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Fully discrete finite element approaches for time-dependent Maxwell's equations

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Summary. A fully discrete finite element method is used to approximate the electric field equation derived from time-dependent Maxwell's equations in three dimensional polyhedral domains. Optimal energy-norm error estimates are achieved for general Lipschitz polyhedral domains. Optimal L^2-norm error estimates are obtained for convex polyhedral domains.

Résumé. On résout, dans un domaine polyédrique, les équations de Maxwell temporelles. Une méthode par éléments finis discrète en temps et en espace est proposée pour calculer le champ électrique. Une estimation d'ordre optimal est obtenue pour l'erreur en norme-énergie dans le cas général. Pour la norme L^2, on obtient une estimation optimale dans le cas d'un polyèdre convexe.

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1. Introduction

Many problems in sciences and industry involve the solutions of Maxwell's equations, for example, problems arising in plasma physics, microwave devices, diffraction of electromagnetic waves. In this paper, we are interested in the numerical solution of time-dependent Maxwell's equations in a bounded polyhedral domain in three dimensions. In the literature, one can find a great deal of work on numerical approximations to time-dependent Maxwell's

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equations and also analyses on the convergence of numerical schemes for stationary Maxwell's equations and related models. We refer readers to Raviart [21], Assous et al [5], Hewett-Nielson [16], Degond-Raviart [11], Ambrosiano-Brandon-Sonnendrücker [2] and Ciarlet-Zou [8], etc. But to our knowledge, it seems that there are few existing works on the convergence analysis for semi-discrete or fully discrete numerical methods for the time-dependent Maxwell systems. In [18], Monk obtained error estimates for a semi-discrete finite element approximation to the time-dependent Maxwell's equations using Nédélec's elements, from which our current paper was initiated. Furthermore, in [17] Makridakis-Monk proposed a fully discrete finite element scheme and obtained the error estimates under strong regularities on the solutions. This scheme involves solving coupled non-symmetric and indefinite linear algebraic systems of both electric and magnetic fields.

The purpose of the current paper is to analyse the convergence of a simple fully discrete finite element scheme for the electric field equation derived from Maxwell's equations by eliminating the magnetic field. The scheme is a fully discrete version of the semi-discrete scheme studied in [18], and it is constructed in a way that involves only solving a symmetric and positive definite linear algebraic system. One of our major interests here is to investigate the convergence order of the fully discrete scheme without making use of strong regularities on the solutions, which is certainly of practical importance. Under appropriate assumptions on the regularity of the continuous solutions, we derive for the concerned fully discrete scheme the optimal energy-norm error estimates for general polyhedral domains and optimal L^2-norm error estimates for convex polyhedral domains.

We now introduce the Maxwell's equations to be considered in the paper. Let Ω be a bounded Lipschitz continuous polyhedral domain in $\mathbb{R}^3$, $\boldsymbol{E}(\boldsymbol{x},t)$ and $\boldsymbol{H}(\boldsymbol{x},t)$ the electric and magnetic fields respectively. Then Maxwell's equations can be formulated as follows:

$$\varepsilon \boldsymbol{E}_t + \sigma \boldsymbol{E} - \operatorname{curl} \boldsymbol{H} = \boldsymbol{J} \quad \text{in } \Omega \times (0,T), \tag{1}$$

$$\mu \boldsymbol{H}_t + \operatorname{curl} \boldsymbol{E} = 0 \quad \text{in } \Omega \times (0,T), \tag{2}$$

where $\varepsilon(\boldsymbol{x})$ and $\sigma(\boldsymbol{x})$ are the dielectric constant and the conductivity of the medium respectively, while $\mu(\boldsymbol{x})$ and $\boldsymbol{J}(\boldsymbol{x},t)$ are the magnetic permeability of the material in Ω and the applied current density respectively. Here, the subscript t denotes the time derivative. It is assumed that these coefficients are piecewise smooth, real, bounded and positive, that is, there exist $\varepsilon_0 > 0$ and $\mu_0 > 0$ such that, for all $\boldsymbol{x} \in \overline{\Omega}$,

$$\varepsilon_0 \le \varepsilon(\boldsymbol{x}), \quad \mu_0 \le \mu(\boldsymbol{x}), \quad \text{and} \quad 0 \le \sigma(\boldsymbol{x}). \tag{3}$$

Moreover, these coefficients $\varepsilon(\boldsymbol{x})$, $\mu(\boldsymbol{x})$ and $\sigma(\boldsymbol{x})$ may be discontinuous. We assume that the boundary of the domain Ω is a perfect conductor, that is,

$$\boldsymbol{E} \times \boldsymbol{n} = 0 \quad \text{on} \quad \partial\Omega \times (0,T). \tag{4}$$

We supplement Maxwell's equations with the initial conditions

$$\boldsymbol{E}(\boldsymbol{x},0) = \boldsymbol{E}_0(\boldsymbol{x}) \quad \text{and} \quad \boldsymbol{H}(\boldsymbol{x},0) = \boldsymbol{H}_0(\boldsymbol{x}), \quad \boldsymbol{x} \in \Omega.$$

Instead of solving the coupled system (1)-(2) with both the electric and magnetic fields as unknowns, we eliminate the magnetic field $\boldsymbol{H}$, by taking the time derivative of (1) and using (2), to obtain the second order electric field equation:

$$\varepsilon \boldsymbol{E}_{tt} + \sigma \boldsymbol{E}_t + \mathrm{curl}\,(\frac{1}{\mu}\mathrm{curl}\,\boldsymbol{E}) = \boldsymbol{J}_t, \quad \text{in} \quad \Omega \times (0,T), \tag{5}$$

with the boundary condition still being (4) but the previous initial conditions being replaced by

$$\boldsymbol{E}(\boldsymbol{x},0) = \boldsymbol{E}_0(\boldsymbol{x}) \quad \text{and} \quad \boldsymbol{E}_t(\boldsymbol{x},0) = \boldsymbol{E}_1(\boldsymbol{x}), \tag{6}$$

where $\boldsymbol{E}_1(\boldsymbol{x}) = \varepsilon^{-1}(\boldsymbol{J}(\boldsymbol{x},0) + \mathrm{curl}\,\boldsymbol{H}_0(\boldsymbol{x}) - \sigma(\boldsymbol{x})\boldsymbol{E}_0(\boldsymbol{x}))$.

Remark 1.1 We have implicitly assumed that the electromagnetic field is generated by a current with density $\boldsymbol{J}$, without any charge density: i.e. the medium is locally electrically neutral, and $\mathrm{div}\,\boldsymbol{J} = 0$. In the more general case, the charge conservation equation reads:

$$\rho_t + \mathrm{div}\,\boldsymbol{J} = 0,$$

where ρ is the charge density.

Therefore, if $\sigma = 0$, we derive from (1) and (2) that

$$\mathrm{div}\,(\varepsilon\boldsymbol{E}) = -\rho, \quad \text{and} \quad \mathrm{div}\,(\mu\boldsymbol{H}) = 0,$$

when these relations hold for the initial data. In this case, one may consider a saddle-point approach, like in Raviart [21] or Ciarlet-Zou [8], where Darwin's model of approximation to Maxwell's equations was studied.

We end this section with the introduction of some notations used in the paper. We define

$$\begin{aligned}
H(\mathrm{div}\,;\Omega) &= \{\mathbf{v} \in (L^2(\Omega))^3;\ \mathrm{div}\,\mathbf{v} \in L^2(\Omega)\},\\
H(\mathrm{div}\,0;\Omega) &= \{\mathbf{v} \in H(\mathrm{div}\,;\Omega);\ \mathrm{div}\,\mathbf{v} = 0\},\\
H(\mathrm{curl}\,;\Omega) &= \{\mathbf{v} \in (L^2(\Omega))^3;\ \mathrm{curl}\,\mathbf{v} \in (L^2(\Omega))^3\},\\
H^\alpha(\mathrm{curl}\,;\Omega) &= \{\mathbf{v} \in (H^\alpha(\Omega))^3;\ \mathrm{curl}\,\mathbf{v} \in (H^\alpha(\Omega))^3\},\\
H_0(\mathrm{curl}\,;\Omega) &= \{\mathbf{v} \in H(\mathrm{curl}\,;\Omega);\ \mathbf{v} \times \boldsymbol{n} = 0 \text{ on } \Gamma\},
\end{aligned}$$

where α is a nonnegative real number. $H(\mathrm{div}\,;\Omega)$, $H(\mathrm{curl}\,;\Omega)$ and $H^\alpha(\mathrm{curl}\,;\Omega)$ are equipped with the norms

$$||\mathbf{v}||_{0,\mathrm{div}} = \Big(||\mathbf{v}||_0^2 + ||\mathrm{div}\,\mathbf{v}||_0^2\Big)^{1/2},$$
$$||\mathbf{v}||_{0,\mathrm{curl}} = \Big(||\mathbf{v}||_0^2 + ||\mathrm{curl}\,\mathbf{v}||_0^2\Big)^{1/2},$$
$$||\mathbf{v}||_{\alpha,\mathrm{curl}} = \Big(||\mathbf{v}||_\alpha^2 + ||\mathrm{curl}\,\mathbf{v}||_\alpha^2\Big)^{1/2}.$$

Here and in the sequel of the paper, $\|\cdot\|_0$ will always mean the $(L^2(\Omega))^3$-norm (or $L^2(\Omega)$-norm if only scalar functions are involved). And in general, we will use $\|\cdot\|_\alpha$ and $|\cdot|_\alpha$ to denote the norm and semi-norm in the Sobolev space $(H^\alpha(\Omega))^3$ (or $H^\alpha(\Omega)$ if only scalar functions are involved). We refer to Adams [1] and Grisvard [15] for more details on Sobolev spaces. C will always denote a generic constant which is independent of both the time step τ and the finite element mesh size h.

2. Fully discrete finite element schemes

We consider discretizing the electric field Cauchy problem (5)-(6) by the implicit backward difference scheme in time together with Nédélec's finite elements in space.

Let us first triangulate the space domain Ω and assume that $\mathcal{T}^h$ is a shape regular triangulation of Ω with a mesh size h made of tetrahedra. An element of $\mathcal{T}^h$ is denoted by K, and the diameters of K and its inscribed ball are denoted by h_K and ρ_K respectively. As usual, we let $h = \max_{K\in\mathcal{T}^h} h_K$. As the triangulation is shape regular, we have $h_K/\rho_K \le C$ (cf. Ciarlet [6]). We then introduce the following Nédélec's $H(\mathrm{curl}\,;\Omega)$-conforming finite element space

$$V_h = \{\mathbf{v}_h \in H(\mathrm{curl}\,;\Omega); \quad \mathbf{v}_h|_K \in (\mathcal{P}_1)^3, \quad \forall\, K \in \mathcal{T}^h\}$$

where $\mathcal{P}_1$ is the space of linear polynomials. It was proved in Nédélec [20] that any function $\mathbf{v}$ in V_h can be uniquely determined by the degrees of freedom in the moment set $M_{\boldsymbol{E}}(\mathbf{v})$ on each element $K \in \mathcal{T}^h$. Here $M_{\boldsymbol{E}}(\mathbf{v})$ is defined as follows:

$$M_{\boldsymbol{E}}(\mathbf{v}) = \{\int_e (\mathbf{v}\cdot\tau)\, q\, ds; \quad \forall\, q \in \mathcal{P}_1(e) \text{ on any edge } e \text{ of } K\},$$

where τ is the unit vector along the edge e.

From [3], Lemma 4.7, we know that the integrals required in the definition of $M_{\mathrm{E}}(\mathbf{v})$ make sense for any $\mathbf{v} \in X_p(K)$, with $p > 2$, where

$$X_p(K) = \{\mathbf{v} \in (L^p(K))^3;\ \mathrm{curl}\,\mathbf{v} \in (L^p(K))^3;\ \mathbf{v}\times\boldsymbol{n} \in (L^p(\partial K))^3\}.$$

Thus we can define, for any $\mathbf{v} \in H^{1/2+\delta}(\mathrm{curl}\,;\Omega)^3$ with $\delta > 0$ (which implies that $\mathrm{curl}\,\mathbf{v} \in (L^{p_\delta}(K))^3$ and $\mathbf{v} \in (L^{p_\delta}(\partial K))^3$ for some $p_\delta > 2$ which depends on δ), an interpolation $\Pi_h\mathbf{v}$ of $\mathbf{v}$ such that $\Pi_h\mathbf{v} \in V_h$ and $\Pi_h\mathbf{v}$ has the same degrees of freedom (defined by $M_E(\mathbf{v})$) as $\mathbf{v}$ on each $K \in \mathcal{T}^h$.

