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MATHEMATICAL AND NUMERICAL STUDY OF A THREE-DIMENSIONAL INVERSE EDDY CURRENT PROBLEM*

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Abstract. We study an inverse problem associated with an eddy current model. We first address the ill-posedness of the inverse problem by proving the compactness of the forward map with respect to the conductivity and the nonuniqueness of the recovery process. Then by virtue of nonradiating source conceptions, we establish a regularity result for the tangential trace of the true solution on the boundary, which is necessary to justify our subsequent mathematical formulation. After that, we formulate the inverse problem as a constrained optimization problem with an appropriate regularization and prove the existence and stability of the regularized minimizers. To facilitate the numerical solution of the nonlinear nonconvex constrained optimization, we introduce a feasible Lagrangian and its discrete variant. Then the gradient of the objective functional is derived using the adjoint technique. By means of the gradient, a nonlinear conjugate gradient method is formulated for solving the optimization system, and a Sobolev gradient is incorporated to accelerate the iterative process. Numerical examples are provided to demonstrate the feasibility of the proposed algorithm.

Key words. inverse eddy current, regularity, ill-posedness, stability, Lagrangian, adjoint problem

AMS subject classifications. 35R30, 35B30

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1. Introduction.

Eddy current inversion is a challenging mathematical and numerical process, but it is one of the most popular nondestructive detection techniques. The inversion technique has attracted great attention in various important applications, such as geophysical prospecting, flaw detection, safety inspection, and biomedical imaging [1, 2, 3, 14, 19, 20, 23, 25]. The eddy current method is based on the low frequency approximation of Maxwell's equation and is much more sensitive to the conductivity of materials when compared with the inversion by using the full electromagnetic Maxwell system. There are two advantages to using the low frequency electromagnetic data in detection. First, a low frequency electromagnetic wave can penetrate deeply in the lossy medium such as a metal structure and the earth. It is well known that the intensity of an electromagnetic wave will decay exponentially in lossy medium with respect to the penetration depth, and the intensity of a higher frequency wave will decay faster [16]. Second, the forward problem needs to be solved repeatedly in most inversion methods. While the full Maxwell's equations are difficult to solve numerically and efficiently themselves, the eddy current approximation of Maxwell's equations is a diffusion equation which can be solved with fast algorithms [7, 15]. Therefore, the eddy current inversion method is widely used in nondestructive testing [20, 22] and geophysical prospecting [14, 25].

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Most inverse problems are known to be ill-posed. We will study two important questions before we formulate our inverse model, i.e., the uniqueness and stability of the recovery. The analyses of these basic issues are very different with different inverse problems; see, e.g., [13] for the time domain inverse Maxwell problem, [9] for the parameter identification problem with elliptic systems, and [1, 5, 23] for inverse Maxwell's source problems and inverse eddy current source problems. To the best of our knowledge, the uniqueness and stability analysis of the inverse eddy current problem have not been studied yet. We shall investigate these two fundamental issues, then formulate and analyze the underlying constrained optimization problem as well as to propose some numerical method for the minimization. We start with the well-posedness of the forward eddy current problem and establish a regularity result for the tangential trace of the true solution on the boundary by virtue of nonradiating source conceptions. This regularity is important to justify our usage of an appropriate selected misfit functional. We then prove the compactness of the forward operator mapping the conductivity to the electric field and study the nonuniqueness of the inverse eddy current problem. With these preparations, we will formulate the ill-posed eddy current inverse problem into a nonlinear and nonconvex constrained minimization with an appropriate regularization and show the existence and stability of the regularized minimizers. To facilitate the numerical solution of the nonlinear nonconvex optimization constrained with the complex-valued eddy current model, we introduce a feasible Lagrangian and its discrete variant in terms of both real and imaginary parts of the constrained PDE. Then we derive the gradient of the objective functional with the adjoint technique. For solving the nonlinear PDE constrained optimization, we formulate a nonlinear conjugate gradient (NLCG) method, with the step size for the descent direction computed by a quadratic approximation to the state field. As the usual NLCG method converges very slowly, we incorporate a Sobolev preconditioner to improve the NLCG iteration.

The outline of the paper is as follows. In section 2, we introduce the forward eddy current problem, present the well-posedness of the forward problem, and prove the regularity of the tangential trace of the true solution. In section 3, an inverse problem with a well-defined misfit functional is formulated and the ill-posedness of the inversion problem is investigated. Then we add a regularization term to the optimization objective functional and prove the existence and stability of the minimizers. In section 4, we first introduce a Lagrangian associated with the regularized optimization problem, then introduce the gradient of the objective functional with adjoint technique, and further study the properties of the adjoint state equation. Moreover, the finite element discretization of the optimization problem is also formulated and studied in the same section, and a nonlinear conjugate gradient method is proposed for the optimization system. We show some numerical examples in section 5 to illustrate the feasibility of the proposed algorithm and present some concluding remarks in section 6.

2. The forward problem. In this section, we introduce the forward model for eddy current inversion and present some necessary preliminaries. The eddy current equation is the low frequency approximation of Maxwell's equation. As we mentioned in the introduction, the eddy current field can penetrate deeply in conducting materials. Moreover, as an electromagnetic method, this method can distinguish the conductor (metal, water) from the insulator (oil, rock) and is an important modality in nondestructive detection. The eddy current problem has been studied extensively

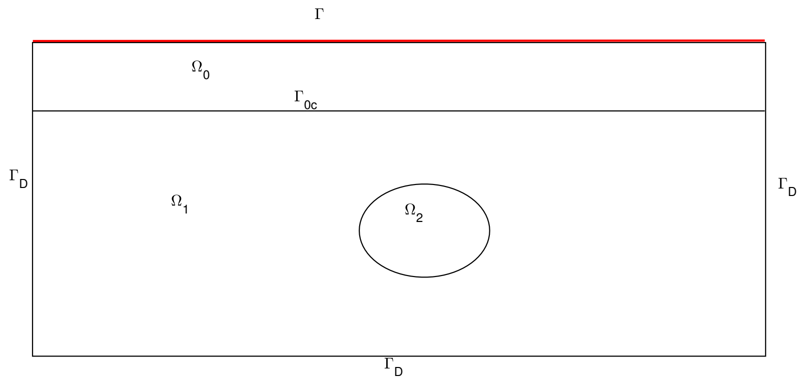

FIG. 1. *The geometric setting of the problem.*

in the literature [24]. The governing equations for the forward problem read

$$\begin{cases} \nabla \times \mathbf{E} = i\omega\mu\mathbf{H} & \text{in } \mathbb{R}^3, \\ \nabla \times \mathbf{H} = \sigma'\mathbf{E} + \mathbf{J}_s & \text{in } \mathbb{R}^3, \end{cases}$$

where $\mathbf{E}$, $\mathbf{H}$ are electric and magnetic fields, respectively, μ is the magnetic permeability, σ' is the conductivity of the medium, and $\mathbf{J}_s$ is the source current. While the equations hold in the whole space $\mathbb{R}^3$, we consider the problem in a bounded domain $\Omega \subset \mathbb{R}^3$ as in many applications and theories, and boundary conditions are specified later to form a well-posed problem.

Now we start with some assumptions for the further consideration of the eddy current model. In the rest of this paper, we concentrate on the electric acquisition case, that is, the measurement data is collected for the tangential components of the electric field on Γ, part of the boundary $\partial\Omega$. We assume that Ω is a convex domain, with a piecewise smooth boundary and a simply connected subdomain Ω_0 occupied by air, hence the conductivity σ' vanishing in Ω_0. Then by the electrical Gauss' law, we have that

$$\nabla \cdot (\varepsilon\mathbf{E}) = 0 \text{ in } \Omega_0, \tag{2.1}$$

where ε is the electric permittivity in the air and is reasonably assumed to be a constant. A typical geometric setting of the problem in a two-dimensional (2D) cross-section is shown in Figure 1, where $\overline{\Omega} = \overline{\Omega}_0 \cup \overline{\Omega}_1 \cup \overline{\Omega}_2$. The material parameter is a different function at each subdomain. We write the conductivity $\sigma'(x)$ in Ω as

$$\sigma'(x) = \sigma_0 + \sigma(x),$$

where σ_0 is the constant background conductivity which is supported in $\Omega_1 \cup \overline{\Omega}_2$ and known a priori. $\sigma(x)$ is the abnormal conductivity. Both $\sigma(x)$ and its support Ω_2 are unknown and are our target to recover simultaneously. We shall write $\Omega \setminus \overline{\Omega}_0$ as Ω_c, and then $\sigma_0 + \sigma(x)$ is supported in Ω_c. We further assume that $\sigma(x)$ is compactly supported in Ω_c. The interface between Ω_0 and Ω_c is denoted by Γ_{0c} and assumed to be a simply connected Lipschitz polyhedral surface, with both domains Ω_0 and Ω_c being polyhedrons and simply connected. In our subsequent study, μ is assumed to be piecewise constant physically, and the source $\mathbf{J}_s$ is compactly supported in Ω_0, and $\nabla \cdot \mathbf{J}_s = 0$.

2.1. The E-based eddy current model and its inverse problem. By eliminating $\mathbf{H}$ in the eddy current equations, we derive the electric field system

$$\left\{ \begin{array}{rl} \nabla \times (\mu^{-1} \nabla \times \mathbf{E}) - i\omega(\sigma_0 + \sigma)\mathbf{E} = i\omega \mathbf{J}_s & \text{in } \Omega\,, \\ \nabla \cdot (\varepsilon \mathbf{E}) = 0 & \text{in } \Omega_0\,, \end{array} \right. \tag{2.2}$$

which are complemented by the interface conditions

$$[\mu^{-1}\mathbf{n} \times \nabla \times \mathbf{E}] = 0,\ [\mathbf{n} \times \mathbf{E}] = 0 \quad \text{on} \ \ \Gamma_{0c} \cup \partial\Omega_2 \tag{2.3}$$

and the boundary conditions

$$\mathbf{n} \times \nabla \times \mathbf{E} = 0 \ \text{ on } \ \Gamma\,; \quad \mathbf{n} \cdot \mathbf{E} = 0 \ \text{ on } \ \Gamma\,; \quad \mathbf{n} \times \mathbf{E} = 0 \ \text{ on } \ \Gamma_D = \partial\Omega \setminus \overline{\Gamma}\,. \tag{2.4}$$

Here and in what follows, $\mathbf{n}$ denotes the outward normal on $\partial\Omega$. We add a divergence free equation in the system (2.2) to ensure the uniqueness of the solution since $\sigma' = 0$ in Ω_0. The piecewise constant ε is the electric permittivity in Ω_0. The divergence free condition makes the field $\mathbf{E}$ an electric field in domain Ω_0. The surface Γ is where we measure the data, i.e., the tangential components $\mathbf{n} \times \mathbf{E}$ of the electric field. The inverse eddy current problem of our interest is formulated as follows:

Given the observation data $\mathbf{n} \times \mathbf{E}^{obs}$ on the measurement surface Γ, recover the conductivity distribution $\sigma(x)$ and its support Ω_2.

2.2. The weak formulation and regularity of the solution. For the variational formulation of the electric field problem (2.2) and its well-posedness, we introduce the Sobolev spaces,

$$\begin{aligned} H_\Gamma(\mathbf{curl};\Omega) &= \left\{\mathbf{u} \in L^2(\Omega)^3 \;\middle|\; \nabla \times \mathbf{u} \in L^2(\Omega)^3,\ \mathbf{n} \times \mathbf{u} = 0 \text{ on } \Gamma_D\right\}, \\ H^1_\Gamma(\Omega_0) &= \left\{v \in L^2(\Omega_0) \;\middle|\; \nabla v \in L^2(\Omega_0)^3,\ v|_{\partial\Omega_0 \setminus \Gamma} = 0\right\}, \\ \mathbf{Y} &= \left\{\mathbf{u} \in H_\Gamma(\mathbf{curl};\Omega) \;\middle|\; (\varepsilon \mathbf{u}, \nabla\phi) = 0 \ \ \forall\, \phi \in H^1_\Gamma(\Omega_0)\right\}, \end{aligned}$$

and the tangential trace space of $H_\Gamma(\mathbf{curl};\Omega)$ on Γ,

$$H^{-1/2}(\mathrm{Div};\Gamma) = \left\{\mathbf{f} \in H^{-1/2}(\Gamma)^3 \;\middle|\; \exists\ \mathbf{u} \in H_\Gamma(\mathbf{curl};\Omega) \text{ such that } \mathbf{n} \times \mathbf{u} = \mathbf{f}\right\},$$

or equivalently [18],

$$H^{-1/2}(\mathrm{Div};\Gamma) = \left\{\mathbf{f} \in H^{-1/2}(\Gamma)^3 \;\middle|\; \mathbf{n} \cdot \mathbf{f} = 0 \ a.e. \text{ on } \Gamma;\ \mathrm{Div}_\tau \mathbf{f} \in H^{-1/2}(\Gamma)\right\}.$$

Here Div_τ is the surface divergence operator which will be formally defined on a smooth surface in section 4. We define a sesquilinear form $a : H_\Gamma(\mathbf{curl};\Omega) \times H_\Gamma(\mathbf{curl};\Omega) \to \mathbb{C}$ as

$$a(\mathbf{E},\mathbf{F}) = \int_\Omega \mu^{-1}\nabla \times \mathbf{E} \cdot \nabla \times \overline{\mathbf{F}} - i\omega(\sigma + \sigma_0)\mathbf{E} \cdot \overline{\mathbf{F}} dx \quad \forall\, \mathbf{E}, \mathbf{F} \in H_\Gamma(\mathbf{curl};\Omega), \tag{2.5}$$

where $\overline{\mathbf{F}}$ denotes the vector-valued complex conjugate of $\mathbf{F}$. Then the weak formulation of problem (2.2) is as follows: Find $\mathbf{E} \in \mathbf{Y}$ such that

$$a(\mathbf{E},\mathbf{F}) = i\omega \int_\Omega \mathbf{J}_s \cdot \overline{\mathbf{F}} dx \ \ \forall\, \mathbf{F} \in \mathbf{Y}. \tag{2.6}$$

The following lemma implies the well-posedness of the problem (2.6).

LEMMA 2.1. *The problem* (2.6) *has a unique solution* $\mathbf{E} \in \mathbf{Y}$.

