(\mu)}.$$ □

Remark 4.1. If (4.8) is solvable, the function $\mathbf{R}_{2,0}[\boldsymbol{\nu}]$ should contain only the normal component, since the functions $\mathbf{R}_{1,0}[\boldsymbol{\nu}]$ and $\mathbf{R}_{1,2}[\boldsymbol{\nu}]$ contain only the normal components. Indeed, this is also physically justifiable. We can see from the fourth equation in (3.6) that it is natural to require the leading term of the traction of the elastic wave outside the bubble D to contain only the normal component in order to strengthen the resonance, since the pressure in the bubble has only the normal component. This property depends heavily on the geometry of the domain D.

Remark 4.2. Since $\delta \ll 1$ and $\mu \ll 1$, we can readily see from (4.9) and (4.10) in Proposition 4.1 that enhanced resonance effects can be achieved.

Remark 4.3. We apply Proposition 4.1 to the case when the bubble D is a unit ball. In such a case, one has that for $\mathbf{x} \in \partial D$,

$$S_{\partial D,0}^{-1}[1](\mathbf{x}) = -1 \quad \text{and} \quad S_{\partial D,0}[\boldsymbol{\nu}](\mathbf{x}) = -\frac{1}{3}\boldsymbol{\nu}.$$

Thus we can obtain from (4.2) that

$$\varphi_0 = \varphi_1 = -1.$$

We first calculate the parameter c_0 defined in (4.5). A direct calculation shows that

$$\int_{\partial D} K^*_{\partial D,2}[-1](\mathbf{x})ds(\mathbf{x}) = -4\pi/3.$$

Hence, one obtains

$$c_0 = \frac{\int_{\partial D} K^*_{\partial D,2}[\varphi_0](\mathbf{x})ds(\mathbf{x})}{\int_{\partial D} \boldsymbol{\nu}\cdot\mathbf{S}^p_{\partial D,0}[\boldsymbol{\nu}](\mathbf{x})ds(\mathbf{x})} = \frac{-4\pi/3}{-4\pi/(3(\lambda+2\mu))} = \lambda + 2\mu.$$

From the proof in Theorem 4.1, we derive

$$\mathbf{R}_{1,0}[\boldsymbol{\nu}] = -\boldsymbol{\nu}/2.$$

Moreover, by some straightforward but rather tedious calculations, one can obtain that

$$\mathbf{R}_{2,0}[\boldsymbol{\nu}] = \boldsymbol{\nu}/3 \quad \text{and} \quad \mathbf{R}_{1,2}[\boldsymbol{\nu}] = -\boldsymbol{\nu}/3.$$

Therefore (4.8) in Proposition 4.1 can be simplified to be

$$\delta\tau^2\boldsymbol{\nu} + \frac{4}{3}\mu\boldsymbol{\nu} - \frac{1}{3}k^2\tau^2\boldsymbol{\nu} = 0,$$

which shows that k should be chosen as

$$k = \sqrt{3\delta + 4\mu/\tau^2}. \tag{4.12}$$

Substituting the parameters in (3.4) into the last equation shows that the resonance frequency should be

$$\frac{1}{2\pi L}\sqrt{\frac{3\kappa + 4\tilde{\mu}}{\rho_e}}, \tag{4.13}$$

where L is the radius of the sphere, which recovers the physical result in [1, 11].

Remark 4.4. It is definitely very interesting to explore whether the enhanced resonance condition (4.8) can hold for bubbles of more general shapes other than the radial one as discussed in Remark 4.3. However, it is rather impractical to solve (4.8) analytically, even in the case when ∂D has a global parameterization, say an ellipsoid. Instead, we have conducted extensive numerical experiments, and shall present one typical example below for the illustration. We note that the geometrical dependence of the resonance was also investigated numerically for the bubbly elastic materials in [11].

