Numer. Math. 73: 149–167 (1996)

Numerische
Mathematik

Overlapping Schwarz methods on unstructured meshes using non-matching coarse grids

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Received March 23, 1994 / Revised version received June 2, 1995

Summary. We consider two level overlapping Schwarz domain decomposition methods for solving the finite element problems that arise from discretizations of elliptic problems on general unstructured meshes in two and three dimensions. Standard finite element interpolation from the coarse to the fine grid may be used. Our theory requires no assumption on the substructures that constitute the whole domain, so the substructures can be of arbitrary shape and of different size. The global coarse mesh is allowed to be non-nested to the fine grid on which the discrete problem is to be solved, and neither the coarse mesh nor the fine mesh need be quasi-uniform. In addition, the domains defined by the fine and coarse grid need not be identical. The one important constraint is that the closure of the coarse grid must cover any portion of the fine grid boundary for which Neumann boundary conditions are given. In this general setting, our algorithms have the same optimal convergence rate as the usual two level overlapping domain decomposition methods on structured meshes. The condition number of the preconditioned system depends only on the (possibly small) overlap of the substructures and the size of the coarse grid, but is independent of the sizes of the subdomains.

Mathematics Subject Classification (1991): 65N30, 65F10

⋆ The work of the first two authors was partially supported by the National Science Foundation under contract ASC 92-01266, the Army Research Office under contract DAAL03-91-G-0150, and ONR under contract ONR-N00014-92-J-1890
⋆⋆ The work of this author was partially supported by the National Science Foundation under contract ASC 92-01266, the Army Research Office under contract DAAL03-91-G-0150, and subcontract DAAL03-91-C-0047

1. Introduction

Unstructured grids are popular and flexible, since they easily allow for complicated geometries and the resolution of fine scale structure in the solution [1], [19]. However, this flexibility may come with a price. Traditional solvers that exploit the regularity of the mesh may become less efficient on an unstructured mesh. Moreover, efficient vectorization and parallelization may require extra care. Thus, there is a need to adapt and develop current solution techniques for structured meshes so that they can run as efficiently on unstructured meshes.

In this paper, we present Schwarz methods defined for overlapping subdomains, for solving elliptic problems on unstructured meshes in two and three space dimensions. These are extensions of existing domain decomposition methods, constructed in such a way that they can be applied to unstructured meshes, and still retain their optimal efficiency. These methods are designed to possess inherent coarse grain parallelism in the sense that the subdomain problems can be solved independently on different processors.

The theory and methodology of domain decomposition methods for elliptic problems on structured meshes are quite well developed, see, for example, [23], [2], [3], [10], [12]. On a structured mesh, most of the existing theories and algorithms exploit the fact that the space of functions on the coarse mesh is a subspace of that on the fine mesh. Unfortunately, this property may no longer hold on an unstructured mesh. Both the theory and the algorithms need to be developed to accommodate this fact.

In this paper, we continue to develop the theory, begun by Cai [4] and Chan and Zou [6], of overlapping Schwarz methods for elliptic problems in two and three dimensions on unstructured meshes. Our main new results are to (1) prove convergence even when the domains defined by the fine grid, Ω, and the coarse grid, Ω^H, are not identical, for instance, when the coarse grid covers only a (large) portion of the fine grid, and (2) provide a simple proof of convergence when standard finite element interpolation from the coarse to fine grid is used that also holds for non-quasi-uniform triangulations. An important observation is that to obtain these strong results, in general, any Neumann boundary must be covered by the coarse grid. As in the earlier work, the subdomains are allowed to be of arbitrary shapes.

2. The finite element problem

We consider the following self-adjoint elliptic problem:

$$-\sum_{i,j=1}^{d} \frac{\partial}{\partial x_i}\left(a_{ij}\frac{\partial u}{\partial x_j}\right) + bu = f, \qquad \text{in } \Omega \tag{1}$$

with a Dirichlet boundary condition

$$u = 0, \qquad \text{on } \Gamma \tag{2}$$

and with a natural boundary condition

$$\sum_{i,j=1}^{d} a_{ij}\frac{\partial u}{\partial x_j} n_i + \alpha\, u = 0, \qquad \text{on } \partial\Omega \setminus \Gamma. \tag{3}$$

Here $\Omega \subset \mathbb{R}^d$ $(d = 2,3)$, $\big(a_{ij}(x)\big)$ is symmetric, uniformly positive definite, and is allowed to be piecewise smooth, but has no large jumps over the entire domain. The function $b(x) \geq 0$ in Ω, $\alpha(x) \geq 0$ on $\partial\Omega$, and $n = (n_1, n_2, \cdots, n_d)$ is the unit outer normal of the boundary $\partial\Omega$.