Proof. The uniqueness is due to the fact that sesquilinear form $a(\cdot,\cdot)$ is coercive in space $\mathbf{Y}$. The proof of coercivity is similar to [7]. For completeness, we sketch a proof here. For any $\mathbf{E} \in \mathbf{Y}$, $\mathbf{n}\times\mathbf{E}|_{\Gamma_{0c}} \in H^{-1/2}(\mathrm{Div};\Gamma_{0c})$ [6]. Let $H_\Gamma(\mathbf{curl};\Omega_0) = \{\mathbf{u}\in H(\mathbf{curl};\Omega_0) \mid \mathbf{n}\times\mathbf{u} = 0 \text{ on } \partial\Omega_0\setminus\Gamma\}$. By the Lax–Milgram theorem, there exists a unique $\mathbf{B}\in H(\mathbf{curl};\Omega_0)$, $\mathbf{n}\times\mathbf{B} = 0$ on Γ_D, $\mathbf{n}\cdot\mathbf{B} = 0$ on Γ, and $\mathbf{n}\times\mathbf{B} = \mathbf{n}\times\mathbf{E}$ on Γ_{0c}, such that

$$\int_{\Omega_0} \nabla\times\mathbf{B}\cdot\nabla\times\overline{\mathbf{u}}dx + \int_{\Omega_0}\varepsilon\mathbf{B}\cdot\overline{\mathbf{u}}dx = 0 \quad \forall\,\mathbf{u}\in H_\Gamma(\mathbf{curl};\Omega_0).$$

By the trace theorem,

$$\|\mathbf{B}\|_{H(\mathbf{curl};\Omega_0)} \le C\|\mathbf{n}\times\mathbf{B}\|_{H^{-1/2}(\mathrm{Div};\Gamma_{0c})} = C\|\mathbf{n}\times\mathbf{E}\|_{H^{-1/2}(\mathrm{Div};\Gamma_{0c})} \le C\|\mathbf{E}\|_{H(\mathbf{curl};\Omega_c)}.$$

Moreover, we have

$$\nabla\cdot\varepsilon\mathbf{B} = 0 \text{ and } \nabla\cdot\varepsilon(\mathbf{E}-\mathbf{B}) = 0 \text{ in } \Omega_0$$

and $\mathbf{n}\times(\mathbf{E}-\mathbf{B}) = 0$ on $\partial\Omega_0\setminus\Gamma$ and $\mathbf{n}\cdot(\mathbf{E}-\mathbf{B}) = 0$ on Γ. Then we know

$$\|\mathbf{E}-\mathbf{B}\|_{L^2(\Omega_0)} \le C\|\nabla\times(\mathbf{E}-\mathbf{B})\|_{L^2(\Omega_0)},$$

and furthermore,

$$\begin{aligned}\|\mathbf{E}\|_{L^2(\Omega_0)} &\le C(\|\mathbf{E}-\mathbf{B}\|_{L^2(\Omega_0)} + \|\mathbf{B}\|_{L^2(\Omega_0)})\\ &\le C(\|\nabla\times\mathbf{E}\|_{L^2(\Omega_0)} + \|\mathbf{E}\|_{H(\mathbf{curl};\Omega_c)})\\ &\le C(\|\nabla\times\mathbf{E}\|_{L^2(\Omega)} + \|\mathbf{E}\|_{L^2(\Omega_c)}).\end{aligned}$$

This implies that $|a(\mathbf{E},\mathbf{E})| \ge C\|\mathbf{E}\|^2_{H(\mathbf{curl};\Omega)}$ for all $\mathbf{E}\in\mathbf{Y}$. ∎

It is difficult to solve problem (2.6) numerically since it is hard to construct a conforming finite element space of $\mathbf{Y}$. Therefore we reformulate the weak formulation (2.6) as a saddle-point problem by introducing a Lagrange multiplier to deal with the divergence-free condition in domain Ω_0. The saddle-point formulation of equation (2.2) reads as follows: Find $(\mathbf{E},\phi)\in H_\Gamma(\mathbf{curl};\Omega)\times H^1_\Gamma(\Omega_0)$ such that

$$\left\{\begin{array}{ll} a(\mathbf{E},\mathbf{F}) + b(\nabla\phi,\mathbf{F}) = i\omega\int_\Omega \mathbf{J}_s\cdot\overline{\mathbf{F}}dx & \forall\,\mathbf{F}\in H_\Gamma(\mathbf{curl};\Omega),\\ b(\mathbf{E},\nabla\psi) = 0 & \forall\,\psi\in H^1_\Gamma(\Omega_0),\end{array}\right. \tag{2.7}$$

where $b: H_\Gamma(\mathbf{curl};\Omega)\times H_\Gamma(\mathbf{curl};\Omega)\to\mathbb{C}$ is a sesquilinear form given by

$$b(\mathbf{E},\mathbf{F}) = \int_{\Omega_0}\varepsilon\mathbf{E}\cdot\overline{\mathbf{F}}dx \quad \forall\,\mathbf{E},\mathbf{F}\in H_\Gamma(\mathbf{curl};\Omega).$$

LEMMA 2.2 (uniqueness). *There is at most one solution to* (2.7).

Proof. We only need to show that $\mathbf{E} = 0$ in Ω and $\phi = 0$ in Ω_0 provided $\mathbf{J}_s = 0$. First, we take $\mathbf{F}$ as the zero extension of $\nabla\phi$ from Ω_0 to Ω, that is,

$$\mathbf{F} = 0 \quad \text{in} \quad \Omega_c \quad \text{and} \quad \mathbf{F} = \nabla\phi \quad \text{in} \quad \Omega_0,$$

which implies $\mathbf{F}\in H_\Gamma(\mathbf{curl};\Omega)$. We plug $\mathbf{F}$ into the first equation of (2.7), along with $\mathbf{J}_s = 0$, to get

$$\int_{\Omega_0}|\nabla\phi|^2dx = 0.$$

So $\nabla\phi = 0$ in Ω_0, and by the boundary condition on $\partial\Omega_0 \setminus \Gamma$, we have $\phi = 0$. Second, taking $\mathbf{F} = \mathbf{E}$, $\psi = \phi$ in (2.7), we obtain

$$a(\mathbf{E}, \mathbf{E}) = \int_\Omega \mu^{-1} |\nabla \times \mathbf{E}|^2 dx - i\omega \int_{\Omega_c} (\sigma + \sigma_0) |\mathbf{E}|^2 dx = 0.$$

This implies $\nabla \times \mathbf{E} = 0$ in Ω and $\mathbf{E} = 0$ in Ω_c. By the tangential continuity of $\mathbf{E}$, we know that

$$\begin{cases} \nabla \times \mathbf{E} = 0 & \text{in } \Omega_0, \\ \nabla \cdot \varepsilon \mathbf{E} = 0 & \text{in } \Omega_0, \\ \mathbf{n} \times \mathbf{E} = 0 & \text{on } \partial\Omega_0 \setminus \Gamma, \\ \mathbf{n} \cdot \mathbf{E} = 0 & \text{on } \Gamma. \end{cases}$$

By the assumption, ε is constant in the simply connected domain Ω_0. Then there exists $p \in H^1(\Omega_0)$ such that $\mathbf{E} = \nabla p$ and

$$\begin{cases} \Delta p = 0 & \text{in } \Omega_0, \\ \frac{\partial p}{\partial \mathbf{n}} = 0 & \text{on } \Gamma, \\ p = C & \text{on } \partial\Omega_0 \setminus \Gamma \end{cases}$$

for some constant C. It is easy to know that the unique solution of the above system is $p = C$, so $\mathbf{E} = 0$ in Ω_0. This completes our proof. ∎

THEOREM 2.1. *Equation* (2.7) *has a unique solution* $(\mathbf{E}, \phi) \in H_\Gamma(\mathbf{curl}, \Omega) \times H^1_\Gamma(\Omega_0)$ *and* $\mathbf{E}$ *satisfies* (2.6). *Moreover, the following stability estimate holds:*

$$\|\mathbf{E}\|_{H(\mathbf{curl};\Omega)} + \|\phi\|_{H^1(\Omega_0)} \le C \|\mathbf{J}_s\|_{L^2(\Omega)^3}, \tag{2.8}$$

where C *is a constant independent of* $\mathbf{E}$ *and* ϕ.

Proof. The existence can be established by proving the equivalence between (2.6) and (2.7). Let $\mathbf{E}$ be the solution of (2.6), and then it is clear that $\mathbf{E}$ satisfies the second equation of (2.7). If we can prove that there exists $\phi \in H^1_\Gamma(\Omega_0)$ such that $\mathbf{E}$ and ϕ satisfy the first equation of (2.7), by the uniqueness of a solution to (2.7) we may conclude the existence of a solution of (2.7).

Now, for any $\mathbf{F} \in H_\Gamma(\mathbf{curl}, \Omega)$, we can find a ψ that satisfies

$$\int_{\Omega_0} \nabla\psi \cdot \nabla\bar{\xi} dx = \int_{\Omega_0} \mathbf{F} \cdot \nabla\bar{\xi} dx \quad \forall\, \xi \in H^1_\Gamma(\Omega_0),$$

and then $\nabla \cdot (\mathbf{F} - \nabla\psi) = 0$ in Ω_0. Let $\tilde{\psi}$ be an extension of ψ,

$$\tilde{\psi} = \begin{cases} \psi & \text{in } \Omega_0, \\ 0 & \text{otherwise.} \end{cases}$$

Let $A = \mathbf{F} - \nabla\tilde{\psi} \in \mathbf{Y}$. Since $\tilde{\psi}$ is supported in Ω_0, we have

$$\begin{aligned} a(\mathbf{E}, \mathbf{F}) + b(\nabla\phi, \mathbf{F}) &= a(\mathbf{E}, A + \nabla\tilde{\psi}) + b(\nabla\phi, A + \nabla\tilde{\psi}) \\ &= a(\mathbf{E}, A) + a(\mathbf{E}, \nabla\tilde{\psi}) + b(\nabla\phi, A) + b(\nabla\phi, \nabla\psi) \\ &= a(\mathbf{E}, A) + b(\nabla\phi, \nabla\psi) \\ &= i\omega \int_\Omega \mathbf{J}_s \cdot \overline{A} dx + b(\nabla\phi, \nabla\psi). \end{aligned}$$

The right-hand side of the first equation of (2.7) becomes

$$i\omega \int_\Omega \mathbf{J}_s \cdot \overline{\mathbf{F}} dx = i\omega \int_\Omega \mathbf{J}_s \cdot \overline{A} dx + i\omega \int_{\Omega_0} \mathbf{J}_s \cdot \nabla\overline{\psi} dx.$$

Let $\phi \in H^1_\Gamma(\Omega_0)$ be a solution to the variational system

$$b(\nabla\phi, \nabla\psi) = i\omega \int_{\Omega_0} \mathbf{J}_s \cdot \nabla\overline{\psi} dx \quad \forall\, \psi \in H^1_\Gamma(\Omega_0). \tag{2.9}$$

We know there exists a unique ϕ satisfying (2.9). Actually $\phi = 0$ because $\mathbf{J}_s$ is divergence-free and compactly supported in Ω_0. With $\mathbf{E}, \phi$ satisfying (2.6) and (2.9), respectively, we have

$$a(\mathbf{E}, \mathbf{F}) + b(\nabla\phi, \mathbf{F}) = i\omega \int_\Omega \mathbf{J}_s \cdot \overline{\mathbf{F}} dx \quad \forall\, \mathbf{F} \in H_\Gamma(\mathbf{curl}, \Omega)\,.$$

Then $(\mathbf{E}, \phi)$ is a solution to (2.7). We can now conclude the existence and uniqueness of a solution to (2.7) by Lemma 2.2. Furthermore, if $(\mathbf{E}, \phi)$ is a solution to (2.7), we readily see $\mathbf{E}$ is a solution to (2.6). ∎

It is known that the tangential trace space of $H_\Gamma(\mathbf{curl}, \Omega)$ is $H^{-1/2}(\mathrm{Div}; \Gamma)$ [6], i.e., $\mathbf{n} \times \mathbf{E}|_\Gamma \in H^{-1/2}(\mathrm{Div}; \Gamma)$ for all $\mathbf{E} \in H_\Gamma(\mathbf{curl}; \Omega)$. Let $\mathbf{n} \times \mathbf{E}^{obs}$ be the data on Γ, and $\mathbf{n} \times \mathbf{E}$ be the corresponding tangential part of the electric field $\mathbf{E}$ on Γ associated with the conductivity σ. Then a direct choice of the misfit of prediction is $\|\mathbf{n} \times (\mathbf{E}^{obs} - \mathbf{E})\|_{H^{-1/2}(\mathrm{Div};\Gamma)}$. Unfortunately, this trace space is naturally equipped with the norm

$$\|\mathbf{f}\|_{H^{-1/2}(\mathrm{Div};\Gamma)} = \inf_{\mathbf{u} \in H_\Gamma(\mathbf{curl}, \Omega), \mathbf{n} \times \mathbf{u} = \mathbf{f} \text{on } \Gamma} \|\mathbf{u}\|_{H(\mathbf{curl}, \Omega)},$$

which is difficult to realize numerically. It would be very convenient and important numerically if a computable norm, such as the L^2-norm on Γ, can be used for the recovery process. Next, we demonstrate that the true solution $\mathbf{E}$ to the problem (2.2) indeed has a higher regularity, suggesting to us a computable norm on Γ. For this purpose, we need a very useful result from [11, Theorem 6.1], as stated below.

THEOREM 2.2. *Assume that $\mathbf{E}$ is the solution to* (2.6) *with $\sigma = 0$, and the source current satisfies $\nabla \cdot \mathbf{J}_s = 0$ and $\mathbf{J}_s \in H^{s-1}(\Omega_c)^3$. Then it holds that*

$$\begin{aligned} &\mathbf{E}|_{\Omega_0} \in H^\tau(\Omega_0) \quad \forall \tau < \tau_0 = \min\{\tau_1^0, \tau_2 + 1, s + 1\}, \\ &\mathbf{E}|_{\Omega_c} \in H^\tau(\Omega_c) \quad \forall \tau < \tau_c = \min\{\tau_1^c, \tau_2 + 1, s + 1\}, \end{aligned}$$

where $\tau_1^0, \tau_1^c, \tau_2$ represent the edge and corner singularities on interface Γ_{0c} and s represents the regularity of source $\mathbf{J}_s$.

Remark 2.1. Applying Theorem 2.2 to our current setting, we can easily check from the definitions of τ_1^0, τ_1^c, and τ_2 in [11] that $\tau_1^0 > 1/2, \tau_1^c > 1/2, \tau_2 > 0,$ $s = 1$ by noting the facts that Γ_{0c} is a Lipschitz polyhedral interface, σ_0 is a constant, and $\mathbf{J}_s \in L^2(\Omega_c)$. Therefore, we have that $\tau_0 > 1/2$, $\tau_c > 1/2$.

For the convenience of the subsequent analysis, we shall write $\mathbf{E}(\sigma)$ for the solution to (2.2) to emphasize its dependence on the conductivity σ.