From a numerical and practical point of view, instead of solving (4.8) exactly, we consider the following optimization problem:

$$\min_{k\in(0,\alpha)} \int_{\partial D} |\mathbf{g}(k)|\, ds, \tag{4.14}$$

where $\alpha \in \mathbb{R}_+$ is properly chosen, say $\alpha = 1$, and $\mathbf{g}(k)$ is the desired function involved in (4.8), namely

$$\mathbf{g}(k) := \delta\tau^2\boldsymbol{\nu} + \mu\left(\frac{1}{\lambda+2\mu}(\mathbf{R}_{2,0} - 2\mathbf{R}_{1,0})[c_0\boldsymbol{\nu}]\right) + k^2(\tau/c_p)^2\mathbf{R}_{1,2}[c_0\boldsymbol{\nu}].$$

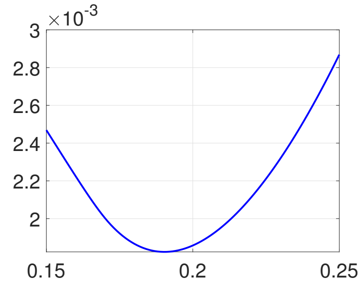

FIG. 1. *The values of* $\int_{\partial D} |\mathbf{g}(k)|\, ds$ *in* (4.14) *versus the parameter* k *when* D *is an oblate spheroid.*

We choose the same parameter configuration as that in [11]. The material parameters for the air in the bubble are

$$\kappa = 1.4 \times 10^5\, Pa, \quad \rho_b = 1.2\, kg/m^3,$$

and the corresponding material parameters for the soft-elastic material are

$$\tilde{\mu} = 6.5 \times 10^5\, Pa, \quad \tilde{\lambda} = 10^9\, Pa, \quad \rho_e = 1042\, kg/m^3.$$

With these physical parameters, one can easily derive that the dimensionless parameters defined in (3.1), (3.3), and (3.4) are given by

$$\delta = 0.0012, \quad \tau = 0.3363, \quad \mu = 5.9 \times 10^{-4}.$$

The volume of the bubble D is maintained to be the constant $3.605 \times 10^{-12}\, m^3$, which is the same as that in [11].

First, we consider the case that D is a central ball whose radius is $9.5 \times 10^{-5}\, m$. From (4.13), we can easily derive that the resonance frequency is 90 kHZ. Next, we consider the case that D is an oblate spheroid with two semiaxes of length $1.5 \times 10^{-4}\, m$ and $3.7 \times 10^{-5}\, m$, respectively. For the spheroidal geometry, we need to solve the optimization problem (4.14) instead of solving (4.8) analytically as previously mentioned. The values of $\int_{\partial D} |\mathbf{g}(k)|\, ds$ versus the parameter k are plotted in Figure 1. From Figure 1, it is easy to locate that the minimum for (4.14) is obtained at $k = 0.194$, which implies that the physical resonance frequency is 70 kHZ, which agrees with the result in [11]. This fact in turn verifies the correctness of identifying the resonance frequency based on solving the optimization problem (4.14) for other geometries of the domain. We see that the enhanced resonance condition (4.8) is not exactly fulfilled in this case; nevertheless, the resonance phenomenon is significantly enhanced, as naturally expected.

Finally, we point out that the enhanced resonance condition (4.8) may still be held if one allows the shape of the embedded bubble to be geometrically and topologically more general, say nonconvex or even nonsimply connected. However, this is beyond the scope of the current work, and it may also lack physical significance since one can always choose to work under the radial geometry from the construction point of view.

We end this section with a generalization of our previous results.