By Green's formula, it is immediate to derive the variational problem corresponding to (1)–(3): Find $u \in H^1_\Gamma(\Omega) = \{v \in H^1(\Omega); v = 0 \text{ on } \Gamma\}$ such that

$$a(u,v) = f(v) \qquad \forall\ v \in H^1_\Gamma(\Omega) \tag{4}$$

with

$$a(u,v) = \int_\Omega \Big(\sum_{i,j=1}^{d} a_{ij}\frac{\partial u}{\partial x_j}\frac{\partial v}{\partial x_i} + b\, uv\Big)dx + \int_{\partial\Omega\setminus\Gamma} \alpha\, uv\, ds, \tag{5}$$

$$f(v) = \int_\Omega f\, v\, dx. \tag{6}$$

We will solve the above variational problem (4) by the finite element method. Suppose we are given a family of triangulations $\{\mathcal{T}^h\}$ on Ω. We will not discuss the effects of approximating Ω but always assume in this paper that the triangulations $\{\mathcal{T}^h\}$ of Ω are exact. So we have $\Omega = \Omega^h \equiv \cup_{\tau\in\mathcal{T}^h}\tau$. Let $h = \overline{h} = \max_{\tau\in\mathcal{T}^h} h_\tau$, $h_\tau =$ diam τ, $\underline{h} = \min_{\tau\in\mathcal{T}^h} h_\tau$, ρ_τ = the radius of the largest ball inscribed in τ. Then we say $\mathcal{T}^h$ is *shape regular* if it satisfies

$$\sup_h \max_{\tau\in\mathcal{T}^h} \frac{h_\tau}{\rho_\tau} \leq \sigma_0\,, \tag{7}$$

and we say $\mathcal{T}^h$ is *quasi-uniform* if it is shape regular and satisfies

$$\overline{h} \leq \gamma\, \underline{h}\,, \tag{8}$$

with σ_0 and γ fixed positive constants; see Ciarlet [8]. In the paper, we only assume that the elements are shape regular, but not necessarily quasi-uniform.

Let V^h be a piecewise linear finite element subspace of $H^1_\Gamma(\Omega)$ defined on $\mathcal{T}^h$ with its basis denoted by $\{\phi_i^h\}_{i=1}^n$, and O_i =supp ϕ_i^h. Later we will use the following simple observations: if $\mathcal{T}^h$ is shape regular, then there exist a positive constant C and an integer ν, both depending only on σ_0 appearing in (7) and independent of h, so that, for $i = 1, 2, \cdots, n$,

$$\text{diam}\, O_i \leq C\, h_\tau,\ \forall\ \tau \subset O_i\,, \tag{9}$$

$$\text{card}\,\{\tau \in \mathcal{T}^h;\ \tau \subset O_i\} \leq \nu. \tag{10}$$

Our finite element problem is: Find $u^h \in V^h$ such that

(11) $$a(u^h, v^h) = f(v^h) \qquad \forall\ v^h \in V^h.$$

The corresponding linear system is

(12) $$Au = f$$

with $A = (\,a(\phi_i^h, \phi_j^h)\,)_{i,j=1}^n$ being the corresponding stiffness matrix.

Because of the ill-conditioning of the stiffness matrix A, our goal is to construct a good preconditioner M for A by domain decomposition methods to be used in conjunction with the preconditioned conjugate gradient method.

As usual, we decompose the domain Ω into p nonoverlapping subdomains Ω_i such that $\overline{\Omega} = \cup_{i=1}^p \overline{\Omega}_i$, then extend each subdomain Ω_i to a larger one Ω_i' such that the distance between $\partial\Omega_i$ and $\partial\Omega_i'$ is bounded from below by $\delta_i > 0$. We denote the minimum of all δ_i by δ. We assume that $\partial\Omega_i'$ does not cut through any element $\tau \in \mathscr{T}^h$. For the subdomains meeting the boundary we cut off the part of Ω_i' that is outside of $\overline{\Omega}$. No other assumptions will be made on $\{\Omega_i\}$ in this paper except that any point $x \in \Omega$ belongs to only a finite number of subdomains $\{\Omega_i'\}$. This means that we allow each Ω_i to be of quite different size and shape from other subdomains. We define the subspaces of V^h corresponding to the subdomains $\{\Omega_i'\}$, $i = 1, 2, \cdots, p$, by

(13) $$V_i^h = \{v_h \in V^h; v_h = 0 \text{ on } (\Omega \setminus \Omega_i') \cup \big(\partial\Omega \setminus (\partial\Omega \cap \partial\Omega_i')\big)\}.$$

For interior subdomains, and those adjacent to only a Dirichlet boundary,

(14) $$V_i^h = V^h \cap H_0^1(\Omega_i').$$

To develop a two level method, we also introduce a coarse grid $\mathscr{T}^H$, which forms a shape regular triangulation of Ω, but has nothing to do with $\mathscr{T}^h$, i.e., none of the nodes of $\mathscr{T}^H$ need to be nodes of $\mathscr{T}^h$. In general, $\Omega^H \neq \Omega$. Let H be the maximum diameter of the elements of $\mathscr{T}^H$, and $\Omega^H = \cup_{\tau^H \in \mathscr{T}^H} \tau^H$. Moreover, let Γ^H denote the portion of the boundary $\partial\Omega^H$ to which we will apply Dirichlet boundary conditions. (If the original problem is not pure Neumann, we require that the measure of Γ^H be at least the order of one coarse element size.)