THEOREM 2.3. *Assuming that Ω_0 and Ω_c are polyhedral domains and Ω is convex, σ_0 is a constant in Ω_c, and $\mathbf{E}^{obs}$ is the solution to* (2.2) *with the exact conductivity $\sigma_0 + \sigma_e$, then for any σ we have*

$$(\mathbf{E}(\sigma) - \mathbf{E}^{obs})|_{\Omega_0} \in H^{1/2}(\Omega_0). \tag{2.10}$$

Proof. It follows from (2.2) that

$$\nabla \times (\mu^{-1}\nabla \times \mathbf{E}(\sigma)) - i\omega(\sigma_0 + \sigma)\mathbf{E}(\sigma) = i\omega \mathbf{J}_s,$$
$$\nabla \times (\mu^{-1}\nabla \times \mathbf{E}^{obs}) - i\omega(\sigma_0 + \sigma_e)\mathbf{E}^{obs} = i\omega \mathbf{J}_s,$$

from which we can easily deduce

$$\nabla \times (\mu^{-1}\nabla \times (\mathbf{E}(\sigma) - \mathbf{E}^{obs})) - i\omega\sigma_0(\mathbf{E}(\sigma) - \mathbf{E}^{obs}) = \mathbf{J}_e, \tag{2.11}$$

where $\mathbf{J}_e = i\omega\sigma\mathbf{E}(\sigma) - i\omega\sigma_e\mathbf{E}^{obs}$. For $\mathbf{J}_e$, we have the following decomposition,

$$\mathbf{J}_e = \mathbf{J}_0 + \nabla\phi,$$

where $\nabla\cdot\mathbf{J}_0 = 0$ and $\Delta\phi = \nabla\cdot\mathbf{J}_e$, $\phi \in H_0^1(\Omega_c)$. Then we let $\mathbf{E}(\sigma)-\mathbf{E}^{obs} = \mathbf{E}_r+\mathbf{E}_\phi$ and $\mathbf{E}_r$, $\mathbf{E}_\phi$ satisfy the following two systems, respectively, with the interface conditions (2.3) and boundary conditions (2.4), i.e.,

$$\begin{cases} \nabla \times (\mu^{-1}\nabla \times \mathbf{E}_r) - i\omega\sigma_0\mathbf{E}_r = \mathbf{J}_0 & \text{in } \Omega, \\ \nabla \cdot \varepsilon\mathbf{E}_r = 0 & \text{in } \Omega_0, \end{cases}$$

and

$$\begin{cases} \nabla \times (\mu^{-1}\nabla \times \mathbf{E}_\phi) - i\omega\sigma_0\mathbf{E}_\phi = \nabla\phi & \text{in } \Omega, \\ \nabla \cdot \varepsilon\mathbf{E}_\phi = 0 & \text{in } \Omega_0. \end{cases}$$

By the assumption on Ω_0 and Ω_c, we know that Γ_{0c} is a Lipschitz interface. With the help of Theorem 2.2, we find that $\mathbf{E}_r|_{\Omega_0} \in H^{1/2}(\Omega_0)$. As for $\mathbf{E}_\phi$, with the arguments in Theorem 3.1 of section 3, we know that $\nabla\phi$ is a nonradiating source, and then $\mathbf{E}_\phi|_{\Omega_0} = 0$. Then we complete the proof by noting that $\mathbf{E}(\sigma) - \mathbf{E}^{obs} = \mathbf{E}_r$ on Ω_0. ∎

Theorem 2.3 implies that the regularity of the solution to (2.6) in subdomain Ω_0 is higher than the global regularity. With this result, we further derive the following estimate.

LEMMA 2.3. *With the same assumptions and notation as in Theorem* 2.3, *we have the estimate*

$$\|\mathbf{E}(\sigma) - \mathbf{E}^{obs}\|_{H^{1/2}(\Omega_0)} \le C\|\mathbf{J}_e\|_{L^2(\Omega_c)},$$

where C is independent of σ.

Proof. Recall the definition of $\mathbf{E}_r$ and $\mathbf{E}_\phi$ in the proof of Theorem 2.3. Let us introduce

$$\mathbf{X}(\Omega) = \big\{\mathbf{u} \in H_\Gamma(\mathbf{curl}, \Omega);\ \nabla\cdot\mathbf{u}|_{\Omega_0} \in L^2(\Omega_0), \nabla\cdot\mathbf{u}|_{\Omega_c} \in L^2(\Omega_c), \mathbf{n}\cdot\mathbf{u} = 0 \text{ on } \Gamma, \mathbf{n}\cdot\mathbf{u}|_{\Omega_c} = 0 \text{ on } \Gamma_{0c}\big\},$$

equipped with the graph norm

$$\|\mathbf{u}\|_{\mathbf{X}} = (\|\mathbf{u}\|^2_{H(\mathbf{curl},\Omega)} + \|\nabla\cdot\mathbf{u}\|^2_{L^2(\Omega_0)} + \|\nabla\cdot\mathbf{u}\|^2_{L^2(\Omega_c)})^{1/2}.$$

Then by Theorem 2.2, $\mathbf{X}(\Omega)$ is embedded in $H^{1/2}(\Omega_0)$. It is direct to verify that $\mathbf{E}_r \in \mathbf{X}(\Omega)$ and hence

$$\|\mathbf{E}_r\|_{H^{1/2}(\Omega_0)} \le C(\|\mathbf{E}_r\|^2_{H(\mathbf{curl},\Omega)} + \|\nabla\cdot\mathbf{E}_r\|^2_{L^2(\Omega_0)} + \|\nabla\cdot\mathbf{E}_r\|^2_{L^2(\Omega_0)})^{1/2}.$$

Since $\mathbf{E}_\phi = 0$, $\nabla\cdot\varepsilon\mathbf{E}_r = 0$ in Ω_0, $\nabla\cdot\sigma_0\mathbf{E}_r = 0$ in Ω_c, and ε and σ_0 are constants, we have with the help of the estimate (2.8) that

$$\|\mathbf{E}(\sigma) - \mathbf{E}^{obs}\|_{H^{1/2}(\Omega_0)} = \|\mathbf{E}_r\|_{H^{1/2}(\Omega_0)} \le C\|\mathbf{E}_r\|_{H(\mathbf{curl},\Omega)} \le C\|\mathbf{J}_e\|_{L^2(\Omega_c)^3}.$$ ∎

3. Ill-posedness of the inverse problem. In this section, we investigate the ill-posedness of the eddy current inverse problem. We know the solution $(\mathbf{E}, \phi) \in H_\Gamma(\mathbf{curl};\Omega) \times H^1_\Gamma(\Omega_0)$ to problem (2.7) depends on the conductivity $\sigma_0 + \sigma(x)$. But in the setting of our inverse problem, σ_0 is known, so we shall write $\mathbf{E}(\sigma)$ to emphasize its dependence on σ. The ill-posedness of the eddy current inverse problem is basically determined by the nature of the forward operator $\mathbf{E}(\sigma)$.

3.1. Compactness of the forward operator $\mathbf{n} \times \mathbf{E}(\boldsymbol{\sigma})$. We first present a result about the continuity of $\mathbf{n} \times \mathbf{E}(\sigma)$. For convenience, we introduce the set $\mathbb{K} = \{\sigma \in H^1_0(\Omega_c) \mid t_1 \le \sigma \le t_2 \text{ a.e. in } \Omega_c\}$.

LEMMA 3.1. *For any sequence $\{\sigma_n\} \subset \mathbb{K}$ such that $\sigma_n \to \sigma_*$ in $L^2(\Omega_c)$ as $n \to \infty$, it holds that*

$$\lim_{n\to\infty} \|\mathbf{n} \times (\mathbf{E}(\sigma_n) - \mathbf{E}(\sigma_*))\|_{L^2(\Gamma)^3} = 0. \tag{3.1}$$

Proof. By definition, $(\mathbf{E}(\sigma_n), \phi(\sigma_n))$ and $(\mathbf{E}(\sigma_*), \phi(\sigma_*))$ are the solutions to (2.7) with σ replaced by σ_n and σ_*, respectively. Then letting $\hat{\mathbf{E}}_n = \mathbf{E}(\sigma_n) - \mathbf{E}(\sigma_*)$ and $\hat{\phi}_n = \phi(\sigma_n) - \phi(\sigma_*)$, it is easy to check that $(\hat{\mathbf{E}}_n, \hat{\phi}_n)$ satisfies (2.7) with σ_* and $(\sigma_n - \sigma_*)\mathbf{E}(\sigma_n)$ in place of σ and $\mathbf{J}_s$, respectively.

Then from Theorem 2.1 we can deduce that

$$\|\hat{\mathbf{E}}_n\|_{H(\mathbf{curl};\Omega)} + \|\hat{\phi}_n\|_{H^1(\Omega_0)} \le C\|(\sigma_n - \sigma_*)\mathbf{E}(\sigma_n)\|_{L^2(\Omega)^3}.$$

With the same argument as in Lemma 2.3, we know $\hat{\mathbf{E}}_n|_{\Omega_0} \in H^{1/2}(\Omega_0)$, and hence

$$\|\hat{\mathbf{E}}_n\|_{H^{1/2}(\Omega_0)} \le C\|(\sigma_n - \sigma_*)\mathbf{E}(\sigma_n)\|_{L^2(\Omega_c)^3}.$$

Now, let $\mathbf{B}$ satisfy that $\nabla \times (\mu^{-1}\nabla \times \mathbf{B}) = i\omega\mathbf{J}_s$, and $\nabla \cdot \varepsilon\mathbf{B} = 0$ in Ω and the boundary condition (2.4). Then we know $\mathbf{B} \in H^1(\Omega)$ from [4]. With the same arguments as in Theorem 2.3, we can find that $\mathbf{E}(\sigma_n) - \mathbf{B} \in H^{1/2}(\Omega_0)$. This implies $\mathbf{E}(\sigma_n) \in H^{1/2}(\Omega_0)$ and $\mathbf{n} \times \mathbf{E}(\sigma_n) \in L^2(\Gamma_{0c})$. Therefore we know $\mathbf{E}(\sigma_n)$ satisfies a Maxwell equation in Ω_c with $\mathbf{n} \times \mathbf{E}(\sigma_n) = 0$ on $\Gamma_D \cap \partial\Omega_c$ and $\mathbf{n} \times \mathbf{E}(\sigma_n) \in L^2(\Gamma_{0c})$. Then applying [12, Theorem 7.1]) to $\mathbf{E}(\sigma_n)$, with $\sigma_n \in \mathbb{K}$ here, we conclude that $\mathbf{E}(\sigma_n)$ lies in H^δ in Ω_c for some $0 < \delta < 1$. This further implies that $\mathbf{E}(\sigma_n) \in L^p(\Omega_c)$ for $p = 6/(3 - 2\delta)$ by the Sobolev embedding theorem. Then it follows by the Cauchy–Schwarz inequality that

$$\|\hat{\mathbf{E}}_n\|_{H^{1/2}(\Omega_0)} \le C\|(\sigma_n - \sigma_*)\|_{L^q(\Omega_c)}\|\mathbf{E}(\sigma_n)\|_{L^p(\Omega_c)^3},$$

where $q = \frac{3}{\delta}$ and $\frac{1}{p} + \frac{1}{q} = \frac{1}{2}$. Noting that $\sigma_n \in \mathbf{K}$ and $\sigma_n \to \sigma_*$ in $L^2(\Omega_c)$, we know that $t_1 \le \sigma_* \le t_2$ a.e. in Ω_c, and then we can derive

$$\int_{\Omega_c} |\sigma_n - \sigma_*|^q dx \le |t_2 - t_1|^{q-2} \int_{\Omega_c} |\sigma_n - \sigma_*|^2 dx\,, \tag{3.2}$$

and therefore $\|\sigma_n - \sigma_*\|_{L^q(\Omega_c)} \to 0$. On the other hand, it follows from [12, Theorem 7.1] that

$$\begin{aligned}&\|\mathbf{E}(\sigma_n)\|_{H^\delta(\Omega_c)}\\ &\quad \le C(\|\mathbf{E}(\sigma_n)\|^2_{H(\mathbf{curl},\Omega)} + \|\nabla \cdot \varepsilon\mathbf{E}(\sigma_n)\|^2_{L^2(\Omega_0)} + \|\nabla \cdot (\sigma_0 + \sigma_n)\mathbf{E}(\sigma_n)\|^2_{L^2(\Omega_c)})^{1/2}.\end{aligned}$$

Since $\mathbf{J}_s$ is compactly supported in Ω_0, we know from (2.7) that $\nabla \cdot \varepsilon\mathbf{E}(\sigma_n) = 0$ in Ω_0 and $\nabla \cdot (\sigma_0 + \sigma_n)\mathbf{E}(\sigma_n) = 0$ in Ω_c. By the Sobolev embedding theory and Theorem 2.1 we have

$$\|\mathbf{E}(\sigma_n)\|_{L^p(\Omega_c)^3} \le C\|\mathbf{E}(\sigma_n)\|_{H^\delta(\Omega_c)} \le C\|\mathbf{J}_s\|_{L^2(\Omega)^3}.$$

Using this, we can see from (3.2) that $\|\hat{\mathbf{E}}_n\|_{H^{1/2}(\Omega_0)} \to 0$ as $n \to \infty$. Now the desired convergence (3.1) follows from the trace theorem. ∎

By Theorem 2.1, given a parameter $\sigma \in H_0^1(\Omega_c)$, there exists a solution $\mathbf{E}(\sigma)$ of (2.7), and $\mathbf{E}(\sigma)$ determines the tangential component $\mathbf{n} \times \mathbf{E}(\sigma)$ on Γ. In this way one can define the forward map $\mathbf{E}(\sigma)$ from $H_0^1(\Omega_c)$ to $L^2(\Gamma)^3$. In the following lemma, we prove that this implicit map from parameter σ to the tangential field $\mathbf{n} \times \mathbf{E}(\sigma)$ is compact, and this implies that the eddy current inverse model that uses the data on Γ to determine σ is ill-posed.

LEMMA 3.2. *The map from $\sigma \in \mathbb{K}$ to $\mathbf{n} \times \mathbf{E}(\sigma)|_\Gamma \in L^2(\Gamma)$ is compact.*

Proof. Let $\{\sigma_n\}_{n=1}^\infty$ be a bounded sequence in $H_0^1(\Omega_c)$. Since $H^1(\Omega_c)$ is compactly imbedded in $L^2(\Omega_c)$, there is a subsequence, still denoted as $\{\sigma_n\}$, that converges to σ_* in $L^2(\Omega_c)$. By Lemma 2.3, for each σ_n, $(\mathbf{E}(\sigma_n) - \mathbf{E}(\sigma_*))|_{\Omega_0} \in H^{1/2}(\Omega_0)^3$. Then it follows from Lemma 2.3 that

$$\|\mathbf{E}(\sigma_n) - \mathbf{E}(\sigma_*)\|_{H^{1/2}(\Omega_0)^3} \leq C\|(\sigma_n - \sigma_*)\mathbf{E}(\sigma_*)\|_{L^2(\Omega_c)}.$$

From the convergence result in Lemma 3.1, we know that $\mathbf{E}(\sigma_n)|_{\Omega_0} \to \mathbf{E}(\sigma_*)|_{\Omega_0}$ in $H^{1/2}(\Omega_0)^3$. Then by trace theorem, $\mathbf{n} \times \mathbf{E}(\sigma_n)|_\Gamma \to \mathbf{n} \times \mathbf{E}(\sigma_*)|_\Gamma$ in $L^2(\Gamma)^3$, which concludes the compactness. ∎

3.2. Nonuniqueness of the recovery of the conductivity. In this subsection, we will study the uniqueness of the recovery of conductivity using the data $\mathbf{n} \times \mathbf{E}(\sigma)$ on the boundary Γ. Some techniques used here are motivated by the uniqueness argument in [23] for an inverse source problem. Let $\mathbf{E}_0$ be the background field which satisfies

$$\begin{cases} \nabla \times (\mu^{-1}\nabla \times \mathbf{E}_0) - i\omega\sigma_0\mathbf{E}_0 = i\omega\mathbf{J}_s & \text{in } \Omega, \\ \nabla \cdot \varepsilon\mathbf{E}_0 = 0 & \text{in } \Omega_0 \end{cases}$$

with boundary conditions (2.4).