In order to take the boundary condition $\boldsymbol{E} \times \boldsymbol{n} = 0$ on $\partial\Omega$ into account, we define a subspace of V_h:

$$V_h^0 = \{\mathbf{v}_h \in V_h; \quad \mathbf{v}_h \times \boldsymbol{n} = 0 \ \text{ on } \partial\Omega\}.$$

This can be done simply by zeroing the degrees of freedom which correspond to the boundary edges.

Next we divide the time interval $(0, T)$ into M equally-spaced subintervals by using nodal points

$$0 = t^0 < t^1 < \cdots < t^M = T$$

with $t^n = n\tau$, and denote the n-th subinterval by $I^n = (t^{n-1}, t^n]$. For a given sequence $\{u^n\}_{n=0}^M \subset L^2(\Omega)$ or $(L^2(\Omega))^3$, we introduce the first and second order backward finite differences:

$$\partial_\tau u^n = \frac{u^n - u^{n-1}}{\tau}, \quad \partial_\tau^2 u^n = \frac{\partial_\tau u^n - \partial_\tau u^{n-1}}{\tau}.$$

For a continuous mapping $u : [0, T] \to L^2(\Omega)$ or $(L^2(\Omega))^3$, written as $u \in C(0, T; (L^2(\Omega))^3)$ subsequently, we define $u^n = u(\cdot, n\tau)$ for $0 \le n \le M$.

Using the above notation, our fully discrete finite element approximation to the electric field equations (5)-(6) is formulated as follows:

$$\boldsymbol{E}_h^0 = \Pi_h \boldsymbol{E}_0, \quad \boldsymbol{E}_h^0 - \boldsymbol{E}_h^{-1} = \tau\, \Pi_h \boldsymbol{E}_1, \tag{7}$$

and for $n = 1, 2, \cdots, M$, find $\boldsymbol{E}_h^n \in V_h^0$ such that

$$\begin{aligned} &(\varepsilon\partial_\tau^2 \boldsymbol{E}_h^n, \mathbf{v}) + (\sigma\partial_\tau \boldsymbol{E}_h^n, \mathbf{v}) + (\frac{1}{\mu}\mathrm{curl}\,\boldsymbol{E}_h^n, \mathrm{curl}\,\mathbf{v}) = (\partial_\tau \boldsymbol{J}^n, \mathbf{v}), \\ &\quad \forall \mathbf{v} \in V_h^0. \end{aligned} \tag{8}$$

Obviously, for each $n = 1, 2, \cdots, M$, it is clear that, by Lax-Milgram theorem, the system (8) has a unique solution $\boldsymbol{E}_h^n$ as its left-hand side defines a symmetric positive definite bilinear form in $H(\mathrm{curl}\,;\Omega)$ with respect to $\boldsymbol{E}_h^n$. In addition, as (8) is symmetric and positive definite, it can be solved by the well-known conjugate gradient method.

Remark 2.1 Instead of the first order backward difference in time used in the fully discrete scheme (7)-(8), one can also use some second order difference approximation in time, e.g. the Crank-Nicolson scheme. In this case, the whole discrete system can be taken as the following:

$$\boldsymbol{E}_h^0 = \Pi_h \boldsymbol{E}_0, \quad \boldsymbol{E}_h^1 - \boldsymbol{E}_h^{-1} = 2\tau\, \Pi_h \boldsymbol{E}_1, \tag{9}$$

and for $n = 0, 1, \cdots, M-1$, find $\boldsymbol{E}_h^{n+1} \in V_h^0$ such that

$$(\varepsilon\, \delta_\tau^2 \boldsymbol{E}_h^n, \mathbf{v}) + (\sigma\, \delta_{2\tau} \boldsymbol{E}_h^n, \mathbf{v}) + (\frac{1}{\mu} \mathrm{curl}\, \bar{\boldsymbol{E}}_h^n, \mathrm{curl}\, \mathbf{v}) = (\delta_{2\tau} \boldsymbol{J}^n, \mathbf{v}), \quad \forall \mathbf{v} \in V_h^0. \tag{10}$$

where $\delta_\tau^2 u^n = (u^{n+1} - 2u^n + u^{n-1})/\tau^2$, $\bar{u}^n = (u^{n+1} + u^{n-1})/2$, $\delta_{2\tau} u^n = (u^{n+1} - u^{n-1})/(2\tau)$. Note that the scheme preserves the symmetry and the positive definiteness. The first unknown $\boldsymbol{E}_h^1$ can be solved by using the initial approximation in (9) and (10) for $n = 0$, and the resultant linear system is also symmetric and positive definite. With this scheme we can achieve similar convergence results as obtained in the paper, see Remark 4.3.

3. Interpolation properties

This section is devoted to some basic approximation properties of the finite element interpolant Π_h defined in Sect. 2, which will be needed in the later error estimates for the finite element scheme (7)-(8). First of all, we know the following properties of Π_h: for any $\mathbf{u} \in (H^2(\Omega))^3$,

$$\|\mathbf{u} - \Pi_h \mathbf{u}\|_0 \le C\, h^2 |\mathbf{u}|_2, \tag{11}$$

while for any $\mathbf{u} \in H^1(\mathrm{curl}\,; \Omega)$, we have

$$\|\mathrm{curl}\,(\mathbf{u} - \Pi_h \mathbf{u})\|_0 \le C\, h\, \|\mathrm{curl}\, \mathbf{u}\|_1. \tag{12}$$

The estimate (11) can be found in Girault [13] (Theorem 3.1) and Nédélec [20] (Proposition 3). The estimate (12) was proved by Monk [18] (Lemma 2.3).

The estimates (11) and (12) stand for functions which are appropriately smooth, i.e. for functions in $(H^2(\Omega))^3$ or $H^1(\mathrm{curl}\,; \Omega)$. But usually the solutions of the Maxwell system considered in the paper may not have such kind of regularity, especially when the domain Ω is not convex and only Lipschitz continuous. Next we are going to present some approximation properties of the interpolant Π_h under weak assumptions on regularity. We first show a similar result to (12) but for the L^2-norm. Comparing with the estimate (12) for the curl operator, we lose one error order. Similar results were obtained in [12] for a different finite element (see Remark 3.3).

Lemma 3.1 *We have*

$$\|\mathbf{u} - \Pi_h \mathbf{u}\|_0 \le C\, h\, ||\mathbf{u}||_{1,\mathrm{curl}}\,, \quad \forall\, \mathbf{u} \in H^1(\mathrm{curl}\,; \Omega).$$

The proof of the lemma is omitted, since it can be inferred from that of Lemmas 3.2 and 3.3 (see [9] for a detailed proof).

Lemma 3.2 *We have, for* $1/2 < \alpha \le 1$*,*

$$\|\mathbf{u} - \Pi_h \mathbf{u}\|_0 \le C\, h^\alpha\, ||\mathbf{u}||_{\alpha,\mathrm{curl}}\,, \quad \forall\, \mathbf{u} \in H^\alpha(\mathrm{curl}\,; \Omega).$$

Remark 3.1 $\alpha > 1/2$ is needed for the definition of the moments in $M_{\boldsymbol{E}}(\mathbf{v}, \phi)$.

Proof. For any element $K \in \mathcal{T}^h$, let $x = B_K \hat{x} + b_K$ be the affine mapping between K and the reference element $\hat{K}$, and we define (cf. Nédélec [20]),

$$\mathbf{u}(x) = (B_K^*)^{-1} \hat{\mathbf{u}}(\hat{x}) \quad \text{or} \quad \hat{\mathbf{u}}(\hat{x}) = B_K^* \mathbf{u}(x), \tag{13}$$

where B_K^* is the transpose of the matrix B_K. Let $\hat{\Pi}$ be the interpolant on the reference element $\hat{K}$, then

$$\|\mathbf{u} - \Pi_h \mathbf{u}\|^2_{L^2(K)} \le |B_K|\, \|(B_K^*)^{-1}\|^2\, \|\hat{\mathbf{u}} - \hat{\Pi}\hat{\mathbf{u}}\|^2_{L^2(\hat{K})}. \tag{14}$$

Throughout the paper, $|A|$ means $|\det(A)|$ for any square matrix A.

Let us now bound $\|\hat{\mathbf{u}} - \hat{\Pi}\hat{\mathbf{u}}\|_{L^2(\hat{K})}$. For that, let $\hat{e}$ (respectively $\hat{F}$) be any edge (respectively face) of $\hat{K}$. For $p > 2$ and p' such that $1/p + 1/p' = 1$, on any edge $\hat{e}$ of $\hat{K}$ we define

$$||\hat{\mathbf{v}}||_{M_{\hat{e}}} = \sup_{\hat{\phi} \in P_1(\hat{e})^3} \frac{|M_{\hat{e}}(\hat{\mathbf{v}}, \hat{\phi})|}{||\hat{\phi}||_{W^{1-1/p',p'}(\hat{e})}} \tag{15}$$

where $M_{\hat{e}}(\hat{\mathbf{v}}, \hat{\phi}) = \int_{\hat{e}} (\hat{\mathbf{v}} \cdot \hat{\tau}) \hat{\phi} ds$. Using the norm equivalence in finite dimensional spaces, we have

$$\begin{aligned}\|\hat{\Pi}\hat{\mathbf{u}}\|_{L^2(\hat{K})} &\le C \sum_{\hat{e} \subset \hat{K}} \|\hat{\Pi}\hat{\mathbf{u}}\|_{M_{\hat{e}}} = C \sum_{\hat{e} \subset \hat{K}} \|\hat{\mathbf{u}}\|_{M_{\hat{e}}} \\ &\le C \Big\{ \|\widehat{\mathrm{curl}}\, \hat{\mathbf{u}}\|_{L^p(\hat{K})} + \sum_{\hat{F} \subset \hat{K}} \|\hat{\mathbf{u}} \times \hat{\boldsymbol{n}}\|_{L^p(\hat{F})} \Big\},\end{aligned}$$

where the last inequality is obtained by integration by parts and the standard extension and lifting techniques (cf. Lemma 4.7 of [3]). This implies

$$\begin{aligned}\|\hat{\mathbf{u}} - \hat{\Pi}\hat{\mathbf{u}}\|_{L^2(\hat{K})} &\le C \Big\{ \|\widehat{\mathrm{curl}}\, \hat{\mathbf{u}}\|_{L^p(\hat{K})} + \|\hat{\mathbf{u}}\|_{L^2(\hat{K})} + \sum_{\hat{F} \subset \hat{K}} \|\hat{\mathbf{u}} \times \hat{\boldsymbol{n}}\|_{L^p(\hat{F})} \Big\} \\ &\le C \left\{ \|\widehat{\mathrm{curl}}\, \hat{\mathbf{u}}\|_{H^\alpha(\hat{K})} + \|\hat{\mathbf{u}}\|_{H^\alpha(\hat{K})} \right\}. \end{aligned} \tag{16}$$

As the left hand side does not change when replacing $\hat{\mathbf{u}}$ by $\hat{\mathbf{u}}$ plus any constant, we have

$$\|\hat{\mathbf{u}} - \hat{\Pi}\hat{\mathbf{u}}\|_{L^2(\hat{K})} \le C\Big\{\|\widehat{\mathrm{curl}}\,\hat{\mathbf{u}}\|_{H^\alpha(\hat{K})} + \inf_{\hat{\mathbf{p}}\in\mathcal{P}_0(\hat{K})^3} \|\hat{\mathbf{u}} + \hat{\mathbf{p}}\|_{H^\alpha(\hat{K})}\Big\}.$$

Note

$$|\hat{\mathbf{w}}|_{H^\alpha(\hat{K})} = \left\{\int_{\hat{K}}\int_{\hat{K}} \frac{||\hat{\mathbf{w}}(\hat{x}) - \hat{\mathbf{w}}(\hat{y})||^2}{||\hat{x} - \hat{y}||^{3+2\alpha}} d\hat{x}d\hat{y}\right\}^{1/2},$$

it is clear that $|\hat{\mathbf{w}} + \hat{\mathbf{p}}|_{H^\alpha(\hat{K})} = |\hat{\mathbf{w}}|_{H^\alpha(\hat{K})}$ for all $\hat{\mathbf{p}} \in \mathcal{P}_0(\hat{K})^3$. From this point, one can easily adapt the proof of Theorem 14.1 in [6] to obtain the norm equivalence in the quotient space $H^\alpha/\mathcal{P}_0$. Then one has

$$\|\hat{\mathbf{u}} - \hat{\Pi}\hat{\mathbf{u}}\|_{L^2(\hat{K})} \le C\left\{\|\widehat{\mathrm{curl}}\,\hat{\mathbf{u}}\|_{H^\alpha(\hat{K})} + |\hat{\mathbf{u}}|_{H^\alpha(\hat{K})}\right\}. \tag{17}$$