PROPOSITION 4.2. *Consider the same setup as that in Theorem* 4.1. *Further-*

more, if

$$\begin{aligned}\delta\boldsymbol{\nu}\sum_{j=0}^{m}\tau^2k^j\sum_{i=0}^{j}S_{\partial D,i}[\varphi_{j-i}]+\frac{k^2\tau^2}{c_p^2}\sum_{j=0}^{m}\left(\frac{k\tau}{c_p}\right)^j\sum_{i=2}^{j+2}\mathbf{R}_{1,i}[c_{j+2-i}\boldsymbol{\nu}]\\+\frac{\mu}{\lambda+2\mu}\sum_{j=0}^{m}\left(\frac{k\tau}{c_p}\right)^j\sum_{i=0}^{j}(\mathbf{R}_{2,i}-2\mathbf{R}_{1,i})[c_{j-i}\boldsymbol{\nu}]=0\end{aligned}\tag{4.15}$$

is solvable, where φ_j *is defined as in* (4.17)*, then one has that*

$$\|\mathcal{A}(\omega,\delta)[\Phi](\mathbf{x})\|_{\mathcal{H}}=\mathcal{O}(\delta k^{m+1})+\mathcal{O}(\mu k^{m+1})+\mathcal{O}(k^{m+3}).\tag{4.16}$$

Proof. Following the proof of Theorem 4.1, one can construct

$$\Phi=\sum_{j=0}^{\infty}k^j\begin{pmatrix}\varphi_j\\ \boldsymbol{\varphi}_j\end{pmatrix},\tag{4.17}$$

where φ_0,φ_1 are the same as those in (4.2) and

$$\varphi_j=\left(-\frac{I}{2}+K^*_{\partial D}\right)^{-1}\left[\sum_{m=0}^{j-2}(\tau/c_p)^m\boldsymbol{\nu}\cdot\mathbf{S}^p_{\partial D,m}[\boldsymbol{\varphi}_{j-2-m}]-\sum_{m=2}^{j}K^*_{\partial D,m}[\varphi_{j-m}]\right]\text{ for } j\geq 2,$$

$$\boldsymbol{\varphi}_j=c_j\boldsymbol{\nu}\quad\text{for}\quad j\geq 0,$$

with

$$c_j=\frac{\int_{\partial D}\sum_{m=2}^{j+2}K^*_{\partial D,m}[\varphi_{j+2-m}](\mathbf{x})-\sum_{m=1}^{j}(\tau/c_p)^m\boldsymbol{\nu}\cdot\mathbf{S}^p_{\partial D,m}[\boldsymbol{\varphi}_{j-m}](\mathbf{x})ds(\mathbf{x})}{\int_{\partial D}\boldsymbol{\nu}\cdot\mathbf{S}^p_{\partial D,0}[\boldsymbol{\nu}]ds(\mathbf{x})}.$$

We remark here that when calculating Φ in (4.17), one should first calculate c_j to obtain $\boldsymbol{\varphi}_j$ and then calculate φ_{j+2} for $j=0,1,2,\ldots$, since $\varphi_0,\varphi_1\in\mathrm{span}\{S^{-1}_{\partial D,0}[1]\}$. Hence one has that

$$\mathcal{A}(\omega,\delta)[\Phi]_1=0,$$

$$\begin{aligned}\mathcal{A}(\omega,\delta)[\Phi]_2=&\delta\boldsymbol{\nu}\sum_{j=0}^{\infty}\tau^2k^j\sum_{i=0}^{j}S_{\partial D,i}[\varphi_{j-i}]+\frac{k^2\tau^2}{c_p^2}\sum_{j=0}^{\infty}\left(\frac{k\tau}{c_p}\right)^j\sum_{i=2}^{j+2}\mathbf{R}_{1,i}[c_{j+2-i}\boldsymbol{\nu}]\\&+\frac{\mu}{\lambda+2\mu}\sum_{j=0}^{\infty}\left(\frac{k\tau}{c_p}\right)^j\sum_{i=0}^{j}(\mathbf{R}_{2,i}-2\mathbf{R}_{1,i})[c_{j-i}\boldsymbol{\nu}],\end{aligned}$$

where $\mathcal{A}(\omega,\delta)[\Phi]_i$ denotes the ith component of the vectorial function $\mathcal{A}(\omega,\delta)[\Phi]$ and the operators $\mathbf{R}_{i,j}$ with $i=1,2$ and $j\geq 0$, are defined as in (3.16). Thus, if (4.15) is solvable, one can conclude that

$$\|\mathcal{A}(\omega,\delta)[\Phi](\mathbf{x})\|_{\mathcal{H}}=\mathcal{O}(\delta k^{m+1})+\mathcal{O}(\mu k^{m+1})+\mathcal{O}(k^{m+3}).\qquad\square$$