If we know the exact data $\mathbf{n} \times \mathbf{E}(\sigma)$ and the background conductivity σ_0, the recovery problem is reduced to determining σ in Ω_c given $\mathbf{n} \times (\mathbf{E}(\sigma) - \mathbf{E}_0)$ on Γ. By simple calculation, we know that

$$\begin{cases} \nabla \times (\mu^{-1}\nabla \times (\mathbf{E}(\sigma) - \mathbf{E}_0)) - i\omega\sigma_0(\mathbf{E}(\sigma) - \mathbf{E}_0) = i\omega\sigma\mathbf{E}(\sigma) & \text{in } \Omega, \\ \nabla \cdot \varepsilon(\mathbf{E}(\sigma) - \mathbf{E}) = 0 & \text{in } \Omega_0, \end{cases} \tag{3.3}$$

and $\mathbf{E}(\sigma) - \mathbf{E}_0$ satisfies boundary conditions (2.4). Let $\mathbf{J}_e = i\omega\sigma\mathbf{E}(\sigma)$. Then it is clear that $\mathbf{J}_e$ is supported in Ω_c. In the following part of this section, we will consider an inverse source problem related to (3.3). To be more specific, let $\mathbf{E}$ satisfy the equation

$$\begin{cases} \nabla \times (\mu^{-1}\nabla \times \mathbf{E}) - i\omega\sigma_0\mathbf{E} = \mathbf{J}_e & \text{in } \Omega, \\ \nabla \cdot \varepsilon\mathbf{E} = 0 & \text{in } \Omega_0, \end{cases} \tag{3.4}$$

and boundary condition (2.4), and the corresponding inverse source problem is

(3.5) given data $\mathbf{n} \times \mathbf{E}$ on Γ, find the source $\mathbf{J}_e$ supported in Ω_c.

To proceed, we denote

$$W = \big\{\mathbf{u} \in H(\mathbf{curl};\Omega_c) \;\big|\; \nabla \times (\mu^{-1}\nabla \times \mathbf{u}) + i\omega\sigma_0\mathbf{u} = 0 \text{ in } \Omega_c, \mathbf{n} \times \mathbf{u} = 0 \text{ on } \partial\Omega_c \setminus \Gamma_{0c}\big\}.$$

It is easy to find that W is not empty because the boundary value problem

$$\begin{cases} \nabla \times (\mu^{-1}\nabla \times \mathbf{u}) + i\omega\sigma_0\mathbf{u} = 0 & \text{in } \Omega_c, \\ \mathbf{n} \times \mathbf{u} = 0 & \text{on } \partial\Omega_c \setminus \Gamma_{0c}, \\ \mathbf{n} \times \mathbf{u} = \eta & \text{on } \Gamma_{0c} \end{cases}$$

has a unique solution for any $\eta \in H^{-1/2}(\mathrm{Div}; \Gamma_{0c})$. Let

$$L^2(\Omega_c)^3 = W \oplus W^\perp.$$

We know that $W^\perp$ is not trivial either. More precisely, for any $\mathbf{v} \in C_0^\infty(\Omega_c)$, it is easy to find that $\nabla \times \mu^{-1}\nabla \times \mathbf{v} - i\omega\sigma_0\mathbf{v} \in W^\perp$. The next theorem tells us there are nonradiation sources satisfying the inverse source problem (3.5).

THEOREM 3.1. *If $\mathbf{J}_e \in W^\perp$, the corresponding field is denoted by $\mathbf{E}$, and then $\mathbf{n} \times \mathbf{E} = 0$ on Γ_{0c} and Γ; in other words, $\mathbf{J}_e$ is a nonradiating source.*

Proof. For any $\mathbf{u} \in W$, by integration by parts, we have

$$\begin{aligned} 0 = \int_{\Omega_c} \mathbf{J}_e \cdot \overline{\mathbf{u}} dx &= \int_{\Omega_c} (\nabla \times (\mu^{-1}\nabla \times \mathbf{E}) - i\omega\sigma_0\mathbf{E}) \cdot \overline{\mathbf{u}} dx \\ &= \int_{\Omega_c} \mathbf{E} \cdot (\nabla \times (\mu^{-1}\nabla \times \overline{\mathbf{u}}) - i\omega\sigma_0\overline{\mathbf{u}}) dx \\ &\quad + \int_{\partial\Omega_c} \mathbf{n} \times \mu^{-1}\nabla \times \mathbf{E} \cdot \overline{\mathbf{u}} + \mu^{-1}\mathbf{n} \times \mathbf{E} \cdot \nabla \times \overline{\mathbf{u}} ds. \end{aligned}$$

Since $\mathbf{u} \in W$, we obtain

$$\int_{\Gamma_{0c}} \mathbf{n} \times \mu^{-1}\nabla \times \mathbf{E} \cdot \overline{\mathbf{u}} + \mu^{-1}\mathbf{n} \times \mathbf{E} \cdot \nabla \times \overline{\mathbf{u}} ds = 0. \tag{3.6}$$

For any $\eta \in H^{1/2}(\mathrm{Div}; \Gamma_{0c})$, let $\mathbf{w} \in H_\Gamma(\mathbf{curl}; \Omega)$ be the solution to the following interface problem:

$$\begin{cases} \nabla \times (\mu^{-1}\nabla \times \mathbf{w}) + i\omega\sigma_0\mathbf{w} = 0 & \text{in } \Omega_c \cup \Omega_0, \\ \nabla \cdot \varepsilon\mathbf{w} = 0 & \text{in } \Omega_0, \\ \mu^{-1}\mathbf{n} \times \nabla \times \mathbf{w}|_{\Omega_c} = \mu^{-1}\mathbf{n} \times \nabla \times \mathbf{w}|_{\Omega_0} + \eta & \text{on } \Gamma_{0c} \end{cases}$$

and boundary conditions (2.4). With the similar method in section 2, one can prove that the above system of equations is well-posed, i.e., for any $\eta \in H^{-1/2}(\mathrm{Div}; \Gamma_{0c})$, it has a unique solution $\mathbf{w}$ in $H_\Gamma(\mathbf{curl}; \Omega)$. Furthermore, $\mathbf{w} \neq 0$ in Ω_c. If we choose $\mathbf{u} = \mathbf{w}|_{\Omega_c}$ in (3.6), it becomes

$$\begin{aligned} 0 &= \int_{\Gamma_{0c}} \mu^{-1}\mathbf{n} \times \nabla \times \mathbf{E} \cdot \overline{\mathbf{u}} + \mu^{-1}\mathbf{n} \times \mathbf{E} \cdot \nabla \times \overline{\mathbf{u}} ds \\ &= \int_{\Gamma_{0c}} \mu^{-1}\mathbf{n} \times \nabla \times \mathbf{E} \cdot \overline{\mathbf{w}} ds - \int_{\Gamma_{0c}} \mu^{-1}\mathbf{n} \times \nabla \times \overline{\mathbf{w}}|_{\Omega_c} \cdot \mathbf{E} ds \\ &= \int_{\Gamma_{0c}} \mu^{-1}\mathbf{n} \times \nabla \times \mathbf{E} \cdot \overline{\mathbf{w}} ds - \int_{\Gamma_{0c}} \mu^{-1}\mathbf{n} \times \nabla \times \overline{\mathbf{w}}|_{\Omega_0} \cdot \mathbf{E} ds - \int_{\Gamma_{0c}} \overline{\eta} \cdot \mathbf{E} ds. \end{aligned}$$

Using the fact that $\mathbf{n} \times \mathbf{E} = 0$ on Γ_D and $\mathbf{n} \times \nabla \times \mathbf{w} = 0$ on Γ, we have

$$\int_{\Gamma_{0c}} \mu^{-1}\mathbf{n} \times \nabla \times \mathbf{E} \cdot \overline{\mathbf{w}} ds - \int_{\Gamma_{0c}} \overline{\eta} \cdot \mathbf{E} ds - \int_{\partial\Omega_0} \mu^{-1}\mathbf{n} \times \nabla \times \overline{\mathbf{w}} \cdot \mathbf{E} ds = 0. \tag{3.7}$$

By integration by parts,

$$\begin{aligned}\int_{\partial\Omega_0} \mu^{-1}\mathbf{n}\times\nabla\times\overline{\mathbf{w}}\cdot\mathbf{E}ds &= \int_{\Omega_0} \nabla\times(\mu^{-1}\nabla\times\overline{\mathbf{w}}\cdot\mathbf{E}) - \mu^{-1}\nabla\times\overline{\mathbf{w}}\cdot\nabla\times\mathbf{E}dx\\ &= -\int_{\Omega_0}\mu^{-1}\nabla\times\overline{\mathbf{w}}\cdot\nabla\times\mathbf{E}dx\\ &= -\int_{\Omega_0}\overline{\mathbf{w}}\cdot\nabla\times(\mu^{-1}\nabla\times\mathbf{E})dx + \int_{\partial\Omega_0}\mu^{-1}\mathbf{n}\times\nabla\times\mathbf{E}\cdot\overline{\mathbf{w}}ds\\ &= \int_{\partial\Omega_0}\mu^{-1}\mathbf{n}\times\nabla\times\mathbf{E}\cdot\overline{\mathbf{w}}ds.\end{aligned}$$

Substituting the above results into (3.7), we have

$$\int_{\Gamma_{0c}} \eta\cdot\mathbf{E}ds = 0.$$

Then $\mathbf{n}\times\mathbf{E}\times\mathbf{n} = 0$ on Γ_{0c}, and this implies that $\mathbf{E} = 0$ in Ω_0 and $\mathbf{n}\times\mathbf{E} = 0$ on Γ. ◻

With the help of Theorem 3.1, we can give the following theorem about the nonuniqueness recovery property for the inverse eddy current problem.

THEOREM 3.2. *With the measurement satisfying* $\mathbf{n}\times(\mathbf{E}(\sigma) - \mathbf{E}_0) \neq 0$ *on* Γ*, one cannot determine* $\sigma\mathbf{E}(\sigma)$ *uniquely.*

Proof. Since $\mathbf{n}\times(\mathbf{E}(\sigma) - \mathbf{E}) \neq 0$ on Γ, we can conclude that $\sigma\mathbf{E}(\sigma) \notin W^\perp$ by Theorem 3.1. Note that $\sigma\mathbf{E}(\sigma) \in H_0(\mathbf{curl};\Omega_c)$. If $\sigma\mathbf{E}(\sigma) \in W$, we can conclude that $\sigma\mathbf{E}(\sigma)$ is an homogeneous eigenfunction corresponding to imaginary eigenvalue $i\omega\mu\sigma_0$ in Ω_c. It is impossible because the operator $\nabla\times\nabla\times$ is a semipositive operator on space $H_0(\mathbf{curl};\Omega_c)$. So $\sigma\mathbf{E}(\sigma) \notin W$ and we conclude that

$$i\omega\sigma\mathbf{E}(\sigma) = \mathbf{J}_1 + \mathbf{J}_2,\ \text{and } \mathbf{J}_1 \in W,\ \mathbf{J}_2 \in W^\perp,\ \mathbf{J}_i \neq 0,\ i = 1, 2.$$

From $\mathbf{J}_2 \neq 0$, by Theorem 3.1, we finish the proof. ◻

Remark 3.1. We do not know whether the sources belonging to W can be uniquely determined or not. It does not matter because we know from Theorem 3.2 that there is always a nonradiation part of $\sigma\mathbf{E}(\sigma)$.

The secondary source term $\mathbf{J}_e = i\omega\mathbf{E}(\sigma)$ depends on σ nonlinearly. The following corollary tells us that when σ is small enough, $\mathbf{J}_e$ can be approximated by $i\omega\mathbf{E}_0$. Then $\mathbf{J}_e$ depends on σ approximately and linearly.

COROLLARY 3.1. *Assume that* σ_0 *is a constant and* σ *is small enough such that* $\sigma < \sigma_m < \sigma_0$ *for some* $\sigma_m > 0$*, and then*

$$\|\mathbf{E}(\sigma) - \mathbf{E}_0\|_{L^2(\Omega_c)^3} \leq \frac{1}{\sigma_0 - \sigma_m}\|\sigma\mathbf{E}_0\|_{L^2(\Omega_c)^3}.$$

Proof. Let $\hat{\mathbf{E}} = \mathbf{E}(\sigma) - \mathbf{E}_0$. Multiplying both sides of the first equation in (3.3) by any $\mathbf{F} \in H_\Gamma(\mathbf{curl};\Omega)$, and using integration by parts, we have

$$\int_\Omega \mu^{-1}\nabla\times\hat{\mathbf{E}}\cdot\nabla\times\overline{\mathbf{F}} - i\omega\sigma_0\hat{\mathbf{E}}\cdot\overline{\mathbf{F}}dx = i\omega\int_{\Omega_c}\sigma\mathbf{E}(\sigma)\cdot\overline{\mathbf{F}}dx.$$

Let $\mathbf{F} = \hat{\mathbf{E}}$, and then

$$\|\mu^{-1/2}\nabla \times \hat{\mathbf{E}}\|^2_{L^2(\Omega)} - i\omega\sigma_0\|\hat{\mathbf{E}}\|^2_{L^2(\Omega_c)} = i\omega \int_{\Omega_c} \sigma\mathbf{E}(\sigma) \cdot \overline{\hat{\mathbf{E}}}dx.$$

So

$$\begin{cases} \sigma_0\|\hat{\mathbf{E}}\|^2_{L^2(\Omega_c)} = -\mathrm{Re}\int_{\Omega_c} \sigma\mathbf{E}(\sigma) \cdot \overline{\hat{\mathbf{E}}}dx, \\ \|\mu^{-1/2}\nabla \times \hat{\mathbf{E}}\|^2_{L^2(\Omega)} = -\omega\mathrm{Im}\int_{\Omega_c} \sigma\mathbf{E}(\sigma) \cdot \overline{\hat{\mathbf{E}}}dx. \end{cases}$$

From the first equality above, we can finish the proof with the following inequality:

$$\sigma_0\|\hat{\mathbf{E}}\|_{L^2(\Omega_c)^3} \leq \|\sigma\mathbf{E}(\sigma)\|_{L^2(\Omega_c)^3} \leq \|\sigma\mathbf{E}_0\|_{L^2(\Omega_c)^3} + \sigma_m\|\hat{\mathbf{E}}\|_{L^2(\Omega_c)^3}. \qquad \square$$

Remark 3.2. We know that the secondary source $\mathbf{J}_e = i\omega\mu\sigma\mathbf{E}(\sigma) = i\omega\mu\sigma\mathbf{E}_0 + i\omega\mu\sigma\hat{\mathbf{E}}$. When σ is small enough, Corollary 3.1 confirms that the second term is of a high order of σ. If we drop the higher order term into the right-hand side of (3.3) and assume that we can uniquely determine the secondary source, then with $\mathbf{E}_0$ known we can uniquely determine σ, except that $\mathbf{E}_0$ vanishes. Unfortunately, for the same reason as in Theorem 3.2, we know that $\sigma\mathbf{E}_0$ does not lie in either W or $W^\perp$, so we cannot determine $\sigma\mathbf{E}_0$ completely with the measurement on Γ.