There remains to bound the right-hand side in (17). Noting that $||x-y|| \le ||B_K||\,||B_K^{-1}(x-y)||$, we deduce

$$|\hat{\mathbf{u}}|^2_{H^\alpha(\hat{K})} \le ||B_K||^{5+2\alpha}|B_K^{-1}|^2|\mathbf{u}|^2_{H^\alpha(K)}.$$

Similarly we have (see [9] for details)

$$||\widehat{\mathrm{curl}}\,\hat{\mathbf{u}}||^2_{L^2(\hat{K})} \le C||B_K||^4\,|B_K^{-1}|\,||\mathrm{curl}\,\mathbf{u}||^2_{L^2(K)}, \text{ and}$$
$$|\widehat{\mathrm{curl}}\,\hat{\mathbf{u}}|^2_{H^\alpha(\hat{K})} \le C\,||B_K||^{7+2\alpha}|B_K^{-1}|^2|\mathrm{curl}\,\mathbf{u}|^2_{H^\alpha(K)}.$$

This with (17) shows (for $||B_K||$ small)

$$\|\hat{\mathbf{u}} - \hat{\Pi}\hat{\mathbf{u}}\|^2_{L^2(\hat{K})} \le C\,\max(||B_K||^{5+2\alpha}|B_K^{-1}|^2, ||B_K||^4\,|B_K^{-1}|)||\mathbf{u}||^2_{H^\alpha(\mathrm{curl}\,;K)}.$$

Using the bounds on B_K and the shape regularity of $\mathcal{T}^h$, we get from (14) that

$$\|\mathbf{u} - \Pi_h\mathbf{u}\|^2_{L^2(K)} \le C\,h_K^{2\alpha}||\mathbf{u}||^2_{H^\alpha(\mathrm{curl}\,;K)}.$$

□

Remark 3.2 The following lemma is an improvement on the results obtained in Nédélec [20] (Propositions 1 and 2) and Monk [18] (Lemma 2.3), where only integers $\alpha \ge 1$ were considered.

Lemma 3.3 *For* $1/2 < \alpha \le 1$, *we have*

$$\|\mathrm{curl}\,(\mathbf{u} - \Pi_h\mathbf{u})\|_0 \le Ch^\alpha|\mathrm{curl}\,\mathbf{u}|_{H^\alpha(\Omega)}, \quad \forall\,\mathbf{u} \in H^\alpha(\mathrm{curl}\,;\Omega).$$

Proof. We follow Nédélec [20] for the notation used below. Let Θ_h be the $H(\operatorname{div} 0;\Omega)$-conforming space of degree 0:

$$\Theta_h = \{\mathbf{v} \in H(\operatorname{div} 0;\Omega);\ \mathbf{v}|_K \in (\mathcal{P}_0)^3, \quad \forall K \in \mathcal{T}^h\},$$

and let r_h be the corresponding interpolant to Θ_h with the moment set:

$$M'_F(\mathbf{v}) = \{\int_F (\mathbf{v}\cdot\boldsymbol{n})\, q\, d\sigma; \quad \forall\, q \in \mathcal{P}_0(F) \ \text{ on any face } F \text{ of } K\}.$$

There are four degrees of freedom attached to this finite element, but as $\operatorname{div}\mathbf{v} = 0$ by definition, their sum (with $q = 1$) is equal to 0.

We can show

(18) $$r_h(\operatorname{curl}\mathbf{u}) = \operatorname{curl}(\Pi_h\mathbf{u}).$$

Indeed, $\operatorname{curl}(\Pi_h\mathbf{u}) \in (\mathcal{P}_0)^3$, $\operatorname{div}(\operatorname{curl}(\Pi_h\mathbf{u})) = 0$ and

$$\begin{aligned}\int_F \operatorname{curl}(\Pi_h\mathbf{u})\cdot\boldsymbol{n}\, d\sigma &= \int_F \operatorname{curl}_F(\Pi_h\mathbf{u})\, d\sigma = \int_{\partial F} \Pi_h\mathbf{u}\cdot\tau\, ds \\ &= \int_{\partial F} \mathbf{u}\cdot\tau\, ds = \int_F \operatorname{curl}_F(\mathbf{u})\, d\sigma \\ &= \int_F \operatorname{curl}\mathbf{u}\cdot\boldsymbol{n}\, d\sigma.\end{aligned}$$

Hence

(19) $$\|\operatorname{curl}(\mathbf{u} - \Pi_h\mathbf{u})\|_0 = \|(\operatorname{curl}\mathbf{u}) - r_h(\operatorname{curl}\mathbf{u})\|_0,$$

this with the following result (20) gives the lemma.

Next, we show that for any element K and $1/2 < \alpha \le 1$,

(20) $$\begin{aligned}&\|\mathbf{w} - r_h\mathbf{w}\|_{L^2(K)} \le Ch^\alpha |\mathbf{w}|_{H^\alpha(K)}, \\ &\qquad \forall\, \mathbf{w} \in H(\operatorname{div} 0;\Omega) \cap H^\alpha(\Omega)^3.\end{aligned}$$

To prove this, we replace the degrees of freedom in $M'_F(\mathbf{v})$ (cf. [19] or [20]) by

$$\{\frac{2}{|B_K|}\int_F (\mathbf{v}\cdot\boldsymbol{n}) q d\sigma; \quad \forall\, q|_K \in (\mathcal{P}_0)^3, \quad K \in \mathcal{T}^h\}.$$

Then the following transformation

$$x = B_K\hat{x} + b_K, \quad \mathbf{w}(x) = B_K\hat{\mathbf{w}}(\hat{x}),$$

preserves the interpolation and divergence, i.e.

$$\hat{r}\hat{\mathbf{w}}(\hat{x}) = r_h\mathbf{w}(x), \quad \widehat{\operatorname{div}}\,\hat{\mathbf{w}}(\hat{x}) = \operatorname{div}\mathbf{w}(x),$$

where $\hat{r}$ is the reference interpolant on $\hat{K}$.

Now consider the moment $M'_{\hat{F}}(\hat{\mathbf{w}}, \hat{\phi}) = \int_{\hat{F}}(\hat{\mathbf{w}} \cdot \hat{\boldsymbol{n}})\hat{\phi}\, d\hat{\sigma}$. Noting that $\operatorname{div} \mathbf{w} = 0$ implies $\widehat{\operatorname{div}}\, \hat{\mathbf{w}} = 0$, we have by integration by parts

$$
\begin{aligned}
|M'_{\hat{F}}(\hat{\mathbf{w}}, \hat{\phi})| &= \Big| \int_{\hat{K}} \widehat{\operatorname{div}}\, \hat{\mathbf{w}}\, \underline{\hat{\phi}}\, d\hat{x} + \int_{\hat{K}} \hat{\mathbf{w}} \cdot \widehat{\operatorname{grad}} \underline{\hat{\phi}}\, d\hat{x} \Big| \\
&= \Big| \int_{\hat{K}} \hat{\mathbf{w}} \cdot \widehat{\operatorname{grad}} \underline{\hat{\phi}}\, d\hat{x} \Big| \\
&\leq C\, ||\hat{\mathbf{w}}||_{L^p(\hat{K})}\, ||\hat{\phi}||_{W^{1-1/p',p'}(\hat{F})} \qquad (21)
\end{aligned}
$$

where p' is again the conjugate number of p and $\underline{\hat{\phi}}$ the extension by zero from $W^{1-1/p',p'}(\hat{F})$ into $W^{1-1/p',p'}(\partial \hat{K})$ combined with a lifting operator from $W^{1-1/p',p'}(\partial \hat{K})$ onto $W^{1,p'}(\hat{K})$.

Using (21), we can bound

$$
||\hat{\mathbf{w}}||_{M'_{\hat{F}}} = \sup_{\hat{\phi} \in (P_0(\hat{F}))^3} \frac{|M'_{\hat{F}}(\hat{\mathbf{w}}, \hat{\phi})|}{||\hat{\phi}||_{W^{1-1/p',p'}(\hat{F})}}
$$

by $\|\hat{\mathbf{w}}\|_{L^p(\hat{K})}$. This with the norm equivalence in finite dimensional spaces gives

$$
\begin{aligned}
\|\hat{r}\hat{\mathbf{w}}\|_{L^2(\hat{K})} &\leq C \sum_{\hat{F} \in \hat{K}} \|\hat{r}\hat{\mathbf{w}}\|_{M'_{\hat{F}}} = C \sum_{\hat{F} \in \hat{K}} \|\hat{\mathbf{w}}\|_{M'_{\hat{F}}} \\
&\leq C\, \|\hat{\mathbf{w}}\|_{L^p(\hat{K})^3} \leq C\, \|\hat{\mathbf{w}}\|_{H^\alpha(\hat{K})},
\end{aligned}
$$

that implies

$$
\|\hat{\mathbf{w}} - \hat{r}\hat{\mathbf{w}}\|_{L^2(\hat{K})} \leq C \|\hat{\mathbf{w}}\|_{H^\alpha(\hat{K})}. \qquad (22)
$$

As replacing $\hat{\mathbf{w}}$ by $\hat{\mathbf{w}}$ plus any constant does not change the left hand side, we come to

$$
\|\hat{\mathbf{w}} - \hat{r}\hat{\mathbf{w}}\|_{L^2(\hat{K})} \leq C |\hat{\mathbf{w}}|_{H^\alpha(\hat{K})}.
$$

Thus we finally derive

$$
\|\mathbf{w} - r_h \mathbf{w}\|^2_{L^2(K)} \leq \|B_K\|^2 |B_K|\, \|\hat{\mathbf{w}} - \hat{r}\hat{\mathbf{w}}\|^2_{L^2(\hat{K})} \leq C h_K^{2\alpha} |\mathbf{w}|^2_{H^\alpha(K)},
$$

this proves (20). □

Remark 3.3 All the results of this paper are also valid for certain other first order, $H(\operatorname{curl}; \Omega)$-conforming, Nédélec's elements, e.g. the element defined in [19], i.e. each element function has the form $\mathbf{v}_{h|K} \in \mathcal{R}^1(K) = \{\mathbf{a}_K + \mathbf{b}_K \times \boldsymbol{x},\ (\mathbf{a}_K, \mathbf{b}_K) \in \mathbb{R}^6\}$, with the related degrees of freedom. The crucial step for the validity is to establish Lemmas 3.1-3.3 for this element. In fact, Lemma 3.1 was established in ([12], Theorem 3.2). Lemmas 3.2-3.3 can be extended to include this first order element by means of similar techniques as used in the present paper. Note, in addition, that higher order finite elements are not considered here as we do not assume, for the solutions, a stronger regularity than $H^\alpha(\mathrm{curl}\,;\Omega)$ with $1/2 < \alpha \le 1$ in Subsect. 4.1 (energy-norm error estimates), or than $H^2(\Omega)$ in Subsect. 4.2 (L^2-norm error estimates).

4. Finite element error estimates

We are now going to derive the error estimates for the fully discrete finite element scheme (7)-(8) both in the energy-norm and the L^2-norm. Throughout this section, $\boldsymbol{E}^n$ and $\boldsymbol{E}_h^n$ will denote the solutions of the electric field equations (5)-(6) and the finite element approximation (8) at time $t = t^n$.

For the error analysis, we need the solution function $\boldsymbol{E}$ to be defined also in the interval $[-2\tau, T]$ in terms of the time variable t. This can be done by extending $\boldsymbol{E}$ with some regularity from the time interval $[0, T]$ to the interval $[-2\tau, T]$. So we shall always implicitly assume that $\boldsymbol{E}$ is well defined in terms of time variable t on the interval $[-2\tau, T]$. Furthermore, to achieve the optimal energy-norm error estimates for the concerned fully discrete finite element scheme, we introduce an important projection operator $P_h : H_0(\mathrm{curl}\,;\Omega) \to V_h^0$ defined by

$$a(P_h\mathbf{u}, \mathbf{v}) = a(\mathbf{u}, \mathbf{v}), \quad \forall\, \mathbf{v} \in V_h^0 \tag{23}$$

where $a(\mathbf{u}, \mathbf{v})$ is the scalar product associated with $\|\cdot\|_{0,\mathrm{curl}}$. Obviously, P_h is well-defined in $H_0(\mathrm{curl}\,;\Omega)$.