Remark 4.5. It is noted that if (4.15) is solvable, then one should have

$$k=\sqrt{\mathcal{O}(\delta)+\mathcal{O}(\mu)}$$

and

$$\mathbf{R}_{2,i}[\boldsymbol{\nu}]=\psi_i\boldsymbol{\nu}\quad\text{for}\quad 0\leq i\leq m,\tag{4.18}$$

with $\psi_i\in L^2(\partial D)$. The identities (4.18) are unobjectionably reasonable, as explained in Remark 4.1.

Remark 4.6. Proposition 4.1 is a special case of Proposition 4.2 with $m = 0$. Indeed, even though (4.15) could be solved for $m > 0$, it is enough to solve (4.15) with $m = 0$, namely (4.8) in Proposition 4.1, to obtain the resonant frequency. This is because it gives the leading-order term of the resonant frequency. As discussed in Remark 4.4, by solving an equation here we actually mean to solve the corresponding optimization problem (4.14).

Remark 4.7. If (4.15) is solvable for $m = \infty$, then the function Φ defined in (4.17) belongs to the kernel of the operator $\mathcal{A}(\omega, \delta)$; namely the condition (3.9) is fulfilled. In this case, condition (4.18) signifies that $\boldsymbol{\nu}$ should be an eigenfunction of the operator $\mathbf{K}_{\partial D}^{\omega,*}$. In fact, this is the case when the domain D is a ball. In [15, 16], it was proved that $\boldsymbol{\nu}$ is an eigenfunction of the operator $\mathbf{K}_{\partial D}^{\omega,*}$, namely

$$\mathbf{K}_{\partial D}^{\omega,*}[\boldsymbol{\nu}] = \chi_1 \boldsymbol{\nu}, \quad \mathbf{x} \in \partial D, \tag{4.19}$$

where

$$\chi_1 = \frac{4\mathrm{i}\mu R k_p}{(\lambda + 2\mu)} j_1(k_p R) h_1(k_p R) - \mathrm{i} R^2 k_p^2 j_1(k_p R) h_0(k_p R) - \frac{1}{2},$$

with R being the radius of the ball D, and $j_n(|\mathbf{x}|)$ and $h_n(|\mathbf{x}|)$ respectively denoting the spherical Bessel function and spherical Hankel function of the first kind and of order n. Moreover, $\boldsymbol{\nu}$ is also an eigenfunction of the operator $\mathbf{S}_{\partial D}^{\omega}$, namely

$$\mathbf{S}_{\partial D}^{\omega}[\boldsymbol{\nu}](\mathbf{x}) = \frac{-\mathrm{i} R^2 k_p}{(\lambda + 2\mu)} h_1(k_p R) j_1(k_p R) \boldsymbol{\nu}, \quad \mathbf{x} \in \partial D. \tag{4.20}$$

It was also proved in [24] that

$$S_{\partial D}^{k}[1](x) = -\mathrm{i} k R^2 h_0(kR) j_0(kR), \quad \mathbf{x} \in \partial D, \tag{4.21}$$

and

$$K_{\partial D}^{k,*}[1](x) = \frac{1}{2} - \mathrm{i} k^2 R^2 j_0'(kR) h_0(kR), \quad \mathbf{x} \in \partial D. \tag{4.22}$$