3.3. Regularized inverse problem. Since the eddy current inverse problem is ill-posed, we take the following regularization to transform the ill-posed problem to a problem that is at least mathematically stable with respect to the change of the noisy data for numerical solutions:

$$\min_{\sigma\in\mathbb{K}} \Phi_\alpha(\sigma) := \frac{1}{2}\|\mathbf{n} \times (\mathbf{E}(\sigma) - \mathbf{E}^{obs})\|^2_{L^2(\Gamma)} + \frac{\alpha}{2}\|\nabla\sigma\|^2_{L^2(\Omega_c)}, \tag{3.8}$$

where $\mathbf{E}(\sigma)$ satisfies the constrained equation (2.2) or (2.7), and α is the regularization parameter.

We first show the existence of the minimizers of the functional $\Phi_\alpha(\sigma)$.

THEOREM 3.3. *Under the same assumptions as in Theorem* 2.3, *there exists a minimizer σ_α to $\Phi_\alpha(\sigma)$ in $\mathbb{K}$.*

Proof. The proof is quite standard; see, e.g., [9, 13] for the inverse elliptic and Maxwell problems. But for readers' convenience, we give an outline of the proof, showing the main differences from the current eddy current problem. First, we assume that $\{\sigma_n\}$ is a minimizing sequence for $\Phi_\alpha(\sigma)$, i.e.,

$$\lim_{n\to\infty} \Phi_\alpha(\sigma_n) = \inf_{\sigma\in\mathbb{K}} \Phi_\alpha(\sigma).$$

By the convergence of $\{\Phi_\alpha(\sigma_n)\}$, we know that $\{\Phi_\alpha(\sigma_n)\}$ is bounded, and so is $\{\|\nabla\sigma_n\|_{L^2(\Omega_c)}\}$. Therefore $\{\sigma_n\}$ is bounded in $H^1_0(\Omega_c)$, and there is a subsequence, still denoted by $\{\sigma_n\}$, that converges weakly to σ_α. This weak convergence is actually strong, due to the compact imbedding of $H^1(\Omega_c)$ in $L^2(\Omega_c)$. For σ_n and σ_α, we denote by $(\mathbf{E}_n, \phi_n)$ and $(\mathbf{E}_\alpha, \phi_\alpha)$ the solutions to the following two systems:

$$\begin{cases} a_n(\mathbf{E}_n, \mathbf{F}) + b(\nabla\phi_n, \mathbf{F}) = i\omega\int_\Omega \mathbf{J}_s \cdot \overline{\mathbf{F}}dx & \forall\, \mathbf{F} \in H_\Gamma(\mathbf{curl};\Omega), \\ b(\mathbf{E}_n, \nabla\psi) = 0 & \forall\, \psi \in H^1_\Gamma(\Omega_0), \end{cases}$$

$$\begin{cases} a_\alpha(\mathbf{E}_\alpha, \mathbf{F}) + b(\nabla\phi_\alpha, \mathbf{F}) = i\omega\int_\Omega \mathbf{J}_s \cdot \overline{\mathbf{F}}dx & \forall\, \mathbf{F} \in H_\Gamma(\mathbf{curl};\Omega), \\ b(\mathbf{E}_\alpha, \nabla\psi) = 0 & \forall\, \psi \in H^1_\Gamma(\Omega_0), \end{cases}$$

where $a_n(\mathbf{E},\mathbf{F})$ and $a_\alpha(\mathbf{E},\mathbf{F})$ are defined by (2.5) but with σ replaced by σ_n and σ_α, respectively. By Lemmas 2.3 and 3.1, the well-posedness of (2.7), and regularity results in Theorem 2.3, we have

$$\mathbf{E}_n|_{\Omega_0} \to \mathbf{E}_\alpha|_{\Omega_0} \text{ in } H^{1/2}(\Omega_0)\,; \quad \phi_n \to \phi_\alpha \text{ in } H^1_\Gamma(\Omega_0).$$

Then by the trace theorem, $\mathbf{n}\times\mathbf{E}_n \to \mathbf{n}\times\mathbf{E}_\alpha$ in $L^2(\Gamma)$. Using the strong convergence of $\mathbf{n}\times\mathbf{E}_n$ and the weakly lower semicontinuity of $\Phi_\alpha(\sigma)$, we get

$$\Phi_\alpha(\sigma_\alpha) \le \liminf_{n\to\infty} \Phi_\alpha(\sigma_n) = \inf_{\sigma\in\mathbb{K}} \Phi_\alpha(\sigma)\,.$$ ∎

We end this section with the stability of the regularized optimization system (3.8), whose proof can be done by standard arguments, along with some special techniques in the proof of Theorem 3.3; see, e.g., [9, 13] for the inverse elliptic and Maxwell problems.

THEOREM 3.4. *Let $\{\mathbf{E}_n\}$ be a sequence such that $\|\mathbf{n}\times\mathbf{E}_n-\mathbf{n}\times\mathbf{E}^{obs}\|_{L^2(\Gamma)}\to 0$ as $n\to\infty$ and σ_n be the minimizer of Φ_α defined by (3.8) but with the quantity $\mathbf{n}\times\mathbf{E}^{obs}$ replaced by $\mathbf{n}\times\mathbf{E}_n$, and then $\{\sigma_n\}$ has a subsequence which converges strongly to a minimizer of Φ_α in $L^2(\Omega_c)$.*

4. Nonlinear conjugate gradient method.

In this section, we first introduce the Lagrangian of the optimization problem (3.8), then derive the gradient of the objective functional and the Gâteaux derivative of the electric field with respect to parameter σ. Finally, we formulate a nonlinear conjugate gradient method with an approximate scheme for step length.

4.1. Lagrangian for the continuous optimization problem.

In order to calculate the gradient of the objective functional Φ_α with respect to σ, we use the standard adjoint state technique. We first recast the problem (3.8) into an unconstrained optimization by introducing some multipliers to relax the PDE constraint. Since the system of equations (2.2) and its weak formulation (2.7) are complex-valued, we relax the constraint in the real and imaginary parts separately to reformulate them into a real-valued unconstrained optimization. Let $\mathbf{E} = \mathrm{E}_1 + i\mathrm{E}_2$, $\phi = \phi_1 + i\phi_2$, $i\omega\mu\mathbf{J}_s = \mathbf{f}_1 + i\mathbf{f}_2$ and define $a_i : H_\Gamma(\mathbf{curl};\Omega)\times H_\Gamma(\mathbf{curl};\Omega)\to\mathbb{R}$ for $i=1,2$ as

$$a_1(\mathbf{E},\mathrm{F}) = \int_\Omega \mu^{-1}\nabla\times\mathrm{E}_1\cdot\nabla\times\mathrm{F}dx + \int_{\Omega_c}\omega(\sigma_0+\sigma)\mathrm{E}_2\cdot\mathrm{F}dx, \tag{4.1}$$

$$a_2(\mathbf{E},\mathrm{F}) = \int_\Omega \mu^{-1}\nabla\times\mathrm{E}_2\cdot\nabla\times\mathrm{F}dx - \int_{\Omega_c}\omega(\sigma_0+\sigma)\mathrm{E}_1\cdot\mathrm{F}dx. \tag{4.2}$$

By taking the test functions in real function spaces $H_\Gamma(\mathbf{curl};\Omega)$ and $H^1_\Gamma(\Omega_0)$, the complex-valued system (2.7) becomes the following real-valued system for $i=1,2$:

$$\begin{cases} a_i(\mathbf{E},\mathrm{F}_i) + b(\nabla\phi_i,\mathrm{F}_i) = \int_\Omega \mathbf{f}_i\cdot\mathrm{F}_i dx & \forall\,\mathrm{F}_i\in H_\Gamma(\mathbf{curl};\Omega),\\ b(\mathrm{E}_i,\psi_i) = 0 & \forall\,\psi_i\in H^1_\Gamma(\Omega_0)\,. \end{cases} \tag{4.3}$$

Accordingly, we rewrite $\Phi_\alpha(\sigma)$ in (3.8) as $\Phi_\alpha(\mathbf{E},\sigma)$, that is,

$$\Phi_\alpha(\mathbf{E},\sigma) = \frac{1}{2}\big(\|\mathbf{n}\times(\mathrm{E}_1-\mathrm{E}_1^{obs})\|^2_{L^2(\Gamma)} + \|\mathbf{n}\times(\mathrm{E}_2-\mathrm{E}_2^{obs})\|^2_{L^2(\Gamma)}\big) + \frac{\alpha}{2}\|\nabla\sigma\|^2_{L^2(\Omega_c)},$$

where E_1^{obs} and E_2^{obs} are the real and imaginary parts of $\mathbf{E}$, respectively. Now we use Σ to denote the product space $H_\Gamma(\mathbf{curl},\Omega)\times H_\Gamma(\mathbf{curl},\Omega)\times H^1_\Gamma(\Omega_0)\times H^1_\Gamma(\Omega_0)$, and then we can define a Lagrangian functional L from $\Sigma \times \Sigma \times \mathbb{K}$ to $\mathbb{R}$:

$$L((\mathrm{E}_1, \mathrm{E}_2, \phi_1, \phi_2), (\mathrm{F}_1, \mathrm{F}_2, \psi_1, \psi_2), \sigma)$$
$$= \Phi_\alpha(\mathbf{E}, \sigma) + \sum_{i=1}^{2} \left(a_i(\mathbf{E}, \mathrm{F}_i) + b(\nabla \phi_i, \mathrm{F}_i) - \int_\Omega \mathbf{f}_1 \cdot \mathrm{F}_i dx + b(\mathrm{E}_i, \psi_i) \right), \tag{4.4}$$

where real functions $\mathrm{F}_1, \mathrm{F}_2, \psi_1, \psi_2$ are Lagrange multipliers. Using the adjoint state technique, we can deduce that

$$\frac{\partial L}{\partial \sigma}(\tilde{\sigma}) = \alpha \int_{\Omega_c} \nabla \sigma \cdot \nabla \tilde{\sigma} dx - \omega \int_{\Omega_c} (\mathrm{E}_1 \cdot \mathrm{F}_2 - \mathrm{E}_2 \cdot \mathrm{F}_1) \tilde{\sigma} dx \quad \forall \tilde{\sigma} \in H_0^1(\Omega_c), \tag{4.5}$$

where $\mathrm{E}_1, \mathrm{E}_2$ and $\mathrm{F}_1, \mathrm{F}_2$ are the solutions of the following systems:

$$\frac{\partial L}{\partial(\mathrm{F}_1, \mathrm{F}_2, \psi_1, \psi_2)}((\tilde{\mathrm{F}}_1, \tilde{\mathrm{F}}_2, \tilde{\psi}_1, \tilde{\psi}_2)) = 0 \quad \forall (\tilde{\mathrm{F}}_1, \tilde{\mathrm{F}}_2, \tilde{\psi}_1, \tilde{\psi}_2) \in \Sigma, \tag{4.6}$$

$$\frac{\partial L}{\partial(\mathrm{E}_1, \mathrm{E}_2, \phi_1, \phi_2)}((\tilde{\mathrm{E}}_1, \tilde{\mathrm{E}}_2, \tilde{\phi}_1, \tilde{\phi}_2)) = 0 \quad \forall (\tilde{\mathrm{E}}_1, \tilde{\mathrm{E}}_2, \tilde{\phi}_1, \tilde{\phi}_2) \in \Sigma. \tag{4.7}$$

We can easily check that (4.6) is exactly the state system (4.3), while (4.7) yields its adjoint-state system

$$\begin{cases} a_j(\hat{\mathbf{F}}, \tilde{\mathrm{E}}_i) + b(\nabla \psi_i, \tilde{\mathrm{E}}_i) = \int_\Gamma \Delta \mathrm{E}_i \cdot \tilde{\mathrm{E}}_i ds & \forall \tilde{\mathrm{E}}_i \in H_\Gamma(\mathbf{curl}, \Omega), \quad j = 2, 1 \text{ for } i = 1, 2, \\ b(\mathrm{F}_i, \tilde{\phi}_i) = 0 & \forall \tilde{\phi}_i \in H_\Gamma^1(\Omega_0), \quad i = 1, 2, \end{cases} \tag{4.8}$$

where $\Delta \mathrm{E}_i = \mathbf{n} \times (\mathrm{E}_i^{obs} - \mathrm{E}_i) \times \mathbf{n}$ for $i = 1, 2$, and $\hat{\mathbf{F}} = \mathrm{F}_2 + i\mathrm{F}_1$.

Let $(\mathbf{E}, \phi_1, \phi_2)$ be the solution of system (4.3) and $\phi = \phi_1 + i\phi_2$, and then $(\mathbf{E}, \phi)$ solves the system

$$\begin{cases} a(\mathbf{E}, \tilde{\mathbf{F}}) + b(\nabla \phi, \tilde{\mathbf{F}}) = \int_\Omega (\mathbf{f}_1 + i\mathbf{f}_2) \cdot \overline{\tilde{\mathbf{F}}} dx & \forall \tilde{\mathbf{F}} \in H_\Gamma(\mathbf{curl}, \Omega), \\ b(\mathbf{E}, \nabla \tilde{\psi}) = 0 & \forall \tilde{\psi} \in H_\Gamma^1(\Omega_0). \end{cases} \tag{4.9}$$

It is easy to find that (4.9) is equivalent to (2.7) because $\mathbf{f}_1 + i\mathbf{f}_2 = i\omega\mu \mathbf{J}_s$.

On the other hand, let $(\hat{\mathbf{F}}, \psi_1, \psi_2)$ be the solution of system (4.8), and let $\mathbf{F} = -i\hat{\mathbf{F}} = \mathbf{F}_1 - i\mathbf{F}_2$, $\psi = \psi_1 - i\psi_2$. Then we can check that $(\mathbf{F}, \psi)$ solves the system

$$\begin{cases} a(\mathbf{F}, \tilde{\mathbf{E}}) + b(\nabla \psi, \tilde{\mathbf{E}}) = \int_\Gamma \mathbf{n} \times \overline{(\mathbf{E}^{obs} - \mathbf{E})} \times \mathbf{n} \cdot \overline{\tilde{\mathbf{E}}} ds & \forall \tilde{\mathbf{E}} \in H_\Gamma(\mathbf{curl}, \Omega), \\ b(\mathbf{F}, \nabla \tilde{\phi}) = 0 & \forall \tilde{\phi} \in H_\Gamma^1(\Omega_0). \end{cases} \tag{4.10}$$

Now we show that the relation (4.5) gives the gradient of $\Phi_\alpha(\sigma)$ in the weak sense. To see this, we define $g(\sigma) \in L^2(\Omega_c)$ with

$$\int_{\Omega_c} g(\sigma) \tilde{\sigma} dx = \alpha \int_{\Omega_c} \nabla \sigma \cdot \nabla \tilde{\sigma} dx + \omega \int_{\Omega_c} \mathrm{Im}(\mathbf{E} \cdot \mathbf{F}) \tilde{\sigma} dx \quad \forall \tilde{\sigma} \in H_0^1(\Omega_c), \tag{4.11}$$

and then we have in the weak sense that

$$\frac{\partial \Phi_\alpha(\sigma)}{\partial \sigma} = g(\sigma). \tag{4.12}$$

We can solve (4.9), (4.10), and (4.11) to calculate the gradient of the objective functional Φ_α with respect to σ.