By the definition of the projection P_h in (23), we easily see that, for $\alpha > 1/2$,

$$\begin{aligned} &\|\mathbf{u} - P_h\mathbf{u}\|_{0,\mathrm{curl}} \le \|\mathbf{u} - \Pi_h\mathbf{u}\|_{0,\mathrm{curl}}\,, \\ &\qquad \forall\, \mathbf{u} \in H_0(\mathrm{curl}\,;\Omega) \cap H^\alpha(\mathrm{curl}\,;\Omega). \end{aligned} \tag{24}$$

Later on, we will need the following identity

$$\sum_{m=1}^{k} (a_m - a_{m-1}) b_m = a_k b_k - a_0 b_0 - \sum_{m=1}^{k} a_{m-1}(b_m - b_{m-1}) \tag{25}$$

and the following estimates for $B = H^1(\mathrm{curl}\,;\Omega)$ or $B = (H^\alpha(\Omega))^3$ with $\alpha \ge 0$,

$$\|\partial_\tau \mathbf{u}^n\|_B^2 \le \frac{1}{\tau}\int_{t^{n-1}}^{t^n} \|\mathbf{u}_t(t)\|_B^2\,dt, \quad \forall\, \mathbf{u} \in H^1(0,T;B), \tag{26}$$

$$\|\partial_\tau^2 \mathbf{u}^n\|_B^2 \le \frac{1}{\tau}\int_{t^{n-2}}^{t^n} \|\mathbf{u}_{tt}(t)\|_B^2\,dt, \quad \forall\, \mathbf{u} \in H^2(0,T;B), \tag{27}$$

$$\|\partial_\tau \mathbf{u}_t^n - \partial_\tau^2 \mathbf{u}^n\|_B^2 \le C\tau \int_{t^{n-2}}^{t^n} \|\mathbf{u}_{ttt}(t)\|_B^2\,dt, \quad \forall\, \mathbf{u} \in H^3(0,T;B). \tag{28}$$

4.1. Energy-norm error estimates

This subsection is devoted to the estimate on the energy-norm error for $\boldsymbol{E}^n - \boldsymbol{E}_h^n$. For the purpose, we first analyse the error $\eta_h^k = \boldsymbol{E}_h^k - P_h\boldsymbol{E}^k$, for $1 \le k \le n$. Once we have estimates for η_h^n, we can easily get the error estimates for $\boldsymbol{E}^n - \boldsymbol{E}_h^n$ by the triangle inequality, the projection properties (24) and the interpolation properties discussed in Sect. 3.

We conduct our analysis only for the constant coefficients case, i.e., we assume $\varepsilon(\boldsymbol{x})$, $\mu(\boldsymbol{x})$ and $\sigma(\boldsymbol{x})$ are all constants. It is straightforward to extend the analysis to the non-constant or elementwise constant case by simply keeping these coefficients inside the integrals or norms and bounding them by taking their maximum or minimum values if necessary.

To analyse η_h^k, we multiply the equation (5) by $\mathbf{v}/\tau \in V_h^0$ and integrate then the resultant over Ω in space and over I^k in time to obtain

$$\varepsilon\,(\partial_\tau \boldsymbol{E}_t^k, \mathbf{v}) + \sigma(\partial_\tau \boldsymbol{E}^k, \mathbf{v}) + \frac{1}{\tau\mu}\Big(\int_{I^k} \operatorname{curl}\boldsymbol{E}\,dt, \operatorname{curl}\mathbf{v}\Big) = (\partial_\tau \boldsymbol{J}^k, \mathbf{v}),$$

$$(29) \quad \forall\, \mathbf{v} \in V_h^0.$$

Now subtracting (29) from (8) and making some rearrangements, we have

$$\begin{aligned}
&\varepsilon\,(\partial_\tau^2\eta_h^k, \mathbf{v}) + \frac{1}{\mu}(\operatorname{curl}\eta_h^k, \operatorname{curl}\mathbf{v}) + \sigma(\partial_\tau\eta_h^k, \mathbf{v})\\
&\quad = \varepsilon\left(\partial_\tau(\boldsymbol{E}_t^k - \partial_\tau P_h\boldsymbol{E}^k), \mathbf{v}\right) + \sigma\Big(\partial_\tau(\boldsymbol{E}^k - P_h\boldsymbol{E}^k), \mathbf{v}\Big)\\
&\qquad + \frac{1}{\tau\mu}\Big(\int_{I^k}\operatorname{curl}(\boldsymbol{E} - P_h\boldsymbol{E}^k)\,dt, \operatorname{curl}\mathbf{v}\Big), \quad \forall\,\mathbf{v} \in V_h^0.
\end{aligned}$$

Then taking $\mathbf{v} = \tau\partial_\tau\eta_h^k = \eta_h^k - \eta_h^{k-1}$ above and using $a(a-b) \ge a^2/2 - b^2/2$, for any real numbers a and b, yield

$$\begin{aligned}
\sigma\tau\,\|\partial_\tau\eta_h^k\|_0^2 \;&+\; \Big(\frac{\varepsilon}{2}\|\partial_\tau\eta_h^k\|_0^2 - \frac{\varepsilon}{2}\|\partial_\tau\eta_h^{k-1}\|_0^2\Big)\\
&+\; \Big(\frac{1}{2\mu}\|\operatorname{curl}\eta_h^k\|_0^2 - \frac{1}{2\mu}\|\operatorname{curl}\eta_h^{k-1}\|_0^2\Big)\\
&\le\; \tau\sigma\Big((\partial_\tau\boldsymbol{E}^k - P_h\partial_\tau\boldsymbol{E}^k), \partial_\tau\eta_h^k\Big)\\
&\quad + \tau\varepsilon\Big(\partial_\tau(\boldsymbol{E}_t^k - \partial_\tau\boldsymbol{E}^k), \partial_\tau\eta_h^k\Big)\\
&\quad + \tau\varepsilon\Big(\partial_\tau^2\boldsymbol{E}^k - P_h\partial_\tau^2\boldsymbol{E}^k), \partial_\tau\eta_h^k\Big)\\
&\quad + \frac{1}{\mu}\Big(\int_{I^k}\operatorname{curl}(\boldsymbol{E} - \boldsymbol{E}^k)\,dt, \operatorname{curl}\partial_\tau\eta_h^k\Big)\\
&\quad + \frac{\tau}{\mu}\Big(\operatorname{curl}(\boldsymbol{E}^k - P_h\boldsymbol{E}^k), \operatorname{curl}\partial_\tau\eta_h^k\Big)
\end{aligned}$$

$$\equiv: \sum_{i=1}^{5} (\mathrm{I})_i. \tag{30}$$

Next, we will estimate $(\mathrm{I})_i$ for $i = 1, 2, 3, 4, 5$ one by one.
First for $(\mathrm{I})_1$, using Cauchy-Schwarz inequality, we have

$$\begin{aligned}
(\mathrm{I})_1 &\le \frac{1}{2}\sigma\tau\|\partial_\tau\eta_h^k\|_0^2 + \frac{1}{2}\,\sigma\tau\,\|\partial_\tau\boldsymbol{E}^k - P_h\partial_\tau\boldsymbol{E}^k\|_0^2 \\
&\le \frac{1}{2}\sigma\tau\|\partial_\tau\eta_h^k\|_0^2 + C\,\sigma\,\tau\,\left(\|\partial_\tau\boldsymbol{E}^k - \Pi_h\partial_\tau\boldsymbol{E}^k\|_{0,\mathrm{curl}}^2\right) \text{ (by (24))} \\
&\le \frac{1}{2}\sigma\tau\|\partial_\tau\eta_h^k\|_0^2 + C\,\sigma\,\tau\,h^2\,\|\partial_\tau\boldsymbol{E}^k\|_{1,\mathrm{curl}}^2 \text{ (by (12) and Lemma 3.1)} \\
&\le \frac{1}{2}\sigma\tau\|\partial_\tau\eta_h^k\|_0^2 + C\,\sigma\,h^2\int_{I^k}\|\boldsymbol{E}_t\|_{1,\mathrm{curl}}^2\,dt \text{ (by (26)).}
\end{aligned}$$

For the estimation of $(\mathrm{I})_2$, by writing $\partial_\tau(\cdot)^k$ into the integral of form $\int_{I^k}(\cdot)_t\,dt$ and using Cauchy-Schwarz inequality, we easily come to

$$(\mathrm{I})_2 \le \frac{1}{2}\tau\varepsilon\|\partial_\tau\eta_h^k\|_0^2 + C\,\varepsilon\tau^2\int_{t^{k-2}}^{t^k}\|\boldsymbol{E}_{ttt}\|_0^2\,dt.$$

The estimate for $(\mathrm{I})_3$ is achieved using the same technique as used for $(\mathrm{I})_1$,

$$(\mathrm{I})_3 \le \frac{1}{2}\varepsilon\tau\|\partial_\tau\eta_h^k\|_0^2 + C\,\varepsilon\,h^2\int_{I^k}\|\boldsymbol{E}_{tt}\|_{1,\mathrm{curl}}^2\,dt.$$

To analyse $(\mathrm{I})_4$, we use Green's formula and the boundary condition to derive

$$\begin{aligned}
(\mathrm{I})_4 &= \frac{1}{\mu}\int_{I^k}\left(\mathrm{curl}\,\mathrm{curl}\,(\boldsymbol{E} - \boldsymbol{E}^k), \partial_\tau\eta_h^k\right)dt \\
&= -\frac{1}{\mu}\int_{I^k}\int_t^{t^k}\left(\mathrm{curl}\,\mathrm{curl}\,\boldsymbol{E}_t, \partial_\tau\eta_h^k\right)dt'\,dt,
\end{aligned}$$

then by Cauchy-Schwarz inequality we obtain

$$(\mathrm{I})_4 \le \frac{1}{2}\tau\varepsilon\|\partial_\tau\eta_h^k\|_0^2 + \frac{\tau^2}{2\varepsilon\mu^2}\int_{I^k}\|\mathrm{curl}\,\mathrm{curl}\,\boldsymbol{E}_t\|_0^2\,dt.$$

Finally, we estimate $(\mathrm{I})_5$. By the definition of P_h in (23), we have

$$(\mathrm{I})_5 = \frac{\tau}{\mu}\left(\mathrm{curl}\,(\boldsymbol{E}^k - P_h\boldsymbol{E}^k), \mathrm{curl}\,\partial_\tau\eta_h^k\right) = -\frac{\tau}{\mu}\left(\boldsymbol{E}^k - P_h\boldsymbol{E}^k, \partial_\tau\eta_h^k\right),$$

then applying the Cauchy-Schwarz inequality and (12), (24) and Lemma 3.1, we come to

$$\begin{aligned}(\mathrm{I})_5 &\le \frac{\tau}{\mu}\|\boldsymbol{E}^k - P_h\boldsymbol{E}^k\|_0\,\|\partial_\tau\eta_h^k\|_0\\ &\le \frac{\tau}{\mu^2}\|\partial_\tau\eta_h^k\|_0^2 + C\,\tau\,h^2\,\|\boldsymbol{E}^k\|_{1,\mathrm{curl}}^2\,.\end{aligned}$$

This completes all the estimates for $(\mathrm{I})_i$ in (30). Now summing both sides in (30) over $k = 1, 2, \cdots, n$ and making use of the previous estimates for $(\mathrm{I})_i$ $(1 \le i \le 5)$, lead to

$$\begin{aligned}&\frac{\varepsilon}{2}\|\partial_\tau\eta_h^n\|_0^2 + \frac{1}{2\mu}\|\mathrm{curl}\,\eta_h^n\|_0^2 \le C\,m_0(\boldsymbol{E})(\tau^2+h^2) + d(\eta_h^0)\\ &\quad + C\,\tau\sum_{k=1}^{n}\Big(\|\partial_\tau\eta_h^k\|_0^2 + \|\mathrm{curl}\,\eta_h^k\|_0^2\Big),\end{aligned} \tag{31}$$

where C is a constant depending on the coefficients ε, σ and μ, and $m_0(\boldsymbol{E})$ is an a priori bound of $\boldsymbol{E}$ of the following form

$$\begin{aligned}m_0(\boldsymbol{E}) &= \max_{0\le t\le T}\|\boldsymbol{E}(t)\|_{1,\mathrm{curl}}^2 + \int_0^T(\|\boldsymbol{E}_{tt}\|_{1,\mathrm{curl}}^2 + \|\mathrm{curl}\,\mathrm{curl}\,\boldsymbol{E}_t\|_0^2)\,dt\\ &\quad + \int_{-\tau}^T\|\boldsymbol{E}_{ttt}\|_0^2\,dt,\end{aligned}$$

while $d(\eta_h^0)$ is the initial error

$$d(\eta_h^0) = \frac{\varepsilon}{2}\|\partial_\tau\eta_h^0\|_0^2 + \frac{1}{2\mu}\|\mathrm{curl}\,\eta_h^0\|_0^2,$$

which can be analysed as follows:

First by the definitions of $\boldsymbol{E}_h^0$ and the projection P_h, we have

$$\begin{aligned}\|\mathrm{curl}\,\eta_h^0\|_0 &= \|\mathrm{curl}\,(\Pi_h\boldsymbol{E}^0 - P_h\boldsymbol{E}^0)\|_0 = \|\mathrm{curl}\,P_h(\Pi_h\boldsymbol{E}^0 - \boldsymbol{E}^0)\|_0\\ &\le \|\Pi_h\boldsymbol{E}^0 - \boldsymbol{E}^0\|_{0,\mathrm{curl}} \le C\,h\,\|\boldsymbol{E}(0)\|_{1,\mathrm{curl}}\,.\end{aligned}$$

Then for the first term in $d(\eta_h^0)$, by definition of η_h^0, $\boldsymbol{E}_h^0$ and $\boldsymbol{E}_h^{-1}$, we know

$$\begin{aligned}\partial_\tau\eta_h^0 &= \tau^{-1}(\tau\Pi_h\boldsymbol{E}_t(0) - P_h\boldsymbol{E}(0) + P_h\boldsymbol{E}(-\tau))\\ &= \tau^{-1}P_h(\boldsymbol{E}(-\tau) - \boldsymbol{E}(0) + \tau\boldsymbol{E}_t(0)) + P_h(\Pi_h\boldsymbol{E}_t(0) - \boldsymbol{E}_t(0)),\end{aligned}$$

using the property of the projection P_h, we derive

$$\begin{aligned}\|\partial_\tau\eta_h^0\|_0 &\le \tau^{-1}\|\boldsymbol{E}(-\tau) - \boldsymbol{E}(0) + \tau\boldsymbol{E}_t(0)\|_{0,\mathrm{curl}}\\ &\quad + \|\Pi_h\boldsymbol{E}_t(0) - \boldsymbol{E}_t(0)\|_{0,\mathrm{curl}}\\ &\le C\,\tau\sup_{(-\tau,0)}\|\boldsymbol{E}_{tt}\|_1 + C\,h\,\|\boldsymbol{E}_t(0)\|_{1,\mathrm{curl}}\,.\end{aligned}$$

Therefore, we get the estimates for the initial error $d(\eta_h^0)$:

$$d(\eta_h^0) \le C\,\tau^2 \sup_{(-\tau,0)} \|\boldsymbol{E}_{tt}\|_1^2 + C\,h^2\,\left(\|\boldsymbol{E}(0)\|_{1,\mathrm{curl}}^2 + \|\boldsymbol{E}_t(0)\|_{1,\mathrm{curl}}^2\right).$$

Then substituting this into (31) and applying the well-known discrete Gronwall's inequality, we conclude that

$$\max_{1\le n\le M}\left(\|\partial_\tau(\boldsymbol{E}_h^n - P_h\boldsymbol{E}^n)\|_0^2 + \|\mathrm{curl}\,(\boldsymbol{E}_h^n - P_h\boldsymbol{E}^n)\|_0^2\right) \le C\,m_0(\boldsymbol{E})\,(\tau^2 + h^2).$$

Finally, applying the triangle inequality to

$$\partial_\tau\boldsymbol{E}_h^n - \boldsymbol{E}_t^n = (\partial_\tau\boldsymbol{E}_h^n - P_h\partial_\tau\boldsymbol{E}^n) + (P_h\partial_\tau\boldsymbol{E}^n - \partial_\tau\boldsymbol{E}^n) + (\partial_\tau\boldsymbol{E}^n - \boldsymbol{E}_t^n)$$

and

$$\boldsymbol{E}_h^n - \boldsymbol{E}^n = (\boldsymbol{E}_h^n - P_h\boldsymbol{E}^n) + (P_h\boldsymbol{E}^n - \boldsymbol{E}^n),$$

we have proved the following energy-norm error estimates

Theorem 4.1 *Let $\boldsymbol{E}$ and $\boldsymbol{E}_h^n$ be the solutions of the electric field equations (5)-(6) and the finite element approximation (7)-(8) at time $t = t^n$, respectively. Assume that*

$$\boldsymbol{E} \in H^2(0,T;H_0(\mathrm{curl}\,;\Omega)\cap H^1(\mathrm{curl}\,;\Omega)\,)\cap H^3(0,T;(L^2(\Omega))^3\,).$$

Then we have

$$\max_{1\le n\le M}\left(\|\partial_\tau\boldsymbol{E}_h^n - \boldsymbol{E}_t^n\|_0^2 + \|\mathrm{curl}\,(\boldsymbol{E}_h^n - \boldsymbol{E}^n)\|_0^2\right) \le C\,(\tau^2 + h^2)$$

where C is a constant independent of both the time step τ and the meshsize h.

Remark 4.1 The error estimate in Theorem 4.1 is optimal both in terms of time step size τ and mesh size h as we have used only the $H^1(\mathrm{curl}\,;\Omega)$-regularity in space and $H^3(0,T)$-regularity in time.

If the solution $\boldsymbol{E}$ has no so much regularity in space as in Theorem 4.1 (cf. Costabel [10], Assous et al [4]), we then have the following weaker error estimates

Theorem 4.2 *Let $\boldsymbol{E}$ and $\boldsymbol{E}_h^n$ be the solutions of the electric field equations (5)-(6) and the finite element approximation (7)-(8) at time $t = t^n$, respectively. Assume that for some $1/2 < \alpha < 1$,*

$$\boldsymbol{E} \in H^2(0,T;H_0(\mathrm{curl}\,;\Omega)\cap H^\alpha(\mathrm{curl}\,;\Omega)\,)\cap H^3(0,T;(L^2(\Omega))^3\,).$$

Then we have

$$\max_{1\le n\le M}\left(\|\partial_\tau\boldsymbol{E}_h^n - \boldsymbol{E}_t^n\|_0^2 + \|\mathrm{curl}\,(\boldsymbol{E}_h^n - \boldsymbol{E}^n)\|_0^2\right) \le C\,(\tau^2 + \tau^2 h^{2(\alpha-1)} + h^{2\alpha}).$$

Proof. The proof is almost identical to the one for Theorem 4.1 by replacing $H^1(\mathrm{curl}\,;\Omega)$ by $H^\alpha(\mathrm{curl}\,;\Omega)$, Lemma 3.1 by Lemmas 3.2-3.3, (12) by Lemma 3.3. The only remaining term we have to re-estimate is the term $(\mathrm{I})_4$ in (30) as we now have no regularity $\mathrm{curl}\,(\mathrm{curl}\,\boldsymbol{E}_t)$ in $(L^2(\Omega))^3$ as used in that proof. Instead we can bound the term as follows: by assumption we have $\boldsymbol{E} \in H^1(0,T;H^\alpha(\mathrm{curl}\,;\Omega)\,)$, so $\mathrm{curl}\,\boldsymbol{E} \in H^1(0,T;(H^\alpha(\Omega))^3\,)$ and then $\mathrm{curl}\,(\mathrm{curl}\,\boldsymbol{E}) \in H^1(0,T;(H^{\alpha-1}(\Omega))^3\,)$, where $H^{\alpha-1}(\Omega)$ is the dual space of $H^{1-\alpha}(\Omega)$ (note that this is true as, when $0 < 1-\alpha < 1/2$, $H^{1-\alpha}(\Omega) = H_0^{1-\alpha}(\Omega)$). Therefore, by Green's formula and the inverse inequality we have

$$
\begin{aligned}
(\mathrm{I})_4 &= \frac{1}{\mu}\int_{I^k} \left\langle \mathrm{curl}\,\mathrm{curl}\,(\boldsymbol{E}-\boldsymbol{E}^k),\ \partial_\tau \eta_h^k \right\rangle_{H^{\alpha-1},H^{1-\alpha}} dt \\
&= -\frac{1}{\mu}\int_{I^k}\int_t^{t^k} \left\langle \mathrm{curl}\,(\mathrm{curl}\,\boldsymbol{E}_t),\ \partial_\tau \eta_h^k \right\rangle_{H^{\alpha-1},H^{1-\alpha}} dt'dt \\
&\le \frac{\tau}{\mu}\int_{I^k} \|\mathrm{curl}\,(\mathrm{curl}\,\boldsymbol{E}_t)\|_{H^{\alpha-1}(\Omega)} \|\partial_\tau \eta_h^k\|_{H^{1-\alpha}(\Omega)} dt \\
&\le C\tau\, h^{\alpha-1}\|\partial_\tau \eta_h^k\|_0 \int_{I^k} \|\mathrm{curl}\,(\mathrm{curl}\,\boldsymbol{E}_t)\|_{H^{\alpha-1}(\Omega)} dt \\
&\le \frac{1}{4}\tau\varepsilon\|\partial_\tau \eta_h^k\|_0^2 + C\tau^2 h^{2(\alpha-1)} \int_{I^k} \|\mathrm{curl}\,(\mathrm{curl}\,\boldsymbol{E}_t)\|_{H^{\alpha-1}(\Omega)}^2 dt.
\end{aligned}
$$

That completes the proof of Theorem 4.2. □

Remark 4.2 It is easy to see that if we take τ to be the same magnitude as h, then the error estimate in Theorem 4.2 is of optimal order, i.e. $O(h^{2\alpha})$, in the sense of the space regularity we have used.

Remark 4.3 Theorems 4.1-4.2 can be extended to the Crank-Nicolson scheme (9)-(10). The major difference in proving the related theorems is to derive a similar equation as (29) and to choose an appropriate test function $\mathbf{v}$. For the former, we may multiply equation (5) at time $t = t^n$ by $\mathbf{v}/\tau \in V_h^0$ and then integrate the resulting equation over Ω. For the latter, we may choose the test function $\mathbf{v} = \eta_h^{n+1} - \eta_h^{n-1} = (\eta_h^{n+1} - \eta_h^n) + (\eta_h^n - \eta_h^{n-1})$.

4.2. L^2-norm error estimates

This subsection is dedicated to the derivation of the L^2-norm error estimates. The basic technique used here is borrowed from Girault [13] and Monk [18], where Nédélec's finite element methods were applied for stationary Navier-Stokes equations and semi-discrete schemes with Nédélec's finite elements for time-dependent Maxwell's equations, respectively.

As usual, we assume the domain Ω is a convex polyhedron in order to achieve the optimal L^2-norm error estimates. If the domain Ω is a general Lipschitz polyhedron, we have the following L^2-norm error estimates: for $\boldsymbol{E} \in H^3(0,T;(L^2(\Omega))^3)$ and $\boldsymbol{E} \in H^2(0,T;H_0(\mathrm{curl}\,;\Omega)\cap H^1(\mathrm{curl}\,;\Omega))$,

$$\max_{1\le n\le M} \|\boldsymbol{E}_h^n - \boldsymbol{E}^n\|_0 \le C\,(\tau + h),$$

which can be obtained immediately by applying the triangle inequality to the relation

$$\begin{aligned}\boldsymbol{E}_h^n - \boldsymbol{E}^n &= (\Pi_h\boldsymbol{E}_0 - \boldsymbol{E}_0) + \tau\sum_{k=1}^{n}(\partial_\tau\boldsymbol{E}_h^k - \partial_\tau\boldsymbol{E}^k)\\ &= (\Pi_h\boldsymbol{E}_0 - \boldsymbol{E}_0) + \tau\sum_{k=1}^{n}(\partial_\tau\boldsymbol{E}_h^k - \boldsymbol{E}_t^k) + \tau\sum_{k=1}^{n}(\boldsymbol{E}_t^k - \partial_\tau\boldsymbol{E}^k)\end{aligned}$$

and the results in Theorem 4.1.