By V^H we denote a subspace of $H^1_{\Gamma^H}(\Omega^H)$ consisting of piecewise polynomials defined on $\mathscr{T}^H$; by $\{\psi_i^H\}_{i=1}^m$ we denote its basis functions related to the nodes $\{q_i^H\}_{i=1}^m$. Let $O_i^H = \text{supp } \psi_i^H$. We note that V^H need not necessarily be piecewise linear; for example, it may be defined by bilinear (2-D), trilinear (3-D), or higher order elements. Thus we do not necessarily have the usual condition: $V^H \subset V^h$. We need to impose one important constraint on the coarse grid:

(A1): $$\partial\Omega \setminus \Gamma \subset \overline{\Omega^H},$$

i.e., the coarse grid covers all of the Neumann boundary, (see Fig. 1).

For technical reasons, we make two further, less restrictive assumptions on the coarse grid:

(A2): $$\tau^H \cap \Omega \neq \emptyset \quad \text{for all } \tau^H \in \mathscr{T}^H,$$

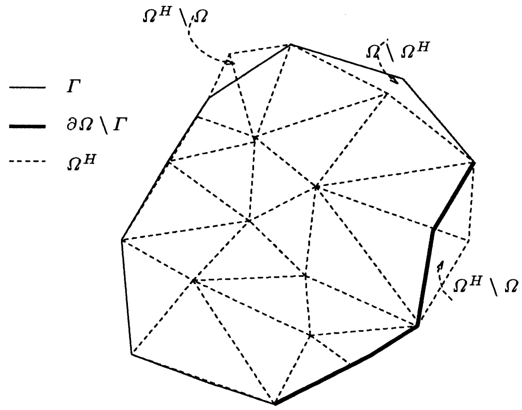

Fig. 1. Non-matching coarse grid

i.e., no coarse grid element lies completely outside the fine grid. For the complementary set $\Omega \setminus \Omega^H$, let S be the set of all vertices q_i^H of Ω^H, and let $B_p(r)$ be a ball centered at the point p with radius r. We assume that

(A3): $\Omega \setminus \Omega^H \subset \cup_{q_i^H \in S} B_{q_i^H}(\text{diam } O_i^H)$,

namely, the coarse grid must cover a significant part of the fine grid.

To overcome the difficulty that $V^H \not\subset V^h$, in both the theory and the algorithms, we need a way of mapping values from V^H to V^h. For the coarse space to be effective, this mapping must possess the properties of H^1-stability and L^2 optimal approximation; see Chan and Zou [6] and Mandel [18]. In this paper we mainly consider two such mappings. The first is the standard finite element interpolation Π_h defined in terms of the fine grid basis functions $\{\phi_i^h\}_{i=1}^n$. The second is the local, L^2-like projection, $\mathscr{R}_h$ used in Chan and Zou [6].

Throughout the paper, we use $||\cdot||_{m,\Omega}$ and $|\cdot|_{m,\Omega}$ to denote the norm and semi-norm of the usual Sobolev space $H^m(\Omega)$ for any integer $m \geq 0$. In addition, $||\cdot||_{m,r,\Omega}$ and $|\cdot|_{m,r,\Omega}$ will denote the norm and semi-norm of the spaces $W^{m,r}(\Omega)$ for any integer $m \geq 0$ and real number $r \geq 1$.

3. Two level overlapping Schwarz algorithms

Based on the finite element spaces V_i^h and V^H given in the preceding section, we derive the two level overlapping Schwarz methods for nonnested grids. Schwarz methods are preconditioners for the linear system $Au = f$ that are built by using local and coarse grid solvers. We first define these solves. From these we may write down the preconditioners using matrix notations.

The local solves are defined as in Dryja and Widlund [11] and in Bramble, Pasciak, Wang, and Xu [3]. Define the H^1-projection operators $P_i : V^h \to V_i^h$, $i = 1, \cdots, p$, such that for any $u \in V^h$, $P_i u \in V_i^h$ satisfies

$$a(P_i u,\, v_i) = a(u,\, v_i) \qquad \forall\, v_i \in V_i^h. \tag{15}$$

The coarse grid projection-like operator must be defined slightly differently than in Dryja and Widlund [11], due to the non-nestedness of the coarse grid space. Let $\mathscr{I}_h$ be any linear operator that maps V^H into a subspace $\mathscr{I}_h V^H$ of V^h. It may be chosen as the modified standard finite element interpolation operator Π_h or the locally defined operator $\mathscr{R}_h$; see Sect. 5 for more details.

In *Method 1*, we define $\tilde{P}_0$ by first defining $P_H u \in V^H$ on the original coarse grid space by

$$a(P_H u, v) = a(u, \mathscr{I}_h v) \qquad u \in V^h, \quad \forall v \in V^H \tag{16}$$

and then define $\tilde{P}_0 = \mathscr{I}_h P_H \colon V^h \to V_0^h$. The subspace $V_0^h \subset V^h$ is defined by $\mathscr{I}_h V^H$.