4.2. Adjoint-state equations. The adjoint system of equations (4.10) looks similar to the state system (4.9) formally, but they are quite different in terms of their corresponding differential equations, which below we derive explicitly and explain their main differences. To do so, for the solution $(\mathbf{F}, \psi)$ to (4.10), we can derive by integration by parts that

$$\int_\Omega (\nabla \times (\mu^{-1} \nabla \times \mathbf{F}) - i\omega(\sigma_0 + \sigma)\mathbf{F}) \cdot \tilde{\mathbf{E}} dx + \int_{\Omega_0} \varepsilon \nabla \psi \cdot \tilde{\mathbf{E}} dx - \int_\Gamma \mu^{-1} \mathbf{n} \times \nabla \times \mathbf{F} \cdot \tilde{\mathbf{E}} ds$$
$$= \int_\Gamma (\mathbf{n} \times \overline{(\mathbf{E}^{obs} - \mathbf{E})} \times \mathbf{n}) \cdot \tilde{\mathbf{E}} ds.$$

On the other hand, for any function $\phi \in H^1_\Gamma(\Omega_0)$, we extend it to Ω by zero, and then choosing $\tilde{\mathbf{E}} = \nabla \phi$ in (4.10), we obtain

$$\int_{\Omega_0} \varepsilon \nabla \psi \cdot \nabla \phi dx = \int_\Gamma \mathbf{n} \times \overline{(\mathbf{E}^{obs} - \mathbf{E})} \times \mathbf{n} \cdot \nabla \phi ds.$$

This gives the corresponding differential equation for ψ:

$$\left\{ \begin{aligned} \nabla \cdot (\varepsilon \nabla \psi) &= 0 && \text{in } \Omega_0, \\ \varepsilon \frac{\partial \psi}{\partial \mathbf{n}} &= \mathrm{Div}_\tau(\mathbf{n} \times \overline{(\mathbf{E}^{obs} - \mathbf{E})} \times \mathbf{n}) && \text{on } \Gamma, \\ \psi &= 0 && \text{on } \partial\Omega_0 \setminus \Gamma. \end{aligned} \right. \tag{4.13}$$

If we choose $\tilde{\mathbf{E}} \in H_\Gamma(\mathbf{curl}, \Omega)$ and $\mathbf{n} \times \tilde{\mathbf{E}} = 0$ on Γ, then $\mathbf{F}$ needs to satisfy

$$\nabla \times (\mu^{-1} \nabla \times \mathbf{F}) - i\omega(\sigma_0 + \sigma)\mathbf{F} + \varepsilon \psi = 0 \text{ in } \Omega.$$

If we choose $\mathbf{n} \times \tilde{\mathbf{E}} \neq 0$ on Γ, we can derive the following boundary condition that $\mathbf{F}$ needs to satisfy:

$$\mu^{-1} \mathbf{n} \times \nabla \times \mathbf{F} = -\mathbf{n} \times \overline{(\mathbf{E}^{obs} - \mathbf{E})} \times \mathbf{n} \text{ on } \Gamma.$$

Together with the second equation in (4.10), we have derived the system of differential equations for the solution $\mathbf{F}$ to (4.10):

$$\left\{ \begin{aligned} \nabla \times (\mu^{-1} \nabla \times \mathbf{F}) - i\omega(\sigma_0 + \sigma)\mathbf{F} &= -\varepsilon \nabla \psi && \text{in } \Omega, \\ \nabla \cdot \varepsilon \mathbf{F} &= 0 && \text{in } \Omega_0, \\ \mu^{-1} \mathbf{n} \times \nabla \times \mathbf{F} &= -\mathbf{n} \times \overline{(\mathbf{E}^{obs} - \mathbf{E})} \times \mathbf{n} && \text{on } \Gamma, \\ \mathbf{n} \cdot \mathbf{F} &= 0 && \text{on } \Gamma, \\ \mathbf{n} \times \mathbf{F} &= 0 && \text{on } \Gamma_D. \end{aligned} \right. \tag{4.14}$$

This, along with (4.13), provides the differential equations of the solution $(\mathbf{F}, \psi)$ to (4.10).

Now we study a special case when $\mathrm{Div}_\tau(\mathbf{n} \times (\mathbf{E}^{obs} - \mathbf{E}) \times \mathbf{n}) = 0$ on Γ. Then we know $\psi = 0$ from (4.13), and (4.14) reduces to

$$\left\{ \begin{aligned} \nabla \times (\mu^{-1} \nabla \times \mathbf{F}) - i\omega(\sigma_0 + \sigma)\mathbf{F} &= 0 && \text{in } \Omega, \\ \nabla \cdot \varepsilon \mathbf{F} &= 0 && \text{in } \Omega_0, \\ \mu^{-1} \mathbf{n} \times \nabla \times \mathbf{F} &= -\mathbf{n} \times \overline{(\mathbf{E}^{obs} - \mathbf{E})} \times \mathbf{n} && \text{on } \Gamma, \\ \mathbf{n} \cdot \mathbf{F} &= 0 && \text{on } \Gamma, \\ \mathbf{n} \times \mathbf{F} &= 0 && \text{on } \Gamma_D. \end{aligned} \right. \tag{4.15}$$

By the definition of the surface divergence [6, 18], we have

$$\mathrm{Div}_\tau(\mathbf{n}\times(\mathbf{E}^{obs}-\mathbf{E})\times\mathbf{n}) = -\mathbf{n}\cdot\nabla\times((\mathbf{E}^{obs}-\mathbf{E})\times\mathbf{n})|_\Gamma. \tag{4.16}$$

Using the relation that $\nabla\times(\mathbf{u}\times\mathbf{v}) = \mathbf{u}(\nabla\cdot\mathbf{v}) - (\mathbf{u}\cdot\nabla)\mathbf{v} + (\mathbf{v}\cdot\nabla)\mathbf{u} - \mathbf{v}(\nabla\cdot\mathbf{u})$ for all $\mathbf{u},\mathbf{v}$ and the normal $\mathbf{n}=(0,0,1)$ of Γ in our case, we get

$$\nabla\times((\mathbf{E}^{obs}-\mathbf{E})\times\mathbf{n}) = (\mathbf{n}\cdot\nabla)(\mathbf{E}^{obs}-\mathbf{E}) - \mathbf{n}(\nabla\cdot(\mathbf{E}^{obs}-\mathbf{E})).$$

Assuming $\mathbf{E}^{obs}$ is the solution to system (2.2) with the true conductivity, we then have $\nabla\cdot(\mathbf{E}^{obs}-\mathbf{E})=0$ in Ω_0. This leads to

$$\nabla\times((\mathbf{E}^{obs}-\mathbf{E})\times\mathbf{n}) = (\mathbf{n}\cdot\nabla)(\mathbf{E}^{obs}-\mathbf{E}).$$

Therefore we deduce from (4.16) that

$$\begin{aligned}\mathrm{Div}_\tau(\mathbf{n}\times(\mathbf{E}^{obs}-\mathbf{E})\times\mathbf{n}) &= -\mathbf{n}\cdot(\mathbf{n}\cdot\nabla)(\mathbf{E}^{obs}-\mathbf{E})|_\Gamma\\ &= -(\mathbf{n}\cdot\nabla)(\mathbf{E}^{obs}-\mathbf{E})\cdot\mathbf{n}|_\Gamma\\ &= -\frac{\partial((\mathbf{E}^{obs}-\mathbf{E})\cdot\mathbf{n})}{\partial\mathbf{n}}.\end{aligned}$$

We know that the above relation is valid for any constant vector $\mathbf{n}$. For the current case with $\mathbf{n}=(0,0,1)$ and ε being a constant on Γ, the above derivation can be simplified. Let $\mathbf{E}_x,\mathbf{E}_y,\mathbf{E}_z$ be the components of $\mathbf{E}$ along the x,y,z-axis, respectively, and then

$$\mathrm{Div}_\tau(\mathbf{n}\times(\mathbf{E}^{obs}-\mathbf{E})\times\mathbf{n}) = \frac{\partial(\mathbf{E}^{obs}-\mathbf{E})_x}{\partial x} + \frac{\partial(\mathbf{E}^{obs}-\mathbf{E})_y}{\partial y} = -\frac{\partial(\mathbf{E}^{obs}-\mathbf{E})_z}{\partial z}.$$

So the condition that $\mathrm{Div}_\tau(\mathbf{n}\times(\mathbf{E}^{obs}-\mathbf{E})\times\mathbf{n})=0$ is equivalent to

$$\frac{\partial(\mathbf{E}^{obs}-\mathbf{E})_z}{\partial z} = 0 \text{ on } \Gamma.$$

In general, the condition that $\mathrm{Div}_\tau(\mathbf{n}\times(\mathbf{E}^{obs}-\mathbf{E})\times\mathbf{n})=0$ is not true, so we do not have $\psi=0$. Comparing with (2.2), we can see that the adjoint equation has a special source $\varepsilon\nabla\psi$, where ψ solves (4.13). Provided that $\mathrm{Div}_\tau(\mathbf{n}\times\overline{(\mathbf{E}^{obs}-\mathbf{E})}\times\mathbf{n})\in H^{-1/2}(\Gamma)$, (4.13) is well-posed, and hence the adjoint system (4.10) is well-posed, due to the well-posedness of (2.7).

Remark 4.1. Generally speaking, if $\mathbf{E}^{obs}-\mathbf{E}\in H_\Gamma(\mathbf{curl};\Omega)$, we have that $\mathbf{n}\times(\mathbf{E}^{obs}-\mathbf{E})\times\mathbf{n}\in H^{-1/2}(\mathrm{Curl};\Gamma)$, the dual space of $H^{-1/2}(\mathrm{Div};\Gamma)$ [6]. Then $\mathrm{Div}_\tau(\mathbf{n}\times(\mathbf{E}^{obs}-\mathbf{E})\times\mathbf{n})$ may not belong to $H^{-1/2}(\Gamma)$. But with the discussion of the regularity in section 2 and the fact that Γ is part of the boundary of a convex domain, we have that $\mathbf{n}\times(\mathbf{E}^{obs}-\mathbf{E})\times\mathbf{n}\in L^2(\Gamma)$ and $\mathrm{Div}_\tau(\mathbf{n}\times(\mathbf{E}^{obs}-\mathbf{E})\times\mathbf{n})\in H^{-1/2}(\Gamma)$.

4.3. Gâteaux derivative of the electric field E. In this subsection we derive the Gâteaux derivative of the electric field with respect to the conductivity σ. The derivative is needed to compute at each iteration of the nonlinear conjugate gradient algorithm (cf. subsection 4.5).

For any $\sigma\in\mathbb{K}$, we write $\sigma=\sigma_a+\sigma_b$ with $\sigma_a,\sigma_b\in\mathbb{K}$ and decompose the corresponding solution $\mathbf{E}(\sigma)$ to the system (2.2) as $\mathbf{E}=\mathbf{E}_0+\mathbf{E}_1+\mathbf{E}_2$, where $\mathbf{E}_0:=$

$\mathbf{E}_0(\sigma_0+\sigma_a)$, $\mathbf{E}_1 := \mathbf{E}_1(\sigma_0+\sigma_a;\sigma_b)$, and $\mathbf{E}_2 := \mathbf{E}_2(\sigma_0+\sigma;\sigma_b)$ solve the following systems, respectively, along with boundary conditions (2.4):

$$\begin{cases} \nabla\times(\mu^{-1}\nabla\times\mathbf{E}_0) - i\omega(\sigma_0+\sigma_a)\mathbf{E}_0 = i\omega\mathbf{J}_s & \text{in } \Omega, \\ \nabla\cdot\varepsilon\mathbf{E}_0 = 0 & \text{in } \Omega_0, \end{cases}$$

$$\begin{cases} \nabla\times(\mu^{-1}\nabla\times\mathbf{E}_1) - i\omega(\sigma_0+\sigma_a)\mathbf{E}_1 = i\omega\mu\sigma_b\mathbf{E}_0 & \text{in } \Omega, \\ \nabla\cdot\varepsilon\mathbf{E}_1 = 0 & \text{in } \Omega_0, \end{cases} \tag{4.17}$$

$$\begin{cases} \nabla\times(\mu^{-1}\nabla\times\mathbf{E}_2) - i\omega(\sigma_0+\sigma)\mathbf{E}_2 = i\omega\mu\sigma_b\mathbf{E}_1 & \text{in } \Omega, \\ \nabla\cdot\varepsilon\mathbf{E}_2 = 0 & \text{in } \Omega_0\,. \end{cases}$$

With the help of Lemma 3.1, we know that $\|\mathbf{E}_1(\sigma_0+\sigma_a;\sigma_b)\|_{H(\mathbf{curl};\Omega)} \le C\|\sigma_b\|_{L^q(\Omega_c)}$ for small σ_b with $q=\frac{3}{\delta}$, and a simple integration by parts gives the following estimate of $\mathbf{E}_2$:

$$\|\mathbf{E}_2\|_{H(\mathbf{curl};\Omega)} \le C\|\sigma_b\|_{L^q(\Omega_c)}\|\mathbf{E}_1\|_{L^p(\Omega)} \le C\|\sigma_b\|^2_{L^q(\Omega_c)}.$$

This leads to

$$\lim_{\|\sigma\|_{L^q(\Omega_c)}\to 0} \frac{\|\mathbf{E}(\sigma_0+\sigma)-\mathbf{E}_0(\sigma_0+\sigma_a)-\mathbf{E}_1(\sigma_0+\sigma_a;\sigma_b)\|_{H(\mathbf{curl};\Omega)}}{\|\sigma\|_{L^q(\Omega_c)}} = 0\,,$$

and hence we know $\mathbf{E}_1(\sigma_0+\sigma_a;\sigma_b)$ gives the Gâteaux derivative of $\mathbf{E}$ along the direction σ_b at $\sigma_0+\sigma_a$. Since $\mathbf{E}_1$ depends on σ_b linearly, we have for given σ_b and small γ that

$$\mathbf{E}(\sigma_0+\sigma_a+\gamma\sigma_b) = \mathbf{E}_0(\sigma_0+\sigma_a) + \gamma\mathbf{E}_1(\sigma_0+\sigma_a;\sigma_b) + o(|\gamma|\|\sigma_b\|_{L^q(\Omega_c)}). \tag{4.18}$$

We note that the first two terms in the right-hand side above are the linear approximation of the electric field $\mathbf{E}(\sigma_0+\sigma_a+\gamma\sigma_b)$. With this approximation, let

$$\Psi(\gamma) = \frac{1}{2}\|\mathbf{n}\times(\mathbf{E}_0+\gamma\mathbf{E}_1-\mathbf{E}^{obs})\|^2_{L^2(\Gamma)} + \frac{\alpha}{2}\|\nabla(\sigma_a+\gamma\sigma_b)\|^2_{L^2(\Omega_c)}, \tag{4.19}$$

and we have

$$\Phi_\alpha(\mathbf{E},\sigma_0+\sigma_a+\gamma\sigma_b) = \Psi(\gamma) + o(|\gamma|\|\sigma_b\|_{L^q(\Omega_c)}).$$

It is easy to find that $\Psi(\gamma)$ is a quadratic function with respect to γ, which we use to help us compute the descent step size in our iterative Algorithm 4.1.