Following the asymptotic expansions for the functions $j_n(|\mathbf{x}|)$ and $h_n(|\mathbf{x}|)$, $n = 0, 1$, with $|\mathbf{x}| \ll 1$ (cf. [13]), one can obtain the expressions of $S_{\partial D,i}[1]$, $\mathbf{R}_{1,i}[\boldsymbol{\nu}]$, and $\mathbf{R}_{2,i}[\boldsymbol{\nu}]$ for $i \geq 0$, respectively. Next we only present the first few terms:

$$\delta\tau^2 + \frac{4}{3}\mu - \mathrm{i}k \left(\frac{3\tau^3\delta + 4\tau\mu}{3\sqrt{\lambda + 2\mu}} \right) - k^2 \left(\frac{1}{3}\tau^2 + \frac{1}{6}\tau^2\delta \right) + k^3 \frac{\mathrm{i}\tau^3\delta}{6\sqrt{\lambda + 2\mu}} + \cdots .$$

One can readily see that (4.15) is reduced to solving a polynomial equation with respect to k of an infinity order. By a truncation and approximation, we solve the following equation:

$$\delta\tau^2 + \frac{4}{3}\mu - \mathrm{i}k \left(\frac{3\tau^3\delta + 4\tau\mu}{3\sqrt{\lambda + 2\mu}} \right) - k^2 \left(\frac{1}{3}\tau^2 + \frac{1}{6}\tau^2\delta \right) = 0, \tag{4.23}$$

whose roots are given by

$$k_{d3\pm} = \frac{\pm\sqrt{(3\tau^2\delta + 4\mu)(4(\lambda + \mu) - 3\tau^2\delta)} - (3\tau^2\delta + 4\mu)\mathrm{i}}{2\tau\sqrt{\lambda + 2\mu}}. \tag{4.24}$$

One can verify directly that the root (4.12) is the positive part of the roots (4.24), neglecting the infinitesimal part. In fact, the critical values obtained in (4.24) exhibit excellent accuracy for the resonant frequencies; see Remarks 5.1 and 5.2 in what follows.

5. Minnaert resonances in two dimensions. In this section, we derive the Minnaert resonances for the system (3.6) in two dimensions when the domain D is a unit disk. The extension of the low-frequency analysis from three dimensions to two dimensions is technically not straightforward. A major difficulty comes from the fact that the asymptotic expansions of the fundamental solutions $G^k(\mathbf{x})$ in two dimensions and three dimensions defined in (2.8) are of a different nature. In fact, the expansion of the fundamental solution $G^k(\mathbf{x})$ in three dimensions is the summation of $\varphi_j(\mathbf{x})k^j$ with $j = 0, 1, \ldots$. However, in two dimensions the asymptotic expansion is the summation of $\phi(\mathbf{x})(c_j + \ln(k))k^j$, with $j = 0, 1, \ldots$ (cf. [2]), which significantly increases the complexity of solving the counterpart equation (3.9) in the two-dimensional case. Hence, for this technical reason, we shall only derive the Minnaert resonances for the system (3.6) in two dimensions for a disk domain D. Indeed, as can be seen from Theorem 5.1, even though the domain D is a disk in two dimensions, one cannot derive the explicit expression of the resonant frequency.

In what follows, we let $J_n(|\mathbf{x}|)$ and $H_n(|\mathbf{x}|)$ respectively denote the Bessel function of order n and the Hankel function of the first kind of order n. When the argument $k \ll 1$, the functions J_n and H_n, $n = 0, 1$, enjoy the following asymptotic expansions (cf. [9]):

$$J_0(k) = 1 - \frac{k^2}{4} + \frac{k^4}{64} + \mathcal{O}(k^6), \qquad J_1(k) = \frac{k}{2} - \frac{k^3}{16} + \mathcal{O}(k^5), \tag{5.1}$$

$$H_0(k) = \frac{\mathrm{i}(\gamma + 2\ln(k))}{\pi} + \frac{\mathrm{i}(-2 + \gamma + 2\ln(k))k^2}{4\pi} + \mathcal{O}((1 + \ln(k))k^3), \tag{5.2}$$

and

$$H_1(k) = -\frac{2\mathrm{i}}{k\pi} + \frac{\mathrm{i}(-1 + \gamma + 2\ln(k))k}{2\pi} + \mathcal{O}((1 + \ln(k))k^3), \tag{5.3}$$

with $\gamma = 2E_c - \mathrm{i}\pi - 2\ln 2$, and E_c being Euler's constant.