In *Method 2*, we define P_0 by calculating the projection directly onto the subspace V_0^h,

$$a(P_0 u, v) = a(u, v) \qquad u \in V^h, \quad \forall v \in V_0^h, \tag{17}$$

where $P_0 u \in V_0^h$.

Remark 1. We note here that for the left-hand side in (16), $a(u_H, v_H)$ for any $u_H, v_H \in V^H$, is not an integral over the original domain Ω, but one over the coarse domain Ω^H, i.e.

$$a(u_H, v_H) = \int_{\Omega^H} \Big(\sum_{i,j=1}^{d} a_{ij} \frac{\partial u}{\partial x_j} \frac{\partial v}{\partial x_i} + b\, u v \Big) dx + \int_{\partial\Omega^H \setminus \Gamma^H} \alpha\, u v\, ds\,. \tag{18}$$

Thus in the sequel we always assume that the coefficient functions a_{ij}, b, α are extended into $\Omega^H \setminus \Omega$ properly in the sense that (1) the extended $a_{ij}(x)$ is still symmetric, uniformly positive definite over the closure of $\Omega \cup \Omega^H$, (2) the largest and smallest eigenvalues of the extended $a_{ij}(x)$ can be bounded by the largest and smallest eigenvalues of the original $a_{ij}(x)$ respectively, (3) the extended b and α can be bounded by the bounds of original b and α, and (4) the extended α is still non-negative over $\partial\Omega^H \setminus \Gamma^H$. Below, we will use $||\cdot||_a$ and $||\cdot||_{a,\Omega^H}$ to denote the energy norms $a(\cdot,\cdot)$ over Ω and Ω^H, respectively.

We now derive the matrix representation of the operators P_i and $\tilde{P}_0$. Using these, we can write down both the additive and multiplicative Schwarz preconditioners. For the rest of this section only, we will use u^h to denote finite practice, the element functions and u to denote the vector of coefficients of that finite element function, that is, $u^h = \sum u_k \phi_k$. A purely algebraic of Schwarz algorithms may be found in Hackbusch [15].

Let $\{\phi_{i,j}^h\}_{j=1}^{n_i} \subset \{\phi_k^h\}_{k=1}^{n}$ be the set of nodal basis functions of V_i^h, $i = 1, 2, \cdots, p$. For each i, we define a matrix extension operator R_i^{T} as follows: For any $u_i^h \in V_i^h$, we denote by u_i the coefficient vector of u_i^h in the basis $\{\phi_{i,j}^h\}_{j=1}^{n_i}$, and we let $R_i^{\mathrm{T}} u_i$ be the coefficient vector of u_i^h in the basis $\{\phi_i^h\}_{i=1}^{n}$.

It is immediate to check that

$$A_i = R_i A R_i^{\mathrm{T}}, \tag{19}$$

where A and A_i, $i = 1, 2, \cdots, p$, are the stiffness matrices corresponding to the fine subspace V^h and the subspaces V_i^h, $i = 1, 2, \cdots, p$. And from (15) it follows that for any $u^h \in V^h$, the coefficient vector of $P_i u^h$ in the basis $\{\phi_i^h\}_{i=1}^n$ is

$$R_i^{\mathrm{T}} A_i^{-1} R_i A u, \tag{20}$$

where u denotes the coefficient vector of u^h in the basis $\{\phi_i^h\}_{i=1}^n$.

Since $\{\psi_i^H\}_{i=1}^m$ is the set of basis functions of V^H, then $\{\mathscr{I}_h \psi_i^H\}_{i=1}^m$ is the set of basis functions of V_0^h. We define a matrix extension operator R_0^{T} as follows: For any $u_0^h \in V_0^h$, we denote by u_0 the coefficient vector of u_0^h in the basis $\{\mathscr{I}_h \psi_i^H\}_{i=1}^m$, and we define $R_0^{\mathrm{T}} u_0$ as the coefficient vector of u_0^h in the basis $\{\phi_j^h\}_{j=1}^n$. Then $R_{0_{ij}} = \mathscr{I}_h \psi_i^H(q_j)$ where q_j is the nodal vertex of ϕ_j^h. When $\mathscr{I}_h = \Pi_h$, then $R_{0_{ij}}$ is simply given by $\psi_i^H(q_j)$.

We first note that the coefficient vector of a function $v \in V^H$ in the basis $\{\psi_i^H\}_{i=1}^m$ is exactly the same as the one for the function $\mathscr{I}_h v$ in the basis $\{\mathscr{I}_h \psi_i^H\}_{i=1}^m$. So from (16) we find that the coefficient vector of $P_H u^h$ in the basis $\{\psi_i^H\}_{i=1}^m$ is

$$A_H^{-1} R_0 A u, \tag{21}$$

where A_H is the stiffness matrix corresponding to the original coarse space V^H, with elements $A_{H_{ij}} = a(\psi_j^H, \psi_i^H)$. Now the coefficient vector of $\tilde{P}_0 u^h = \mathscr{I}_h P_H u^h \in V_0^h$ in the basis $\{\mathscr{I}_h \psi_i^H\}_{i=1}^m$ is also $A_H^{-1} R_0 A u$. Therefore, by the definition of R_0^{T}, $R_0^{\mathrm{T}} A_H^{-1} R_0 A u$ is the coefficient vector of $\tilde{P}_0 u^h$ in the basis $\{\phi_i^h\}_{i=1}^n$.