4.4. Finite element discretization of the minimization problem. In this section we discuss the edge element approximation of the optimization system (3.8). For this purpose, we partition the domain Ω into a set of tetrahedral elements $\mathcal{M}_h$, with each element $K\in\mathcal{M}_h$ lying completely in Ω_c or Ω_0. Let $\mathcal{M}_h^0$ and $\mathcal{M}_h^c$ be the unions of elements contained in Ω_0 and Ω_c, respectively. Then we define the Nédélec edge element space

$$\mathbf{X}_h = \left\{\mathbf{u}_h \in H_\Gamma(\mathbf{curl};\Omega) \;\middle|\; \mathbf{u}_h|_K = a_K + b_K\times\mathbf{x}, a_K, b_K\in\mathbb{R}^3\right\},$$

and $U_h\subset H^1_\Gamma(\Omega_0)$ and $V_h\subset H^1_0(\Omega_c)$ are the standard continuous piecewise linear finite element spaces over $\mathcal{M}_h^0$ and $\mathcal{M}_h^c$, respectively. For ease of presentation, we use the notation $\Sigma_h = \mathbf{X}_h\times\mathbf{X}_h\times U_h\times U_h$ and $\mathbb{K}_h = V_h\cap\mathbb{K}$ in what follows. With these preparations, we propose the approximation of the optimization (3.8):

$$\min_{\sigma_h\in\mathbb{K}_h} \Phi_\alpha(\sigma_h) = \frac{1}{2}\|\mathbf{n}\times(\mathbf{E}_h(\sigma_h)-\mathbf{E}^{obs})\|^2_{L^2(\Gamma)} + \frac{\alpha}{2}\|\nabla\sigma_h\|^2_{L^2(\Omega_c)}, \tag{4.20}$$

where $\mathbf{E}_h(\sigma_h)$ solves

$$(4.21) \qquad \begin{cases} a_h(\mathbf{E}_h, \mathbf{F}_h) + b(\nabla\phi_h, \mathbf{F}_h) = i\omega \int_\Omega \mathbf{J}_s \cdot \overline{\mathbf{F}_h} dx & \forall\, \mathbf{F}_h \in \mathbf{X}_h, \\ b(\mathbf{E}_h, \nabla\psi_h) = 0 & \forall\, \psi_h \in U_h, \end{cases}$$

where a_h is given by the sesquilinear operator $a_h(\mathbf{E}, \mathbf{F}) = \int_\Omega \mu^{-1}\nabla \times \mathbf{E}_h \cdot \nabla \times \overline{\mathbf{F}_h} - i\omega(\sigma_0 + \sigma_h)\mathbf{E}_h \cdot \overline{\mathbf{F}_h} dx$ for all $\mathbf{E}$, $\mathbf{F} \in \mathbf{X}_h$. By writing the space

$$\mathbf{Y}_h = \big\{\mathbf{u}_h \in \mathbf{X}_h \;\big|\; b(\mathbf{u}_h, \nabla\phi_h) = 0 \;\; \forall\, \phi_h \in U_h\big\},$$

we know (4.21) is a saddle-point problem that is equivalent to the problem $\mathbf{E}_h \in \mathbf{Y}_h$ satisfying

$$(4.22) \qquad a_h(\mathbf{E}_h, \mathbf{F}_h) = i\omega \int_\Omega \mathbf{J}_s \cdot \overline{\mathbf{F}_h} dx \;\; \forall\, \mathbf{F}_h \in \mathbf{Y}_h.$$

We can easily see that $\mathbf{Y}_h$ is not a subspace of $\mathbf{Y}$, so we cannot deduce the well-posedness of (4.22) from that of the continuous weak problem (2.6). Instead the well-posednss of (4.22) can be achieved from that of (2.7) by using the fact that Σ_h is a subspace of Σ and following the arguments in [7, 8] for the magneto-static problem and field/circuit coupling problem.

Similarly to the proof of Theorem 3.3, we have the following existence.

THEOREM 4.1. *There exists at least one minimizer to the discrete optimization problem* (4.20).

Now we introduce a discrete Lagrangian on $\Sigma_h \times \Sigma_h \times \mathbb{K}_h$ associated with (4.20). To do so, we first define $a_{i,h}(\cdot,\cdot)$ for $i = 1, 2$ to be the same bilinear form as $a_i(\cdot,\cdot)$ defined in (4.1)–(4.2), but with σ replaced by σ_h. Then we define the discrete Lagrangian as

$$\begin{aligned} &L((\mathbf{E}_h, \phi_h), (\mathbf{F}_h, \psi_h), \sigma_h) \\ &\quad = \Phi_\alpha(\mathbf{E}_h, \sigma_h) + \sum_{i=1}^{2} \left(a_{i,h}(\mathbf{E}_h^i, \mathrm{F}_h^i) + b(\nabla\phi_h^i, \mathrm{F}_h^i) - \int_\Omega \mathbf{f}_1 \cdot \mathrm{F}_h^i dx + b(\mathrm{E}_h^i, \psi_h^i) \right). \end{aligned}$$

Let $g_h(\sigma_h) := \frac{\partial\Phi_\alpha(\sigma_h)}{\sigma_h}$, and then we can derive a similar relation to the continuous one (4.11),

$$(4.23) \quad \int_{\Omega_c} g_h\tilde{\sigma}_h dx = \alpha \int_{\Omega_c} \nabla\sigma_h \cdot \nabla\tilde{\sigma}_h dx - \omega \int_{\Omega_c} (\mathrm{E}_h^1 \cdot \mathrm{F}_h^2 - \mathrm{E}_h^2 \cdot \mathrm{F}_h^1)\tilde{\sigma}_h dx \;\; \forall\, \tilde{\sigma}_h \in V_h,$$

where $\mathbf{E}_h = \mathrm{E}_h^1 + i\mathrm{E}_h^2$ and $\hat{\mathbf{F}}_h = \mathrm{F}_h^2 + i\mathrm{F}_h^1$ solve the following state and adjoint systems, respectively,

$$(4.24) \qquad \begin{cases} a_{i,h}(\mathbf{E}_h, \tilde{\mathrm{F}}_h^i) + b(\nabla\phi_h^i, \tilde{\mathrm{F}}_h^i) = \int_\Omega \mathbf{f}_1 \cdot \tilde{\mathrm{F}}_h^i dx & \forall\, \tilde{\mathrm{F}}_h^i \in \mathbf{X}_h, i = 1, 2, \\ b(\mathrm{E}_h^i, \tilde{\psi}_h^i) = 0 & \forall\, \tilde{\psi}_h^i \in U_h, i = 1, 2, \end{cases}$$

(4.25)

$$\begin{cases} a_{j,h}(\hat{\mathbf{F}}, \tilde{\mathrm{E}}_h^i) + b(\nabla\psi_h^i, \tilde{\mathrm{E}}_h^i) = \int_\Gamma \mathbf{n} \times (\mathbf{E}_i^{obs} - \mathrm{E}_h^i) \times \mathbf{n} \cdot \tilde{\mathbf{E}}_h^1 ds & \\ & \forall\, \tilde{\mathrm{E}}_h^i \in \mathbf{X}_h, j = 2, 1 \text{ and } i = 1, 2, \\ b(\mathrm{F}_h^i, \tilde{\phi}_h^i) = 0 & \forall\, \tilde{\phi}_h^i \in U_h, i = 1, 2. \end{cases}$$

If we write $\phi_h = \phi_h^1 + i\phi_h^2$, $\mathbf{F}_h = \mathbf{F}_h^1 - i\mathbf{F}_h^2$, $\psi_h = \psi_h^1 - i\psi_h^2$, then (4.24)–(4.25) are just the discrete versions of (4.3)–(4.8), with their corresponding complex-valued systems given by

$$\text{(4.26)}\qquad \begin{cases} a_h(\mathbf{E}_h,\tilde{\mathbf{F}}_h)+b(\nabla\phi_h,\tilde{\mathbf{F}}_h)=\int_\Omega(\mathbf{f}_1+i\mathbf{f}_2)\cdot\overline{\tilde{\mathbf{F}}_h}dx & \forall\,\tilde{\mathbf{F}}_h\in X_h,\\ b(\mathbf{E}_h,\nabla\tilde{\psi}_h)=0 & \forall\,\tilde{\psi}_h\in U_h, \end{cases}$$

(4.27)

$$\begin{cases} a_h(\mathbf{F}_h,\tilde{\mathbf{E}}_h)+b(\nabla\psi_h,\tilde{\mathbf{E}}_h)=\int_\Gamma \mathbf{n}\times\overline{(\mathbf{E}^{obs}-\mathbf{E}_h)}\times\mathbf{n}\cdot\overline{\tilde{\mathbf{E}}_h}ds & \forall\,\tilde{\mathbf{E}}_h\in X_h,\\ b(\mathbf{F}_h,\nabla\tilde{\phi}_h)=0 & \forall\,\tilde{\phi}_h\in U_h. \end{cases}$$

In addition, we can see that (4.23) can be simplified as

$$\text{(4.28)}\qquad \int_{\Omega_c} g_h\tilde{\sigma}_h dx=\alpha\int_{\Omega_c}\nabla\sigma_h\cdot\nabla\tilde{\sigma}_h dx+\int_{\Omega_c}\omega\mathrm{Im}(\mathbf{E}_h\cdot\mathbf{F}_h)\tilde{\sigma}_h dx\quad\forall\,\tilde{\sigma}_h\in V_h.$$

4.5. A nonlinear conjugate gradient method. With the derivations in the previous subsections, we can now formulate the following nonlinear conjugate gradient algorithm for solving the discrete optimization problem (4.20).

ALGORITHM 4.1 (NLCG method). *Given the observation data* $\mathbf{n}\times\mathbf{E}^{obs}$ *on* Γ, *the background medium* σ_0, *the initial guess* σ_h^0, *set* $k=0$.

1. *Solve problem* (4.26) *with* $\sigma_h=\sigma_h^k$ *to get* $\mathbf{E}_h^k$ *and* ϕ_h^k.
2. *Solve problem* (4.27) *with* $\sigma_h=\sigma_h^k$ *to get* $\mathbf{F}_h^k$ *and* ψ_h^k.
3. *Solve problem* (4.28) *to get the gradient* g_h^k.
4. *Update the descent direction* $d_k=-g_h^k+\beta_k d_{k-1}$, *with the step size* β_k *computed by*
$$\beta_k=\begin{cases}\frac{\|g_h^k\|^2_{L^2(\Omega_c)}}{\|g_h^{k-1}\|^2_{L^2(\Omega_c)}} & \textit{for } k>0,\\ 0 & \textit{for } k=0.\end{cases}$$
5. *Solve problem* (4.17) *with* $\sigma_a=\sigma_h^k$ *and* $\sigma_b=d_k$ *for the solution* $\mathbf{E}_h^g$.
6. *Compute*
$$\gamma_k=-\frac{\int_\Gamma\Re((\mathbf{E}_h^k-\mathbf{E}^{obs})\times\mathbf{n}\cdot\overline{(\mathbf{E}_h^g\times\mathbf{n})})ds+\alpha(\nabla\sigma_h^k,\nabla d_k)_{\Omega_c}}{\|\mathbf{E}_h^g\times\mathbf{n}\|^2_{L^2(\Gamma)}+\alpha\|\nabla d_k\|^2_{L^2(\Omega_c)}}.$$
7. *Update* $\sigma_h^{k+1}=\sigma_h^k+\gamma_k d_k$; *set* $k:=k+1$ *and go to step* 1 *until convergence is achieved.*

We note that the step size γ_k in step 6 above is not calculated by the exact line search algorithm, but it is simply computed by using the quadratic approximation of the objective function Φ_α at $\sigma_0+\sigma_h^k$ along direction d_k, namely, the real quadratic function $\Psi(\gamma)$ in (4.19) with $\sigma_a=\sigma_h^k,\sigma_b=d_k$.

4.6. Sobolev gradient. We recall that we have defined and used the weak gradient of the objective functional (4.20) in (4.23) or (4.28) that approximates the continuous gradient in (4.11). It appears that the nonlinear conjugate Algorithm 4.1 converges very slowly for our nonlinear eddy current inverse problem, similarly to its behavior for most other nonlinear inverse problems. Next, we introduce a Sobolev gradient to help improve the convergence as was done in [17]. We can easily see that the weak gradient $g(\sigma)$ in (4.11) is just the weak gradient of the objective functional in the L^2 sense. Now we define a Sobolev gradient of the functional in the H^1 sense, namely, to find an element $g_S(\sigma)\in H_0^1(\Omega_c)$ satisfying

$$\text{(4.29)}\qquad \int_{\Omega_c}\nabla g_S(\sigma)\cdot\nabla\psi+g_S(\sigma)\psi dx=\int_{\Omega_c}g(\sigma)\psi dx\quad\forall\,\psi\in H_0^1(\Omega_c),$$

which is the weak formulation of the elliptic eproblem

$$\begin{cases} -\Delta g_S(\sigma) + g_S(\sigma) = g(\sigma) & \text{in } \Omega_c, \\ g_S(\sigma) = 0 & \text{on } \partial\Omega_c. \end{cases}$$

This suggests to us to compute the Sobolev gradient g_h^S in the third step of Algorithm 4.1 by solving the following equation:

$$\begin{aligned} &\int_{\Omega_c} \nabla g_h^S \cdot \nabla \tilde{\sigma}_h + g_h^S \tilde{\sigma}_h dx \\ &= \alpha \int_{\Omega_c} \nabla \sigma_h \cdot \nabla \tilde{\sigma}_h dx - \omega \int_{\Omega_c} (\mathbf{E}_h^1 \cdot \mathbf{F}_h^2 - \mathbf{E}_h^2 \cdot \mathbf{F}_h^1) \tilde{\sigma}_h dx \quad \forall \tilde{\sigma}_h \in V_h. \end{aligned} \tag{4.30}$$

This can be solved very efficiently by many existing preconditioning-type iterative methods.

5. Numerical experiments.

In this section, we present some numerical examples to illustrate the efficiency of Algorithm 4.1. We take the computational domain $\Omega = [-2, 2] \times [-2, 2] \times [-2, 0.2]$, with the nonconducting subregion $\Omega_0 = [-2, 2] \times [-2, 2] \times [0, 0.2]$ (where the conductivity σ_0 vanishes) and the conducting subregion $\Omega_c = [-2, 2] \times [-2, 2] \times [-2, 0]$. The state and adjoint state equations involved are solved with edge element methods. We implement the algorithm using the parallel hierarchical grid platform (PHG) [21]. The numerical examples are carried out using an Apple laptop with Intel i7 8750h CPU and 16G memory. The data $\mathbf{n} \times \mathbf{E}^{obs}$ is generated by the edge element method [7] and can be written as

$$\mathbf{n} \times \mathbf{E}^{obs}(x)|_\Gamma = \mathbf{n} \times \sum_{e \in \mathcal{E}_\Gamma} (R^e + iI^e)\Phi^e(x),$$

where $\mathcal{E}_\Gamma$ is the union of all edges of the mesh $\mathcal{M}_h$ on the measurement surface Γ (i.e., the plane $z = 0.2$), R^e and I^e are the degrees of freedom on the edge e for real and imaginary parts of the electric field with exact abnormal conductivity, respectively, and $\Phi^e(x)$ is the edge element basis function associated with edge e. To test the algorithm with noisy data, we generate the noisy data by adding the noise in the form

$$\mathbf{n} \times \mathbf{E}^{obs}(x)|_\Gamma = \mathbf{n} \times \sum_{e \in \mathcal{E}_\Gamma} (R^e + iI^e)(1 + \delta\xi)\Phi^e(x),$$

where δ is the noise level, and ξ is a uniformly distributed random variable in $[-1, 1]$.