A similar result, when $\boldsymbol{E}$ is only in $H^2(0,T;H_0(\mathrm{curl}\,;\Omega)\cap H^\alpha(\mathrm{curl}\,;\Omega))$ and $H^3(0,T;(L^2(\Omega))^3)$, for $1/2 < \alpha < 1$, is

$$\max_{1\le n\le M} \|\boldsymbol{E}_h^n - \boldsymbol{E}^n\|_0 \le C\,h^\alpha,$$

by using Theorem 4.2 and taking τ to be of order $O(h)$.

But if the domain Ω is convex and the solution $\boldsymbol{E}$ is more regular, we can expect higher convergence order. We now show this is possible. We will consider only the case that the coefficients ε and μ are constants (we take 1 for simplicity) and $\sigma = 0$. We need the following decomposition (cf. Ciarlet [7])

(32) $$L^2(\Omega)^3 = M \oplus M^\perp$$

where M and $M^\perp$ are two spaces defined by

$$M = \{\mathbf{v} = \nabla z;\ z \in H_0^1(\Omega)\}$$

and

$$M^\perp = \{\mathbf{v} \in L^2(\Omega)^3;\ (\mathbf{v}, \nabla z) = 0 \quad \forall\, z \in H_0^1(\Omega)\} = H(\mathrm{div}\,0;\Omega).$$

The last equality can be proved directly from definition. As Ω is convex, we have $H_0(\mathrm{curl}\,;\Omega) \cap M^\perp \subset H^1(\Omega)^3$ (see [14]).

To introduce a discrete version of (32), we define a finite element space

$$S_h = \{z_h \in H_0^1(\Omega); \quad z_h|_K \in \mathcal{P}_2 \quad \forall\, K \in \mathcal{T}^h\}.$$

An $L^2(\Omega)$ projection Q_h onto S_h will be needed in our later analysis, i.e. we define $Q_h : L^2(\Omega) \mapsto S_h$ as follows

$$(Q_h z,\ v_h) = (z,\ v_h), \quad \forall\, z \in L^2(\Omega), v_h \in S_h.$$

Now we can introduce the following discrete decomposition

$$V_h^0 = M_h \oplus M_h^\perp \tag{33}$$

where M_h and $M_h^\perp$ are two spaces defined by

$$M_h = \{\mathbf{v} = \nabla z_h;\ z_h \in S_h\}$$

and

$$M_h^\perp = \{\mathbf{v} \in V_h^0;\ (\mathbf{v}, \nabla z_h) = 0 \quad \forall\, z_h \in S_h\}.$$

Note that, by definition, M (resp. M_h) is the kernel of the curl operator in $H_0(\mathrm{curl}\,;\Omega)$ (resp. V_h^0).

Before discussing our main results in this section, we introduce some operators which are very useful in the later L^2-norm error estimates. We first define an operator T. For any $\mathbf{u} \in M^\perp$, $T\mathbf{u} \in H_0(\mathrm{curl}\,;\Omega) \cap M^\perp$ and satisfies

$$(\mathrm{curl}\,(T\mathbf{u}),\ \mathrm{curl}\,\mathbf{v}) = (\mathbf{u}, \mathbf{v}), \quad \forall\, \mathbf{v} \in H_0(\mathrm{curl}\,;\Omega) \cap M^\perp.$$

For a given $\mathbf{u}$, $T\mathbf{u}$ can be regarded also as the solution, in variable $\mathbf{w}$, to the following system (cf. Girault [13]):

$$\mathrm{curl}\,(\mathrm{curl}\,\mathbf{w}) = \mathbf{u} \quad \text{in } \Omega, \tag{34}$$

$$\mathrm{div}\,\mathbf{w} = 0 \quad \text{in } \Omega, \tag{35}$$

$$\mathbf{w} \times \boldsymbol{n} = 0 \quad \text{on } \Gamma. \tag{36}$$

The problem (34)-(36) is a particular case of the system in $(\mathbf{w}', p)$:

$$\mathrm{curl}\,(\mathrm{curl}\,\mathbf{w}') + \nabla\, p = \mathbf{g} \quad \text{in } \Omega, \tag{37}$$

$$\mathrm{div}\,\mathbf{w}' = 0 \quad \text{in } \Omega, \tag{38}$$

$$\mathbf{w}' \times \boldsymbol{n} = 0 \quad \text{on } \Gamma, \tag{39}$$

$$p = 0 \quad \text{on } \Gamma, \tag{40}$$

for a given $\mathbf{g}$ in $(L^2(\Omega))^3$. According to Lemma 4.1 in [13], the system (37)-(40) has a unique solution $\mathbf{w}' \in H_0(\mathrm{curl}\,;\Omega) \cap H(\mathrm{div}\,0;\Omega)$ and $p \in H_0^1(\Omega)$. In addition, the following properties hold:

– $\mathbf{w}' \in H^1(\Omega)^3$ and $||\mathbf{w}'||_1 \le C\,||\mathrm{curl}\,\mathbf{w}'||_0 \le C\,||\mathbf{g}||_0$.
– $\mathrm{curl}\,\mathbf{w}' \in H^1(\Omega)^3$ and $||\mathrm{curl}\,\mathbf{w}'||_1 \le C\,||\mathbf{g}||_0$.
– $|p|_1 \le C\,||\mathbf{g}||_0$.

Clearly, if $\mathbf{g} \in M^\perp$, then $p = 0$: this is the case with $\mathbf{g} = \mathbf{u}$.

The discrete version $T_h : M^\perp \to M_h^\perp$ of the operator T can be defined by

$$(\mathrm{curl}\,(T_h\mathbf{u}),\ \mathrm{curl}\,\mathbf{v}_h) = (\mathbf{u}, \mathbf{v}_h), \quad \forall\, \mathbf{u} \in M^\perp,\ \mathbf{v}_h \in M_h^\perp.$$

We now state some properties of the operators T and T_h. The results (i) and (ii) were established by Monk in [18], so we focuse on the proof of (iii) and (iv). The property (iii) is an improvement over the result in [18].

Lemma 4.1 *Let Ω be a convex polyhedron and suppose $T\mathbf{u} \in H^2(\Omega)^3$, for $\mathbf{u} \in H_0(\mathrm{curl}\,;\Omega) \cap H(\mathrm{div}\,0;\Omega)$. Then there exists a constant C independent of h and $\mathbf{u}$ such that*

$$\begin{aligned}
&\text{(i)} \quad \|\mathrm{curl}\,((T-T_h)\mathbf{u})\|_0 \le C\,h\,\|\mathrm{curl}\,(T\mathbf{u})\|_1,\\
&\text{(ii)} \quad \|\mathrm{curl}\,(T_h\mathbf{u})\|_0 \le C\,\|\mathbf{u}\|_0,\\
&\text{(iii)} \quad \|(T-T_h)\mathbf{u}\|_0 \le C\,h^2\|T\mathbf{u}\|_2,\\
&\text{(iv)} \quad \|T_h^{1/2}\mathbf{u}\|_0 \le C\,\|\mathbf{u}\|_0.
\end{aligned}$$

Proof. Notice that by using Weber's result [22], one easily gets that $||\mathrm{curl}\,\mathbf{w}||_0$ defines a norm in $H_0(\mathrm{curl}\,;\Omega) \cap H(\mathrm{div}\,0;\Omega)$ which is equivalent to its canonical norm.

Now, as $p=0$, $\mathbf{w} = T\mathbf{u}$ is the solution to:

$$(\mathrm{curl}\,\mathbf{w},\ \mathrm{curl}\,\mathbf{v}) = (\mathbf{u},\mathbf{v}), \quad \forall\,\mathbf{v} \in H_0(\mathrm{curl}\,;\Omega),$$

and $\mathbf{w}_h = T_h\mathbf{u} \in M_h^\perp$ also satisfies the discrete variational formulation:

$$(\mathrm{curl}\,\mathbf{w}_h,\ \mathrm{curl}\,\mathbf{v}_h) = (\mathbf{u},\mathbf{v}_h), \quad \forall\,\mathbf{v}_h \in V_h^0 \subset H_0(\mathrm{curl}\,;\Omega).$$

Therefore, $\|\mathrm{curl}\,(\mathbf{w}-\mathbf{w}_h)\|_0 \le ||\mathrm{curl}\,\mathbf{w}||_0 \le C\,||\mathbf{u}||_0$.

We first prove the fourth inequality. We see

$$||T_h^{1/2}\mathbf{u}||_0^2 = (T_h\mathbf{u},\mathbf{u}) = (\mathbf{w}_h,\mathbf{u}) = (\mathbf{w}_h - \mathbf{w},\mathbf{u}) + (\mathbf{w},\mathbf{u}).$$

Define $q \in H_0^1(\Omega)$ by

$$(\nabla q, \nabla\mu) = (\mathbf{w}_h, \nabla\mu),\ \forall \mu \in H_0^1(\Omega).$$

Note that ∇q and $\mathbf{u}$ belong to orthogonal subspaces of $L^2(\Omega)^3$. Set $\mathbf{v} = \mathbf{w}_h - \nabla q$: $\mathrm{curl}\,\mathbf{v} = \mathrm{curl}\,\mathbf{w}_h$, $\mathrm{div}\,\mathbf{v} = 0$, $\mathbf{v} \times \boldsymbol{n} = 0$ on Γ. Therefore $\mathbf{w}_h - \nabla q - \mathbf{w} \in H_0(\mathrm{curl}\,;\Omega) \cap H(\mathrm{div}\,0;\Omega)$ and

$$||\mathbf{w}_h - \nabla q - \mathbf{w}||_0 \le C\,||\mathrm{curl}\,(\mathbf{w}_h - \mathbf{w})||_0 \le C\,||\mathrm{curl}\,\mathbf{w}||_0 \le C\,||\mathbf{u}||_0.$$

Also, $||\mathbf{w}||_0 \le C\,||\mathrm{curl}\,\mathbf{w}||_0 \le C\,||\mathbf{u}||_0$, which proves (iv).

Now we turn to the third inequality: take any $\mathbf{g}$ in $L^2(\Omega)^3$ and solve (37)-(40). Then

$$\begin{aligned}
&(\mathbf{g}, \mathbf{w}_h - \mathbf{w})\\
&\quad = (\mathrm{curl}\,\mathbf{w}', \mathrm{curl}\,(\mathbf{w}_h - \mathbf{w})) + (\nabla p, \mathbf{w}_h - \mathbf{w})\\
&\quad = (\mathrm{curl}\,(\mathbf{w}' - \mathbf{w}_h^*), \mathrm{curl}\,(\mathbf{w}_h - \mathbf{w})) + (\nabla(p-p_h), \mathbf{w}_h - \mathbf{w})\\
&\quad = (\mathrm{curl}\,(\mathbf{w}' - \mathbf{w}_h^*), \mathrm{curl}\,(\mathbf{w}_h - \mathbf{w})) + (\nabla(p-p_h), \mathbf{w}_h - \mathbf{y}_h)\\
&\qquad +(\nabla(p-p_h), \mathbf{y}_h - \mathbf{w}),\ \forall\,\mathbf{w}_h^*, \mathbf{y}_h \in V_h^0,\ \forall\,p_h \in S_h.
\end{aligned} \tag{41}$$

Set $\mathbf{z}_h = \mathbf{w}_h - \mathbf{y}_h$ and split up $\mathbf{z}_h$ into $\mathbf{z}_h = \nabla q + \mathbf{v}$, with $q \in H_0^1(\Omega)$ defined by

$$(\nabla q, \nabla \mu) = (\mathbf{z}_h, \nabla \mu), \ \forall \mu \in H_0^1(\Omega).$$

As usual $\mathbf{v} = \mathbf{z}_h - \nabla q$ is an element of $H_0(\mathrm{curl}\,; \Omega) \cap H(\mathrm{div}\, 0; \Omega)$. As $\mathbf{z}_h$ belongs to V_h^0, it has sufficient regularity for defining $\Pi_h \mathbf{z}_h = \mathbf{z}_h$. Therefore, it can also be split up into

$$\mathbf{z}_h = \Pi_h \mathbf{v} + \nabla q_h, \ \text{ with } q_h \in S_h.$$

Remark 4.4 To prove the splitting one can write $\Pi_h(\nabla q)$ as ∇q_h, then use (18) which leads to $\mathrm{curl}\,(\Pi_h(\nabla q)) = 0$: thus there exists $q' \in H^1(\Omega)$ such that $\Pi_h(\nabla q) = \nabla q'$. Moreover, by definition, $\Pi_h(\nabla q)$ is in V_h^0, so its restriction to any tetrahedron K belongs to $(\mathcal{P}_1)^3$. Putting these together shows that q' is an element of S_h.