By the definition of the strong resonance in (3.9) for the system (3.6), we next construct a nontrivial solution Φ such that

$$\mathcal{A}(k, \delta)[\Phi](\mathbf{x}) = 0, \tag{5.4}$$

where $\mathcal{A}(k, \delta)$ is defined as in (3.8). If the domain D is a unit disk, direction calculations show that for $\mathbf{x} \in \partial D$,

$$\mathbf{S}_{\partial D}^{k\tau}[\boldsymbol{\nu}](\mathbf{x}) = \zeta_1 \boldsymbol{\nu} \quad \text{and} \quad \mathbf{K}_{\partial D}^{k\tau,*}[\boldsymbol{\nu}](\mathbf{x}) = \zeta_2 \boldsymbol{\nu}, \tag{5.5}$$

where

$$\zeta_1 = \frac{-\mathrm{i}\pi}{2(\lambda + 2\mu)} J_1(k_p) H_1(k_p),$$

and

$$\zeta_2 = \frac{-\mathrm{i}\pi J_1(k_p)}{2(\lambda + 2\mu)} \left((\lambda + 2\mu)k_p H_1'(k_p) + \lambda H_1(k_p)\right) - \frac{1}{2},$$

with $k_p = k\tau/\sqrt{\lambda + 2\mu}$. Moreover, one has that for $\mathbf{x} \in \partial D$ (cf. [17]),

$$S_{\partial D}^{k}[1](\mathbf{x}) = \zeta_3 \quad \text{and} \quad K_{\partial D}^{k,*}[1](\mathbf{x}) = \zeta_4, \tag{5.6}$$

where

$$\zeta_3 = \frac{-\mathrm{i}\pi}{2} J_0(k) H_0(k) \quad \text{and} \quad \zeta_4 = \frac{1}{2} - \frac{\mathrm{i}\pi}{2} k J_0'(k) H_0(k).$$

Hence the nontrivial solution to (5.4) should have the following form:

$$\Phi = \begin{pmatrix} b_1 \\ b_2 \boldsymbol{\nu} \end{pmatrix}.$$

Substituting the last equation into (5.4) yields that

$$\mathbf{B}\mathbf{b} = 0, \tag{5.7}$$

where

$$\mathbf{B} = \begin{pmatrix} \frac{1}{k^2}\left(-\frac{1}{2} + \zeta_4\right) & -\zeta_1 \\ \delta\tau^2\zeta_3 & \frac{1}{2} + \zeta_2 \end{pmatrix} \quad \text{and} \quad \mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$$

with ζ_i, $i = 1, 2, 3, 4$, defined as in (5.5) and (5.6). To ensure that (5.7) possesses nontrivial solutions, the determinant $\det(\mathbf{B})$ of the matrix $\mathbf{B}$ should vanish. Through some straightforward but rather tedious calculations and with the help of the asymptotic expressions in (5.1), (5.2), and (5.3), we can obtain

$$\begin{aligned}\det(\mathbf{B}) &= \frac{1}{k^2}\left(-\frac{1}{2} + \zeta_4\right)\left(\frac{1}{2} + \zeta_2\right) + \delta\tau^2\zeta_1\zeta_3 \\ &= -(\gamma + 2\ln(k))\left(\frac{(\mu + \delta\tau^2)}{4(\lambda + 2\mu)} + \frac{k^2\tau^2\lambda(\gamma + 2\ln(k\tau/\sqrt{\lambda + 2\mu}))}{16(\lambda + 2\mu)^2}\right) \\ &\quad + o(\mu(\gamma + \ln(k))) + o(\delta(\gamma + \ln(k))) + o(k^2(\gamma + \ln(k))),\end{aligned}$$

where γ is defined as in (5.2). Hence, we readily come to the following conclusion.