In all examples we choose the source $\mathbf{J}_s$ as $(\nabla \cdot \mathbf{J}_s = 0)$

$$\mathbf{J}_s = \nabla \times \sum_{i,j=1}^{9} \delta(\mathbf{x} - \mathbf{x}_{ij})\mathbf{e}_1,$$

where $\mathbf{e}_1$ is the unit vector along the x-axis and $\mathbf{x}_{ij} = (-2.0 + 0.4 * i, -2.0 + 0.4 * j, 0.1)$, i.e., there are 81 point sources on plane $z = 0.1$. We assume the background conductivity $\sigma_0 = 1.0$ in $\Omega_c = \Omega_1 \cup \Omega_2$. By this setting, we apply Algorithm 4.1 to recover the abnormal conductivity σ with the data on boundary Γ. We always choose the initial guess 0 in the NLCG algorithm and take the parameters $\varepsilon = 1.0, \mu = 1.0, \omega = 0.79$, and the regularization $\alpha = 10^{-6}$ unless it is specified otherwise.

5.1. Example 1.

In this example, the domain with abnormal conductivity is $\Omega_2 = [-0.4, 0.4] \times [-0.4, 0.4] \times [-1.2, -0.4]$, where the exact abnormal conductivity

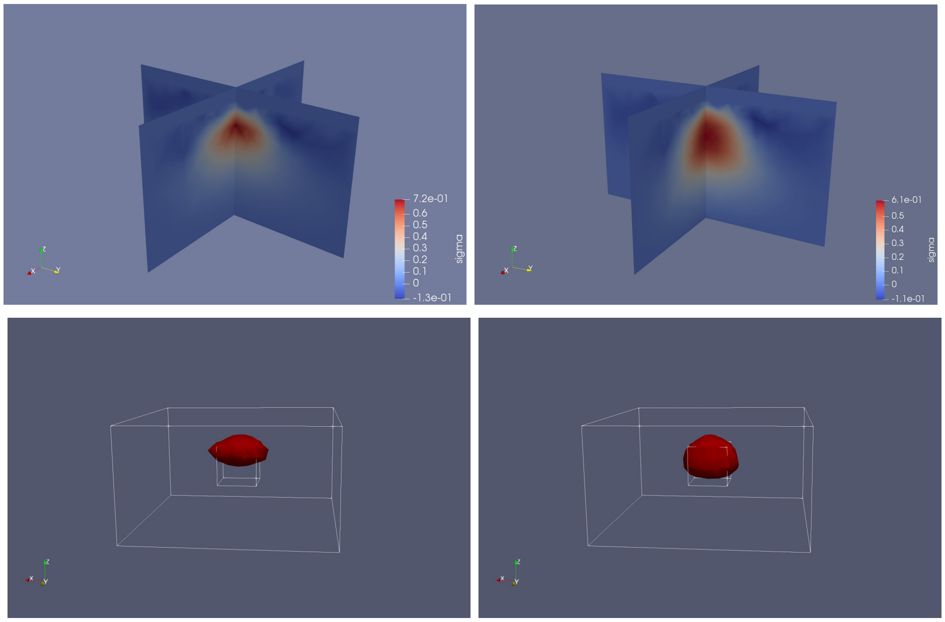

FIG. 2. *The recovery of* σ *after* 100 *iterations (upper left) and* 200 *iterations (upper right). The lower two pictures are the corresponding isosurfaces of the recovered* σ *with isovalue* 0.35.

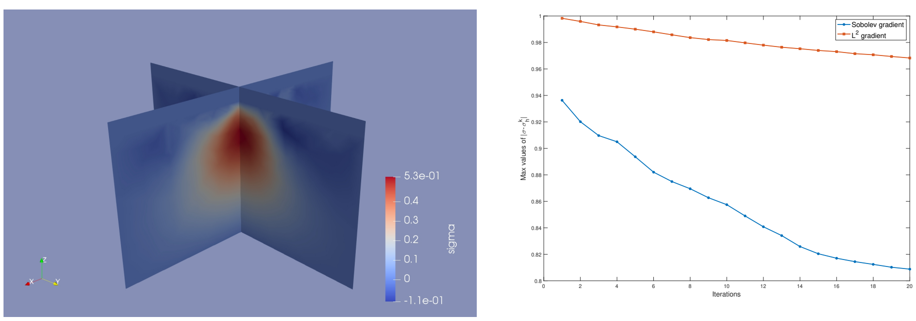

FIG. 3. *The recovery of* σ *after* 20 *iterations (left) and the convergence history (right).*

is given by $\sigma = 1.0$, and σ vanishes in Ω_1. That is, the exact conductivity $\sigma_0 + \sigma$ is constant 0.0, 1.0, and 2.0 in Ω_0, Ω_1, and Ω_2, respectively.

The total degrees of freedom of the edge elements are 213,128. First, we use the L^2 gradient of the objective functional in Algorithm 4.1, and the recovery results are shown in Figure 2, where the left and right pictures give the results in 100 and 200 iterations, respectively. We can find that the recovery is closer to the exact conductivity with more iterations. The recovery result by Algorithm 4.1 using the Sobolev gradient is given in the left of Figure 3 (20 iterations), with the convergence history of the nonlinear CG algorithm by the L^2 and Sobolev gradients, respectively, in the right of Figure 3. We notice that the algorithm with the Sobolev gradient converges much faster. The recovered conductivity by using the Sobolev gradient with 20 iterations is very close to the result with 200 iterations by using the L^2 gradient. We also show the recovery results of the algorithm with different α; see Figure 4. We find that the

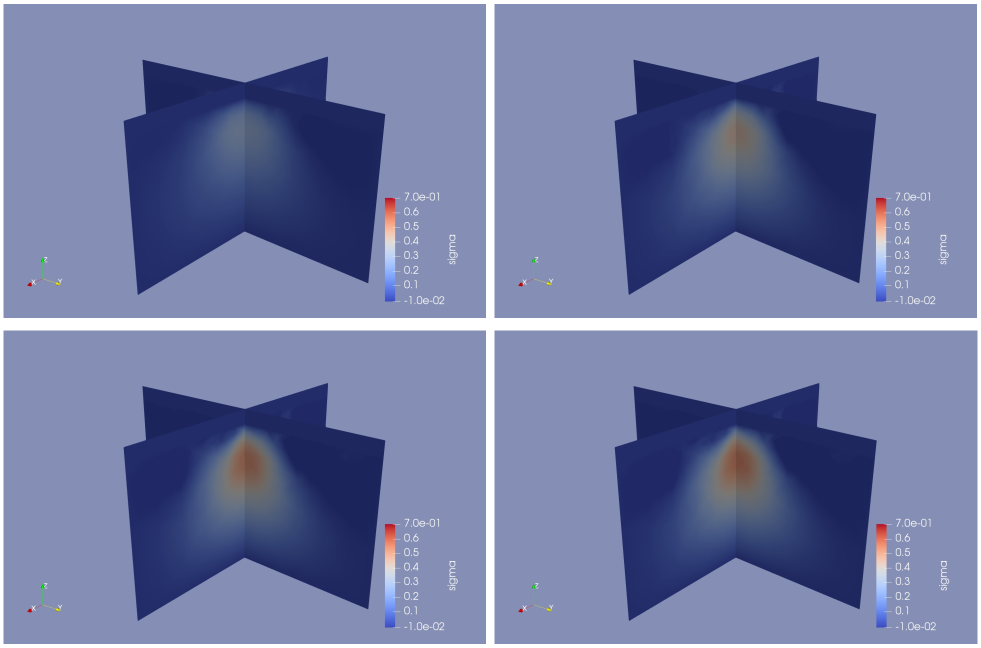

FIG. 4. *The recovery results after* 20 *iterations, with* $\alpha = 10^{-4}, 10^{-5}, 10^{-6}$, *and* 10^{-8} *from left to right, top to bottom. The same color represents the same value of* σ *in all the pictures.*

recovery results with $\alpha = 10^{-6}$ and 10^{-8} are quite similar and are both much more improved than the result with $\alpha = 10^{-5}$, while the result with $\alpha = 10^{-4}$ is very poor. We have observed similar reconstructions in most of our numerical simulations, so we shall report only the results with the regularization parameter $\alpha = 10^{-6}$ for all subsequent examples.

5.2. Example 2. In this example, we consider the case with two abnormal subdomains, Ω_{21} and Ω_{22}. We take $\Omega_{21} = [-1.2, -0.4] \times [-0.4, 0.4] \times [-1.2, -0.4]$, with the exact abnormal conductivity $\sigma = -0.9$, and $\Omega_{22} = [0.4, 1.2] \times [-0.4, 0.4] \times [-1.2, -0.4]$, with the exact abnormal conductivity $\sigma = 1.0$. To be more specific,

$$\sigma_0 + \sigma = \begin{cases} 0 & \text{in } \Omega_0, \\ 1.0 & \text{in } \Omega_1, \\ 0.1 & \text{in } \Omega_{21}, \\ 2.0 & \text{in } \Omega_{22}. \end{cases}$$

The total degrees of freedom of the edge elements of the state and adjoint equations are 266,690. In this example, we show the recovery results only for the Sobolev gradient defined by (4.30). The left and right pictures of Figure 5 present the recovery results in 100 and 200 iterations, respectively. We can see that two abnormal objects are well separated and recovered.

Figure 6 shows the recovered conductivity with the noisy data: the left picture with the noise level $\delta = 0.1\%$ and the right picture with $\delta = 0.4\%$, for both of which the regularization parameter is taken to be $\alpha = 10^{-4}$ and 100 iterations are conducted.

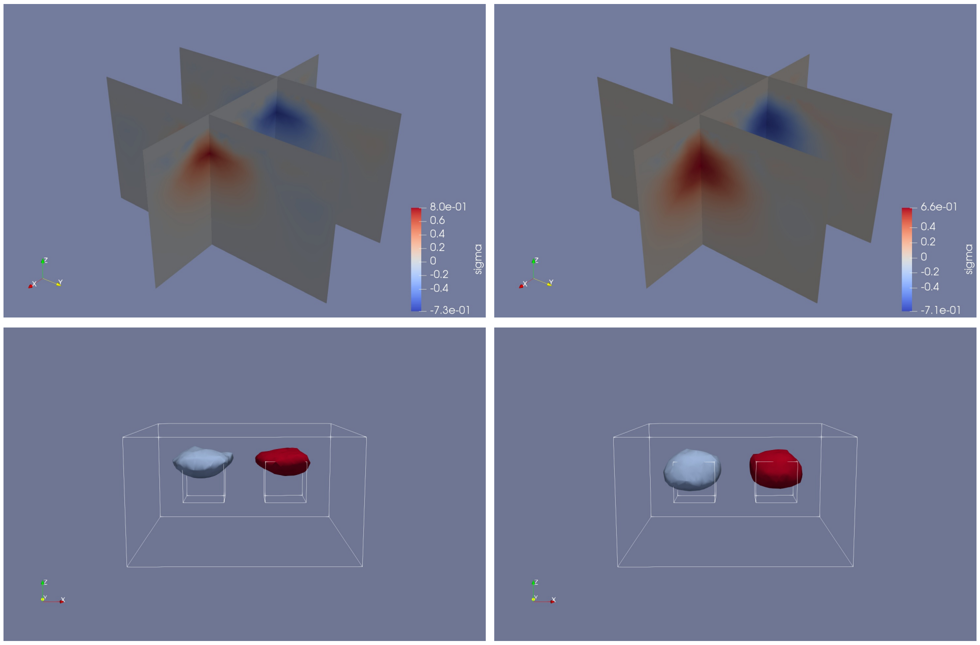

FIG. 5. *The recovered σ after* 100 *iterations (upper left) and* 200 *iterations (upper right). The lower two pictures are the isosurfaces of the recovered σ with isovalues* 0.35 *(lower left) and* -0.35 *(lower right). The small cubes are the real locations of the two anomalies.*

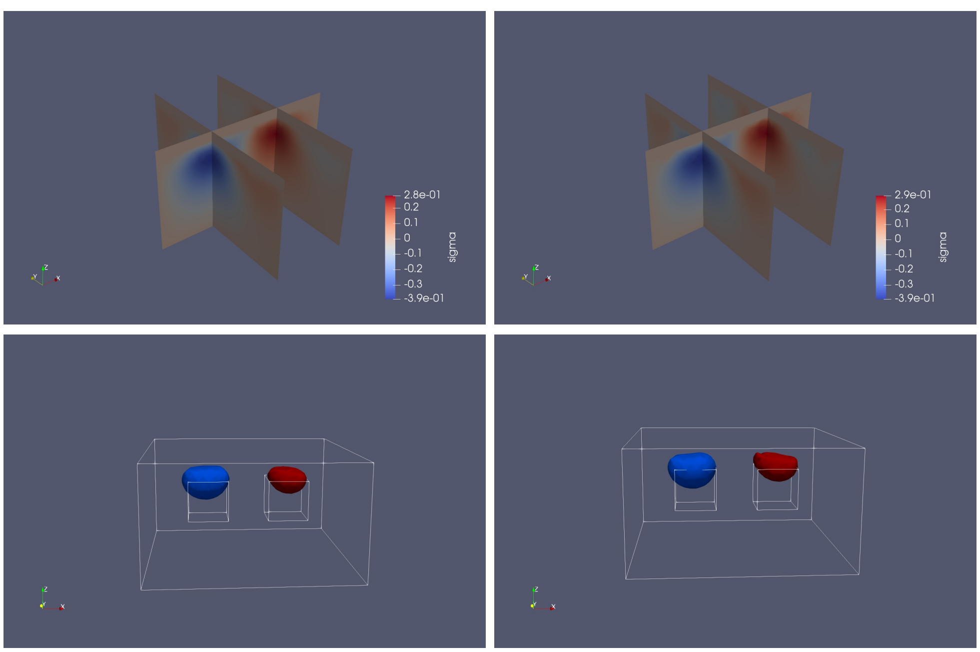

FIG. 6. *Recovery results after* 100 *iterations with noisy data; the noise level is* 0.1% *(upper left) and* 0.4% *(upper right). The lower two pictures are the isosurfaces of recovered σ with isovalues* 0.20 *(lower left) and* -0.25 *(lower right).*

6. Concluding remarks. We have studied an ill-posed eddy current inversion problem mathematically and numerically. We have first investigated the ill-posedness of the inverse eddy current problem, by showing the compactness of the forward operator mapping the conductivity parameter to the tangential trace of the electric field, and the nonuniqueness of the inverse problem. For the nonlinear regularized minimization formulation of the inverse problem, we have explored the existence and stability of the minimizers and the optimality system of its Lagrange formulation in terms of the real and imaginary parts of the PDE constraints, as well as the finite element approximation of the nonlinear regularized minimization system. A nonlinear conjugate gradient method is formulated for solving the discrete nonlinear constrained optimization problem, with its step sizes updated very effectively by a quadratic approximation of the objective function, and a Sobolev gradient introduced to effectively accelerate the iteration. Numerical examples have shown the feasibility and effectiveness of the reconstruction algorithm, which can clearly recover the locations and sizes of separated inclusions in the noisy case.

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