For Method 2 it is straightforward to derive that

$$A_0 = R_0 A R_0^{\mathrm{T}}, \tag{22}$$

where A_0 is the stiffness matrix corresponding to the subspace V_0^h. It follows from (17) that the coefficient vector of $P_0 u^h$ in the basis $\{\phi_i^h\}_{i=1}^n$ is

$$R_0^{\mathrm{T}} A_0^{-1} R_0 A u. \tag{23}$$

From the above, the additive Schwarz preconditioner may be written as

$$M_1 = R_0^{\mathrm{T}} A_H^{-1} R_0 + \sum_{i=1}^{p} R_i^{\mathrm{T}} A_i^{-1} R_i \tag{24}$$

for Method 1 and

$$M_2 = R_0^{\mathrm{T}} A_0^{-1} R_0 + \sum_{i=1}^{p} R_i^{\mathrm{T}} A_i^{-1} R_i \tag{25}$$

for Method 2. These may be thought of as an overlapping block Jacobi method with the addition of a coarse grid correction. The multiplicative Schwarz method is the Gauss-Seidel version of the additive algorithm. We write down the symmetrized version, using Method 1, as

$$M = (I - (I - R_1^{\mathrm{T}} A_1^{-1} R_1 A) \dots (I - R_p^{\mathrm{T}} A_p^{-1} R_p A)(I - R_0^{\mathrm{T}} A_H^{-1} R_0 A)$$
$$(I - R_p^{\mathrm{T}} A_p^{-1} R_p A) \dots (I - R_1^{\mathrm{T}} A_1^{-1} R_1 A))A^{-1}. \qquad (26)$$

In practice, the application of the multiplicative Schwarz preconditioner is carried out directly, not as given in (26).

Remark 2. From the above matrix representations M_1 and M_2 for *Method 1* and *Method 2*, we see that the only difference between them is in the global coarse problem solver. The latter coarse problem (with A_0^{-1}) is conducted on the newly constructed coarse subspace V_0^h, but the former (with A_H^{-1}) is conducted on the original coarse subspace V^H. Since V^H is not necessarily nested to V^h, A_H may not be expressed in terms of the stiffness matrix A as A_0 is in (22).

We give convergence results for the additive algorithm. Similar results may be obtained for the multiplicative algorithm using the techniques in Xu [27] or Dryja, Smith, and Widlund [10].

It is easy to check that

$$\kappa(M_1 A) = \kappa(\tilde{P}_0 + \sum_{i=1}^{p} P_i), \quad \kappa(M_2 A) = \kappa(\sum_{i=0}^{p} P_i). \qquad (27)$$

For these condition numbers, we have the following bounds:

Theorem 1. *Suppose that both triangulations $\mathscr{T}^h$ and $\mathscr{T}^H$ are shape regular (not necessarily quasi-uniform), and satisfy Assumptions (A1)–(A3). Then we have*

$$\kappa(M_1 A),\ \kappa(M_2 A) \le C \left(1 + \frac{H}{\delta}\right)^2. \qquad (28)$$

Theorem 1 will be proved at the end of Sect. 5.

4. Boundedness of the operator $\mathscr{I}_h$

Let W^h and W^H be any two finite element subspaces related to the triangulations $\mathscr{T}^h$ and $\mathscr{T}^H$, respectively. Since $W^H \not\subset W^h$, the convergence proof for the overlapping two level Schwarz methods requires that the operator $\mathscr{I}_h : W^H \to W^h$ possess the following H^1 stability and L^2 optimal approximation properties:

$$|\mathscr{I}_h u|_{1,\Omega} \le C |u|_{1,\Omega^H}, \qquad \forall u \in W^H, \qquad (29)$$

and

$$||\mathscr{I}_h u - u||_{0,\Omega} \le C h |u|_{1,\Omega^H}, \qquad \forall u \in W^H. \qquad (30)$$

There exist many options for the operator $\mathscr{I}_h$, for example, L^2 and quasi-L^2 projection operators Q_h and $\tilde{Q}_h$. For a discussion, we refer to Chan and Zou [6]. In this paper, we consider only the most natural option for $\mathscr{I}_h$, i.e., the standard finite element interpolation operator Π_h and the Clément's local L^2 projection operator $\mathscr{R}_h$.

Generally, for $\mathscr{I}_h = \Pi_h$, (29) and (30) are not true for all $u \in H^1(\Omega)$. Fortunately, they are true for general finite element spaces. We state this fact in the following lemma. Several alternative proofs for this result exist; see, for instance, Cai [4], Zhang [29], and Widlund [25].