Now, the estimate of $(\nabla(p - p_h), \mathbf{z}_h)$ proceeds as in Girault [13]: choose for p_h the H_0^1-projection of p into S_h, hence

$$|(\nabla(p - p_h), \mathbf{z}_h)| \le |p|_1 \, ||\Pi_h \mathbf{v} - \mathbf{v}||_0. \tag{42}$$

We are going to estimate $||\Pi_h \mathbf{v} - \mathbf{v}||_0$. First, following [3],

$$\|\hat{\mathbf{v}} - \hat{\Pi}\hat{\mathbf{v}}\|_{L^2(\hat{K})} \le C\Big\{ \|\widehat{\mathrm{curl}}\,\hat{\mathbf{v}}\|_{L^p(\hat{K})} + \|\hat{\mathbf{v}}\|_{L^2(\hat{K})} + \sum_{\hat{F}\subset\hat{K}} \|\hat{\mathbf{v}} \times \hat{\boldsymbol{n}}\|_{L^p(\hat{F})} \Big\}.$$

But $\mathrm{curl}\,\mathbf{v} = \mathrm{curl}\,\mathbf{z}_h$ and (13) preserves the curl. Therefore, $\widehat{\mathrm{curl}}\,\hat{\mathbf{v}} = \widehat{\mathrm{curl}}\,\hat{\mathbf{z}}_h$ which belongs to a finite dimensional space. Thus,

$$\|\widehat{\mathrm{curl}}\,\hat{\mathbf{v}}\|_{L^p(\hat{K})} \le \hat{C}\, \|\widehat{\mathrm{curl}}\,\hat{\mathbf{v}}\|_{L^2(\hat{K})}.$$

Using the norm equivalence in the quotient space $H^1/\mathcal{P}_0$, we come to

$$\|\hat{\mathbf{v}} - \hat{\Pi}\hat{\mathbf{v}}\|_{L^2(\hat{K})} \le C\left\{ \|\widehat{\mathrm{curl}}\,\hat{\mathbf{v}}\|_{L^2(\hat{K})} + |\hat{\mathbf{v}}|_{H^1(\hat{K})} \right\}.$$

Thanks to the estimates which bound the right-hand side (cf. [9] and [6]), it follows from (14) that

$$\|\mathbf{v} - \Pi_h \mathbf{v}\|_{L^2(K)} \le C\, h_K\, |\mathbf{v}|_{H^1(K)}.$$

Thus

$$\begin{aligned}\|\mathbf{v} - \Pi_h \mathbf{v}\|_0 &\le C\, h\, |\mathbf{v}|_1 = C\, h\, ||\mathrm{curl}\,\mathbf{v}||_0 \\ &\le C\, h\, ||\mathrm{curl}\,\mathbf{z}_h||_0 = C\, h\, ||\mathrm{curl}\,(\mathbf{w}_h - \mathbf{y}_h)||_0.\end{aligned}$$

We then have from (42) that

$$|(\nabla(p - p_h), \mathbf{z}_h)| \le C\, h\, |p|_1 \left\{ ||\mathrm{curl}\,(\mathbf{w}_h - \mathbf{w})||_0 + ||\mathrm{curl}\,(\mathbf{w} - \mathbf{y}_h)||_0 \right\}.$$

Therefore, the last two terms in (41) are bounded by

$$|p|_1 \{C\, h\, (||\mathrm{curl}\,(\mathbf{w}_h - \mathbf{w})||_0 + ||\mathrm{curl}\,(\mathbf{w} - \mathbf{y}_h)||_0) + ||\mathbf{w} - \mathbf{y}_h||_0\}\,, \quad \forall \mathbf{y}_h \in V_h^0.$$

By assumption, $\mathbf{w}$ belongs to $H^2(\Omega)$: choosing $\mathbf{y}_h = \Pi_h \mathbf{w}$ gives

$$\begin{aligned} &\|\mathbf{w} - \mathbf{y}_h\|_0 \le C\, h^2 |\mathbf{w}|_2, \\ &\|\mathrm{curl}\,(\mathbf{w} - \mathbf{y}_h)\|_0 \le C\, h\, \|\mathrm{curl}\, \mathbf{w}\|_1. \end{aligned} \tag{43}$$

And, with the help of (i), we obtain the final bound for the last two terms in (41), that is

$$C\, h^2\, |p|_1\, \|\mathbf{w}\|_2 \le C\, h^2\, ||\mathbf{g}||_0\, \|\mathbf{w}\|_2.$$

To conclude, as $\mathbf{w}'$ belongs to $H^1(\mathrm{curl}\,;\Omega) \cap H_0(\mathrm{curl}\,;\Omega)$, we can choose $\mathbf{w}_h^* = \Pi_h \mathbf{w}'$, and using (43) and (i), we have the final bound for the first term in (41),

$$\left|(\mathrm{curl}\,(\mathbf{w}' - \mathbf{w}_h^*), \mathrm{curl}\,(\mathbf{w}_h - \mathbf{w}))\right| \le C\, h^2\, ||\mathbf{g}||_0\, ||\mathbf{w}||_2.$$

This proves (iii). □

Now we are in a position to derive the main results in this section, i.e. the L^2-norm error estimate. By means of the decomposition (32), for any $\boldsymbol{J}(t) \in L^2(\Omega)$, $t \in (0,T)$, we can write

$$\boldsymbol{J} = \boldsymbol{J}_1 + \nabla z, \quad \boldsymbol{J}_1 \in M^\perp,\ z \in H_0^1(\Omega). \tag{44}$$

This decomposition enables us to separate our error estimates into two parts, i.e. Theorems 4.3 and 4.4.

The general principles of the proofs for these two theorems are similar to those of the proofs for Theorems 4.1 and 4.2 in [18]. But as our scheme is fully discrete, a lot of technical details need to be treated newly. We will give only an outline for each proof but refer to [9] for details.

First we show

Theorem 4.3 *Let $\boldsymbol{E}$ and $\boldsymbol{E}_h^n$ be the solutions to (5)-(6) and (7)-(8). Assume that $\boldsymbol{J}(t) \in M^\perp$ for $t \in (0,T)$, and that $\boldsymbol{E}_0$ and $\boldsymbol{E}_1$ belong to $M^\perp \cap (H^2(\Omega))^3$. Moreover we assume that*

$$\boldsymbol{E} \in C^1(0,T;(H^2(\Omega))^3) \cap C^2(0,T;(H^1(\Omega))^3) \cap H^3(0,T;(L^2(\Omega))^3).$$

Then we have

$$\|\boldsymbol{E}^n - \boldsymbol{E}_h^n\|_0 \le C\,(\tau + h^2) \quad \textit{for } n = 1,2,\cdots,M$$

where the constant C is independent of τ and h but may depend on the a priori bounds on $\boldsymbol{E}$.

Proof. By the assumption, it is easy to derive $\boldsymbol{E}(t) \in M^{\perp}$. Using (33), we can write $\boldsymbol{E}_h^k = \overline{\boldsymbol{E}}_h^k + \nabla z_h^k$ with $\overline{\boldsymbol{E}}_h^k \in M_h^{\perp}$ and $z_h^k \in S_h$, for $1 \le k \le n$. Then

$$\boldsymbol{E}^k - \boldsymbol{E}_h^k = (\boldsymbol{E}^k - \overline{\boldsymbol{E}}_h^k) - \nabla z_h^k \equiv \eta_h^k - \nabla z_h^k, \quad \text{for} \quad 1 \le k \le n. \tag{45}$$

It suffices to estimate the L^2-norms for both η_h^n and ∇z_h^n.

We first analyse ∇z_h^n. Taking $\mathbf{v} = \nabla \partial_\tau z_h^k \in M_h$ for any $z_h^k \in S_h$ in (8) and the Cauchy-Schwarz inequality, we get

$$\|\partial_\tau \nabla z_h^n\|_0^2 \le \|\partial_\tau \nabla z_h^{n-1}\|_0^2 \le \cdots \le \|\partial_\tau \nabla z_h^0\|_0^2.$$

But by orthogonality, we can write

$$\|\partial_\tau \nabla z_h^0\|_0^2 = (\partial_\tau \boldsymbol{E}_h^0, \partial_\tau \nabla z_h^0) = (\partial_\tau \boldsymbol{E}_h^0 - \partial_\tau \boldsymbol{E}^0, \partial_\tau \nabla z_h^0),$$

from which and the initial condition (7) we can show

$$\|\partial_\tau \nabla z_h^0\|_0 \le C(h^2 \|\boldsymbol{E}_1\|_2 + \tau \sup_{t \in (-\tau, 0)} \|\boldsymbol{E}_{tt}(t)\|_1). \tag{46}$$

Similarly, we have $\|\nabla z_h^0\|_0 \le C\, h^2\, |\boldsymbol{E}_0|_2$. Using this, (46) and the identity

$$\nabla z_h^n = \nabla z_h^0 + \tau \sum_{k=1}^{n} \partial_\tau \nabla z_h^k,$$

we derive that

$$\|\nabla z_h^n\|_0 \le C\, h^2 (\|\boldsymbol{E}_1\|_2 + \|\boldsymbol{E}_0\|_2) + C\tau \sup_{t \in (-\tau, 0)} \|\boldsymbol{E}_{tt}(t)\|_1. \tag{47}$$

Next we estimate η_h^k in (45). Taking $\mathbf{v} = T_h \phi_h$ in (8) with $\phi_h \in M_h^{\perp}$, we have, for $1 \le k \le n$,

$$(\partial_\tau^2 T_h \overline{\boldsymbol{E}}_h^k, \phi_h) + (\overline{\boldsymbol{E}}_h^k, \phi_h) = (\partial_\tau T_h \boldsymbol{J}^k, \phi_h), \quad \forall\, \phi_h \in M_h^{\perp}. \tag{48}$$

A similar relation to (48) for the continuous solution $\boldsymbol{E}$ can be derived by multiplying (5) by $\mathbf{v} = \tau^{-1} T\phi$ with $\phi \in M^{\perp}$ and then integrating over I^k in time and over Ω in space:

$$(\partial_\tau T \boldsymbol{E}_t^k, \phi) + (\tilde{\boldsymbol{E}}^k, \phi) = (\partial_\tau T \boldsymbol{J}^k, \phi), \quad \forall\, \phi \in M^{\perp}, \tag{49}$$

where $\tilde{\boldsymbol{E}}^k = \tau^{-1} \int_{I^k} \boldsymbol{E}(t)\, dt$.

Note for any $\phi_h \in M_h^{\perp}$, we can write $\phi_h = \phi + \nabla z_1$ with $\phi \in M^{\perp}$ and $z_1 \in H_0^1(\Omega)$. Using this relation we obtain from (49) that

$$(\partial_\tau T \boldsymbol{E}_t^k, \phi_h) + (\tilde{\boldsymbol{E}}^k, \phi_h) = (\partial_\tau T \boldsymbol{J}^k, \phi_h), \quad \forall\, \phi_h \in M_h^{\perp}. \tag{50}$$

Subtracting (48) from (50) and making some arrangements, we derive for any $\phi_h \in M_h^{\perp}$ that

$$(51)\quad (\partial_\tau^2 T_h \eta_h^k, \phi_h) + (\eta_h^k, \phi_h) \\ = (\,(T - T_h)\partial_\tau \boldsymbol{J}^k, \phi_h) + (\partial_\tau^2 T_h \boldsymbol{E}^k - T\partial_\tau \boldsymbol{E}_t^k, \phi_h) + (\boldsymbol{E}^k - \tilde{\boldsymbol{E}}^k, \phi_h),$$

then taking $\phi_h = \partial_\tau \overline{\boldsymbol{E}}_h^k$ gives

$$\begin{aligned} &(\partial_\tau^2 T_h \eta_h^k, \partial_\tau \eta_h^k) + (\eta_h^k, \partial_\tau \eta_h^k) \\ &= (\,T_h \partial_\tau^2 \eta_h^k, \partial_\tau \boldsymbol{E}^k) + (\eta_h^k, \partial_\tau \boldsymbol{E}^k) + (\,(T_h - T)\partial_\tau \boldsymbol{J}^k, \partial_\tau \overline{\boldsymbol{E}}_h^k) \\ &\quad + (\tilde{\boldsymbol{E}}^k - \boldsymbol{E}^k, \partial_\tau \overline{\boldsymbol{E}}_h^k) + (\partial_\tau T \boldsymbol{E}_t^k - \partial_\tau^2 T_h \boldsymbol{E}^k,\ \partial_\tau \overline{\boldsymbol{E}}_h^k). \end{aligned}$$