THEOREM 5.1. *Consider the system* (3.6) *in two dimensions with D being a central disk. If the parameters are chosen according to* (3.5)*, then strong resonance occurs. Moreover, the leading-order terms of the resonant frequencies are given by the roots of the following equation:*

$$(\gamma + 2\ln(k))\left(\frac{(\mu + \delta\tau^2)}{4(\lambda + 2\mu)} + \frac{k^2\tau^2\lambda(\gamma + 2\ln(k\tau/\sqrt{\lambda + 2\mu}))}{16(\lambda + 2\mu)^2}\right) = 0, \tag{5.8}$$

where τ and γ are as given in (3.3) *and* (5.2)*, respectively.*

Remark 5.1. The method used above in deriving the resonances in two dimensions can be applied to three dimensions as well when the domain D is a central ball. From (4.19) to (4.22), in a similar manner one can calculate the determinant of the matrix $\mathbf{B}$ in three dimensions and determine the critical values k such that

$$\det(\mathbf{B}) = 0$$

holds to ensure the occurrence of the strong resonance.

Remark 5.2. There exist critical values k such that $\det(\mathbf{B})$ vanishes in both two and three dimensions; that is, strong resonance occurs. Since the expression of $\det(\mathbf{B})$ is nonlinear with respect to k, we can resort to computational algorithms to determine these critical values, namely resonant frequencies. Next, for illustrations, we conduct some numerical experiments to find out these critical values. We denote by k_{b2} and

TABLE 1
The critical values of k_{b2}, k_{d2}, k_{b3}, and k_{d3+} with positive real parts.

	$i=2$	$i=3$	$i=4$
k_{b2}	0.110087 − 0.040732i	0.030796 − 0.007347i	0.008681 − 0.001513i
k_{d2}	0.109963 − 0.040294i	0.030790 − 0.007341i	0.008681 − 0.001513i
k_{b3}	0.262065 − 0.034521i	0.083584 − 0.003495i	0.026454 − 0.000349i
k_{d3+}	0.262296 − 0.034655i	0.083592 − 0.003496i	0.026455 − 0.000349i

k_{b3} the critical values by directly solving the equation $\det(\mathbf{B}) = 0$ in two and three dimensions, respectively. As comparisons, we also calculate k_{d3+} defined in (4.24) and solve (5.8). The root of (5.8) is denoted by k_{d2}. The parameters in our numerical experiments are chosen as follows:

$$\lambda = 1, \quad \tau = 1, \quad \mu = \delta = 10^{-i}, \ i = 2, 3, 4.$$

Moreover, the bubble D is a unit disk in $\mathbb{R}^2$ and a unit ball in $\mathbb{R}^3$. We remark that the case $i = 3$ almost indicates the experiment in [11]. The corresponding values, k_{b2}, k_{d2}, k_{b3}, and k_{d3+} with positive real parts, are presented in Table 1. From Table 1, one can conclude that there indeed exist critical values k such that $\det(\mathbf{B}) = 0$ in both two and three dimensions. Moreover, the roots of (4.23) and (5.8) exhibit excellent accuracy agreement with the resonant frequencies. Finally, we would like to point out that the negative imaginary parts in the values computed in Table 1 are a physically reasonable requirement (cf. [2]).

6. Concluding remarks. We have studied the Minnaert resonances for bubble-elastic structures. By delicately and subtly balancing the acoustic and elastic parameters as well as the geometry of the bubble, we have shown that the Minnaert resonance can (at least approximately) occur for rather general constructions. Our study opens up a new direction for the mathematical investigation on bubbly elastic mediums with many potential developments. In the present paper, we have considered only the case that a single bubble is embedded in a soft elastic material. It would be interesting to consider the case with multiple bubbles as well as the corresponding application to the effective realization of elastic metamaterials. Moreover, we have investigated only the case that the resonance is mainly caused by the p-wave, but it would be interesting to investigate more general bubbly elastic structures with more general resonances as well as their applications in elastic metamaterials (cf. [23, 25]). We shall consider these and other related topics in our forthcoming work.

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