Lemma 1. *Assume that $\mathscr{T}^h$ and $\mathscr{T}^H$ are both shape regular, not necessarily quasi-uniform, and W^h and W^H are any two corresponding finite element spaces consisting of continuous piecewise polynomials defined on Ω and Ω^H, respectively. Furthermore, assume that $\Omega \subset \Omega^H$. Then (29) and (30) hold in both two and three dimensions for $\mathscr{I}_h = \Pi_h$.*

Proof. Let $\tau^h \in \mathscr{T}^h$, then (see, for example, Ciarlet [8], Theorem 3.1.5), for $r > 3$ and $s = 0, 1$, we know for any $u \in W^H$

$$|u - \Pi_h u|^2_{s,\tau^h} \le Ch^{2(1-s)} h_\tau^{2d(1/2-1/r)} |u|^2_{1,r,\tau^h}. \tag{31}$$

This implies

$$\sum_{\tau^h \cap \tau^H \ne \emptyset} |u - \Pi_h u|^2_{s,\tau^h} \le C\, h^{2(1-s)} \sum_{\tau^h \cap \tau^H \ne \emptyset} h_\tau^{2d(1/2-1/r)} |u|^2_{1,r,\tau^h}. \tag{32}$$

Now apply the Cauchy inequality

$$\sum_i a_i b_i \le (\sum_i a_i^q)^{1/q} (\sum_i b_i^p)^{1/p}$$

to the right hand side with $p = r/2 > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$, and get

$$\begin{aligned}
\sum_{\tau^h \cap \tau^H \ne \emptyset} |u - \Pi_h u|^2_{s,\tau^h} &\le C\, h^{2(1-s)} (\sum_{\tau^h \cap \tau^H \ne \emptyset} h_\tau^{2d(1/2-1/r)q})^{1/q} (\sum_{\tau^h \cap \tau^H \ne \emptyset} |u|^{2p}_{1,r,\tau^h})^{1/p} \\
&\le C\, h^{2(1-s)} (\sum_{\tau^h \cap \tau^H \ne \emptyset} h_\tau^d)^{1-2/r} (\sum_{\tau^h \cap \tau^H \ne \emptyset} |u|^r_{1,r,\tau^h})^{2/r} \\
&\le C\, h^{2(1-s)} H_\tau^{d(1-2/r)} (\sum_{\overline{\tau}^H \cap \overline{\tau}^{H'} \ne \emptyset} |u|^r_{1,r,\tau^{H'}})^{2/r} \\
&\le C\, h^{2(1-s)} H_\tau^{d(1-2/r)} \left(H_\tau^{d(2/r-1)} \sum_{\overline{\tau}^H \cap \overline{\tau}^{H'} \ne \emptyset} |u|^2_{1,\tau^{H'}} \right). \qquad (33)
\end{aligned}$$

The last line follows from (9) and (10), and a standard local inverse inequality; see, for instance, Ciarlet [8], Theorem 3.2.6. Note that since the sum is over only those elements that are neighbors to τ^H, we do not need the quasi-uniformity assumption, called the inverse assumption by Ciarlet [8], only the shape regular assumption.

Taking the sum over τ^H, we obtain

$$\sum_{\tau^H} \sum_{\tau^h \cap \tau^H \ne \emptyset} |u - \Pi_h u|^2_{s,\tau^h} \le Ch^{2(1-s)} |u|^2_{1,\Omega^H}, \tag{34}$$

which implies (29) and (30). □

For our later use, we introduce a special, locally defined projection operator $\mathscr{R}_h$, which has been used in the domain decomposition context in [7] and [6]. Operators with properties similar to $\mathscr{R}_h$ can be also found in Scott and Zhang [21].

We denote the set of basis functions of V^h by $\{\phi_i^h\}_{i=1}^n$ corresponding to the vertices $\{q_i^h\}_{i=1}^n$. Let O_i =supp ϕ_i^h, $i = 1, 2, \cdots, n$.

Definition 1. The mapping $\mathscr{R}_h^0 : L^2(\Omega) \to V^h$ is defined by

$$\mathscr{R}_h^0 u = \sum_{i=1}^{n} Q_i\, u(q_i^h)\, \phi_i^h \qquad \forall\, u \in L^2(\Omega), \tag{35}$$

where $Q_i\, u \in \mathscr{P}_1(O_i)$ satisfies

$$\int_{O_i} Q_i u\, p\, dx = \int_{O_i} u\, p\, dx \qquad \forall\, p \in \mathscr{P}_1(O_i) \tag{36}$$

where $\mathscr{P}_1(O_i)$ is the space of linear functions defined on O_i.

By using the Poincaré inequality, the definition of $\mathscr{R}_h^0$, and relations (9) and (10), we can show the following properties of $\mathscr{R}_h^0$; see also Clément [9].