Note that, (52) holds actually for any $\phi_h \in V_h^0$ by orthogonality. Then, adding the previous equation to the equation (52) with $\phi_h = \Pi_h \partial_\tau \boldsymbol{E}^k \in V_h^0$, we get

$$\begin{aligned} &(\partial_\tau^2 T_h \eta_h^k, \partial_\tau \eta_h^k) + (\eta_h^k, \partial_\tau \eta_h^k) \\ &= (\,T_h \partial_\tau^2 \eta_h^k, \partial_\tau \boldsymbol{E}^k - \Pi_h \partial_\tau \boldsymbol{E}^k) + (\eta_h^k, \partial_\tau \boldsymbol{E}^k - \Pi_h \partial_\tau \boldsymbol{E}^k) \\ &\quad + (\,(T_h - T)\partial_\tau \boldsymbol{J}^k, \partial_\tau \overline{\boldsymbol{E}}_h^k - \Pi_h \partial_\tau \boldsymbol{E}^k) + (\tilde{\boldsymbol{E}}^k - \boldsymbol{E}^k, \partial_\tau \overline{\boldsymbol{E}}_h^k - \Pi_h \partial_\tau \boldsymbol{E}^k) \\ &\quad + (\partial_\tau T \boldsymbol{E}_t^k - \partial_\tau^2 T_h \boldsymbol{E}^k,\ \partial_\tau \overline{\boldsymbol{E}}_h^k - \Pi_h \partial_\tau \boldsymbol{E}^k) \\ &\equiv: \sum_{i=1}^{5} (\mathrm{I})_i, \end{aligned}$$

which implies, using $a(a-b) \ge \frac{1}{2}a^2 - \frac{1}{2}b^2$, for any $a, b \in \mathbb{R}$,

$$(52)\quad \left\{ \frac{1}{2} \|T_h^{1/2} \partial_\tau \eta_h^k\|_0^2 - \frac{1}{2} \|T_h^{1/2} \partial_\tau \eta_h^{k-1}\|_0^2 \right\} + \left\{ \frac{1}{2} \|\eta_h^k\|_0^2 - \frac{1}{2} \|\eta_h^{k-1}\|_0^2 \right\} \le \tau \sum_{i=1}^{5} (\mathrm{I})_i.$$

The terms $(\mathrm{I})_i$, $i = 1, \cdots, 5$ can be estimated one by one (cf. [9] for details). With these estimates, we then sum up (52) over $k = 1, \cdots, n$ and finally obtain

$$\begin{aligned} \|T_h^{1/2} \partial_\tau \eta_h^n\|_0^2 + \|\eta_h^n\|_0^2 &\le \|T_h^{1/2} \partial_\tau \eta_h^0\|_0^2 + \|\eta_h^0\|_0^2 + C(\boldsymbol{E})(\tau^2 + h^4) \\ &\le C(\boldsymbol{E})(\tau^2 + h^4). \end{aligned}$$

This with (45) and (47) implies the desired estimate of Theorem 4.3. □

Now we turn to the second part of the L^2-norm error estimate, i.e. we are going to prove

Theorem 4.4 *Let $\boldsymbol{E}$ and $\boldsymbol{E}_h^n$ be the solutions to (5)-(6) and (7)-(8). Assume that $\boldsymbol{J}$ is in $H^1(0,T;M\cap(H^2(\Omega))^2)$, and that $\boldsymbol{E}_0$ and $\boldsymbol{E}_1$ belong to $M\cap(H^2(\Omega))^3$. Moreover we assume that*

$$\boldsymbol{E}\in C^2(0,T;(H^2(\Omega))^3)\cap H^3(0,T;(L^2(\Omega))^3).$$

Then we have

$$\|\boldsymbol{E}^n-\boldsymbol{E}_h^n\|_0\le C\,(\tau+h^2)\quad \text{for } n=1,2,\cdots,M$$

where the constant C is independent of τ and h but may depend on the a priori bounds on $\boldsymbol{E}$.

Proof. By the assumption, it is easy to get $\boldsymbol{E}(t)\in M$. So we have $\boldsymbol{E}(t)=\nabla z(t)$ with $z\in H_0^1(\Omega)$. Using (33), we can write $\boldsymbol{E}_h^k=\overline{\boldsymbol{E}}_h^k+\nabla z_h^k$ with $\overline{\boldsymbol{E}}_h^k\in M_h^\perp$ and $z_h^k\in S_h$, for $1\le k\le n$. Note $\boldsymbol{E}_0\in M$, so $\boldsymbol{E}_0=\nabla\overline{z}_0$ for some $\overline{z}_0\in H_0^1(\Omega)$, thus we have $\boldsymbol{E}_h^0=\Pi_h\boldsymbol{E}_0=\Pi_h\nabla\overline{z}_0=\nabla\overline{z}_h^0$ for some $\overline{z}_h^0\in S_h$, i.e. $\overline{\boldsymbol{E}}_h^0=0$. Similarly we have $\overline{\boldsymbol{E}}_h^{-1}=0$.

By means of these decompositions, we may write

$$\begin{aligned}\boldsymbol{E}^n-\boldsymbol{E}_h^n&=(\boldsymbol{E}^0-\boldsymbol{E}_h^0)+\sum_{k=1}^n\int_{t^{k-1}}^{t^k}(\boldsymbol{E}_t-\partial_\tau\boldsymbol{E}_h^k)dt\\&=(\boldsymbol{E}_0-\Pi_h\boldsymbol{E}_0)+\sum_{k=1}^n\int_{t^{k-1}}^{t^k}\nabla(z_t-\partial_\tau z_h^k)dt\\&\quad-\sum_{k=1}^n\int_{t^{k-1}}^{t^k}\partial_\tau\overline{\boldsymbol{E}}_h^k dt.\end{aligned}$$

which, with the fact $\boldsymbol{E}(t)=\nabla z(t)$, thus

$$\|\nabla(z_t-\partial_\tau z_h^k)\|_0\le\tau\Big(\int_0^T\|\boldsymbol{E}_{tt}\|_0^2dt\Big)^{1/2}+\|\nabla(z_t^k-\partial_\tau z_h^k)\|_0,$$

implies

$$\begin{aligned}\|\boldsymbol{E}^n-\boldsymbol{E}_h^n\|_0&\le\tau\sum_{k=1}^n\|\nabla(z_t^k-\partial_\tau z_h^k)\|_0+\tau\sum_{k=1}^n\|\partial_\tau\overline{\boldsymbol{E}}_h^k\|_0\\&\quad+Ch^2\,|\boldsymbol{E}_0|_2+\tau\Big(\int_0^T\|\boldsymbol{E}_{tt}\|_0^2dt\Big)^{1/2}.\end{aligned}\tag{53}$$

It remains to estimate the first and second terms in the right side of (53). Let us first estimate the term $\|\partial_\tau\overline{\boldsymbol{E}}_h^k\|_0$. Taking $\mathbf{v}=\partial_\tau\overline{\boldsymbol{E}}_h^k\in M_h^\perp$ in (8), and using orthogonality and $\boldsymbol{J}(t)=\nabla g(t)$ for some $g\in H_0^1(\Omega)$, we get

$$(\partial_\tau^2\overline{\boldsymbol{E}}_h^k,\ \partial_\tau\overline{\boldsymbol{E}}_h^k)+(\operatorname{curl}\overline{\boldsymbol{E}}_h^k,\ \operatorname{curl}\partial_\tau\overline{\boldsymbol{E}}_h^k)=(\partial_\tau\nabla g^k,\ \partial_\tau\overline{\boldsymbol{E}}_h^k),$$

which implies

$$\left\{\frac{1}{2}\|\partial_\tau \overline{\boldsymbol{E}}_h^k\|_0^2 - \frac{1}{2}\|\partial_\tau \overline{\boldsymbol{E}}_h^{k-1}\|_0^2\right\} + \left\{\frac{1}{2}\|\mathrm{curl}\, \overline{\boldsymbol{E}}_h^k\|_0^2 - \frac{1}{2}\|\nabla \overline{\boldsymbol{E}}_h^{k-1}\|_0^2\right\} \le \tau(\partial_\tau \nabla g^k,\ \partial_\tau \overline{\boldsymbol{E}}_h^k).$$

Summing up the above equations over $k = 1, \cdots, n$, we obtain

$$\begin{aligned}\|\partial_\tau \overline{\boldsymbol{E}}_h^n\|_0^2 + \|\mathrm{curl}\, \overline{\boldsymbol{E}}_h^n\|_0^2 &\le 2\tau \sum_{k=1}^{n} (\partial_\tau \nabla g^k,\ \partial_\tau \overline{\boldsymbol{E}}_h^k) \\ &\le 2\tau \sum_{k=1}^{n} \Big(\nabla(\partial_\tau g^k - Q_h \partial_\tau g^k),\ \partial_\tau \overline{\boldsymbol{E}}_h^k\Big) \\ &\le \tau^2 \sum_{k=1}^{n} \|\partial_\tau \overline{\boldsymbol{E}}_h^k\|_0^2 + Ch^4 \int_0^T \|g_t\|_3^2 dt.\end{aligned}$$

That means by Gronwall's inequality and $\boldsymbol{J}(t) = \nabla g$ that

$$\|\partial_\tau \overline{\boldsymbol{E}}_h^n\|_0^2 + \|\mathrm{curl}\, \overline{\boldsymbol{E}}_h^n\|_0^2 \le Ch^4 \int_0^T \|\boldsymbol{J}_t\|_2^2 dt. \tag{54}$$

Next, we are going to estimate the term $\|\nabla(z_t^k - \partial_\tau z_h^k)\|_0$ in (53). Taking $\mathbf{v} = \nabla y_h^k$ for any $y_h^k \in S_h$ in (29) and then subtracting it from (8) yields

$$(\partial_\tau \nabla(z_t^k - \partial_\tau z_h^k),\ \nabla y_h^k) = 0, \quad \forall\, y_h^k \in S_h,$$

using this, we obtain

$$\begin{aligned}&\frac{1}{2}\|\nabla(z_t^k - \partial_\tau z_h^k)\|_0^2 - \frac{1}{2}\|\nabla(z_t^{k-1} - \partial_\tau z_h^{k-1})\|_0^2 \\ &\le \tau(\partial_\tau \nabla(z_t^k - \partial_\tau z_h^k),\ \nabla(z_t^k - y_h^k)\,).\end{aligned}$$

summing the equations over $k = 1, \cdots, n$ and using (25) gives for any $y_h^k \in S_h$,

$$\begin{aligned}&\|\nabla(z_t^n - \partial_\tau z_h^n)\|_0^2 \\ &\le \|\nabla(z_t^0 - \partial_\tau z_h^0)\|_0^2 + \tau \sum_{k=1}^{n} (\nabla(z_t^k - \partial_\tau z_h^k),\ \nabla \partial_\tau (z_t^k - y_h^k)\,) \\ &\quad + (\nabla(z_t^n - \partial_\tau z_h^n),\ \nabla(z_t^n - y_h^n)\,) - (\nabla(z_t^0 - \partial_\tau z_h^0),\ \nabla(z_t^0 - y_h^0)\,).\end{aligned} \tag{55}$$

Now taking $y_h^k = Q_h \partial_\tau z^k$, the terms in (55) can be bounded as follows:

$$\|\nabla(z_t^k - y_h^k)\|_0^2 \leq C\tau \int_{t^{k-1}}^{t^k} \|\boldsymbol{E}_{tt}\|_0^2 dt + Ch^4 \max_{0\leq t\leq T} |\boldsymbol{E}_t|_2^2 dt,$$

$$\|\nabla\partial_\tau(z_t^k - y_h^k)\|_0^2 \leq C\tau \int_{t^{k-1}}^{t^k} \|\boldsymbol{E}_{ttt}\|_0^2 dt + Ch^4 \max_{0\leq t\leq T} |\boldsymbol{E}_{tt}|_2^2,$$

and

$$\|\nabla(z_t^0 - \partial_\tau z_h^0)\|_0 \leq Ch^2 \, |\boldsymbol{E}_1|_2.$$

Substituting these estimates into (55), we have

$$\|\nabla(z_t^n - \partial_\tau z_h^n)\|_0^2 \leq C(\boldsymbol{E})(\tau^2 + h^4). \tag{56}$$

Theorem 4.4 then follows from (53), (54) and (56). □

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