Lemma 2. *The operator $\mathscr{R}_h^0$ defined by (35) and (36) has the properties*

$$||\mathscr{R}_h^0 u||_{r,\Omega} \le C\, ||u||_{r,\Omega},\ \forall\, u \in H_\Gamma^1(\Omega), r = 0, 1, \tag{37}$$

$$||u - \mathscr{R}_h^0 u||_{0,\Omega} \le C\, h|u|_{1,\Omega},\ \forall\, u \in H_\Gamma^1(\Omega), \tag{38}$$

where the constant C is independent of h.

Remark 3. In Lemma 2, we assume only that $\mathscr{T}^h$ is shape regular, not necessarily quasi-uniform, unlike the usual L^2 projection.

Remark 4. The definition $\mathscr{R}_h^0$ can be generalized to more general finite element spaces V^h, e.g., to bilinear element (2-D), trilinear element (3-D), and higher order elements. In these cases, one needs only to replace $\mathscr{P}_1(O_i)$ in the relations (35) and (36) by $\tilde{P}(O_i)$, which are determined by the types of elements used in V^h, and Lemma 2 will still hold.

5. Partition lemma

In this section, we give a partition lemma for the finite element space V^h. The lemma is essential for the convergence proof of Theorem 1. As denoted previously, let $\{\psi_i^H\}_{i\in N_H}$ be the set of basis functions of V^H with $N_H = \{1, 2, \cdots, m\}$, and let $\{q_i^H\}_{i\in N_H}$ be the corresponding nodes and O_i = supp ψ_i^H.

We introduce an auxiliary subspace $\tilde{V}^H$ of V^H:

$$\tilde{V}^H = \text{span}\ \{\psi_i^H; i \in N_H^0\} \tag{39}$$

with $N_H^0 = \{i \in N_H;\ \psi_i^H = 0 \text{ on } \Gamma\}$. We need $\tilde{V}^H$ only for the proof of our main theorem, we do not require its explicit computation for our algorithms. It is easy to check that $\tilde{V}^H|_{\Omega} \subset H^1_\Gamma(\Omega)$.

By $\tilde{\Omega}^H$ and Ω_{N}^H we denote

$$\tilde{\Omega}^H = \cup_{i\in N_H^0} \operatorname{supp} \psi_i^H, \quad \Omega_{\mathrm{N}}^H = \tilde{\Omega}^H \cup (\Omega \setminus \tilde{\Omega}^H). \tag{40}$$

Let $\mathscr{R}_H$ be defined for V^H similarly to $\mathscr{R}_h$ defined for V^h in (35) and (36), with natural modifications, see Remark 4. Now we define a modified operator $\tilde{\mathscr{R}}_H : L^2(\tilde{\Omega}^H) \to \tilde{V}^H$ as follows

$$\tilde{\mathscr{R}}_H u = \sum_{i\in N_H^0} Q_i\, u(q_i^H)\, \psi_i^H \qquad \forall\ u \in L^2(\tilde{\Omega}^H), \tag{41}$$

where $Q_i\, u \in \mathscr{P}(O_i)$ satisfies

$$\int_{O_i} Q_i u\, p\, dx = \int_{O_i} u\, p\, dx \qquad \forall\ p \in \mathscr{P}(O_i). \tag{42}$$

Here $\mathscr{P}(O_i)$ is determined by the type of elements used in V^H. If V^H consists of piecewise polynomials of degree $\le q$, then $\mathscr{P}(O_i) = \mathscr{P}_q(O_i)$. We note that $\tilde{\mathscr{R}}_H u$ is well-defined on Ω by extending by zero.

For the operator $\tilde{\mathscr{R}}_H$, we have the following lemma.

Lemma 3. *The operator $\tilde{\mathscr{R}}_H$ defined by (41) and (42) has the properties*

$$||u - \tilde{\mathscr{R}}_H u||_{r,\Omega_{\mathrm{N}}^H} \le C\, ||u||_{r,\Omega_{\mathrm{N}}^H},\ \forall\ u \in H^1_\Gamma(\Omega_{\mathrm{N}}^H), r = 0, 1, \tag{43}$$

$$||u - \tilde{\mathscr{R}}_H u||_{0,\Omega_{\mathrm{N}}^H} \le C\, H |u|_{1,\Omega_{\mathrm{N}}^H},\ \forall\ u \in H^1_\Gamma(\Omega_{\mathrm{N}}^H), \tag{44}$$

where the constant C is independent of h and H.

Proof. Analogous to Lemma 2, we can prove that

$$||u - \mathscr{R}_H u||_{r,\Omega^H} \le C\, ||u||_{r,\Omega^H}\ \forall\ u \in H^1(\Omega^H), r = 0, 1, \tag{45}$$

$$||u - \mathscr{R}_H u||_{0,\Omega^H} \le C\, H |u|_{1,\Omega^H},\ \forall\ u \in H^1(\Omega^H). \tag{46}$$

Let $\partial N_H = \{i; i \in N_H \setminus N_H^0\}$. We see that for any $u \in H^1_\Gamma(\Omega_{\mathrm{N}}^H)$

$$u - \tilde{\mathscr{R}}_H u = u - \mathscr{R}_H u + \sum_{i\in\partial N_H} Q_i u(q_i^H)\psi_i^H. \tag{47}$$

Using (45) and (46), we need only to estimate the last term of (47).

For any $i \in \partial N_H$, we have by a local finite element inverse inequality, Poincaré's inequality, (cf. Ladyzhenskaya and Ural'tseva [16]), and the previous assumption (A3) on $\Omega \setminus \Omega^H$, that

$$
\begin{aligned}
||Q_i u(q_i^H)\psi_i^H||_{0,O_i}^2 &\le \sum_{\tau^H \subset O_i} ||\psi_i^H||_{0,\tau^H}^2 \, ||Q_i u||_{L^\infty(\tau^H)}^2 \\
&\le C \sum_{\tau^H \subset O_i} H_\tau^d \left(H_\tau^{-d} ||Q_i u||_{0,\tau^H}^2 \right) \\
&\le C \, ||Q_i u||_{0,O_i}^2 \le C \, ||u||_{0,O_i}^2 \\
&\le C \, ||u||_{0,\tilde{O}_i}^2 \le C \, (\text{diam } O_i)^2 |u|_{1,\tilde{O}_i}^2 \\
&\le C \, H^2 \, |u|_{1,\tilde{O}_i}^2 , \qquad (48)
\end{aligned}
$$

where $\tilde{O}_i$ = the union of O_i with the part of Ω that is outside O_i, (see Fig. 1).

Now (43) with $r = 0$ and (44) follow from (45)–(48). The inequality (43) with $r = 1$ can be proved analogously to the case of $r = 0$ above. □

We choose

$$V_0^h = \mathcal{I}_h V^H , \qquad (49)$$

where $\mathcal{I}_h$ can be any linear operator that maps V^H onto the subspace $\mathcal{I}_h V^H$ of V^h and retains the H^1 stability and L^2- optimal approximation *in any subspace (not necessarily in the whole V^H) of V^H of functions that vanish on Γ*. This essential observation will become very clear when we go through the following proof of Lemma 4. Therefore, $\mathcal{I}_h$ may be chosen as the standard finite element interpolation operator, or local L^2 projection operator $\mathcal{R}_h$ after simple and natural modifications for satisfying the Dirichlet boundary condition on Γ. For example, $\mathcal{I}_h$ may be chosen as $\mathcal{R}_h^0$ defined in Definition 4.1, or as Π_h^0 defined as follows:

$$\Pi_h^0 u = \sum_{i=1}^{n} u(q_i^h)\phi_i^h . \qquad (50)$$

From (49), we require that the coarse grid cover the fine grid Neumann boundary, see (A1). Otherwise, $\mathcal{I}_h$ makes no sense for the part $\Omega \setminus \Omega^H$. But the coarse grid does not need to cover the fine grid Dirichlet boundary, since we impose also homogeneous Dirichlet boundary conditions on the corresponding coarse grid boundary, so $\mathcal{I}_h$ still makes sense by naturally extending the coarse grid functions by zero for $\Omega \setminus \Omega^H$. Our numerical experiments will show that this strategy is important for practical computations.

We now have the following partition lemma for the fine space V^h:

$$V^h = V_0^h + V_1^h + \cdots + V_p^h . \qquad (51)$$

Lemma 4. *Let $\Omega \subset R^d$ ($d = 2, 3$). We assume that both triangulations $\mathcal{T}^h$ and $\mathcal{T}^H$ are shape regular but not necessarily quasi-uniform. Then for any $u \in V^h$, there exists a constant C independent of h, p, H, δ, and $u_i \in V_i^h$, $i = 1, \cdots, p$ and $u_0 = \mathcal{I}_h u_H \in V_0^h$ with $u_H \in V^H$ such that*

$$u = u_0 + u_1 + \cdots + u_p \qquad (52)$$

and

$$\sum_{i=1}^{p} ||u_i||_{1,\Omega}^2 \le C\,\left(1+\frac{H}{\delta}\right)^2 ||u||_{1,\Omega}^2, \tag{53}$$

$$||u_0||_{1,\Omega} \le C\,||u||_{1,\Omega},\quad ||u_H||_{1,\Omega^H} \le C\,||u||_{1,\Omega}. \tag{54}$$

Proof. Let $\hat{\Omega}$ be an open domain in R^d large enough such that $\Omega \subset \Omega_{\rm N}^H \subset\subset \hat{\Omega}$. Then we know, (see Stein [24]), that there exists a linear extension operator $E : H^1(\Omega) \to H^1(\hat{\Omega})$ such that $Eu|_\Omega = u$ and

$$||Eu||_{1,\hat{\Omega}} \le C\,||u||_{1,\Omega}. \tag{55}$$

We note that we do not require Eu for $u \in V^h$ to be a finite element function. For any $u \in V^h$, we choose $u_0 = \mathscr{I}_h u_H$ with $u_H = \tilde{\mathscr{R}}_H \tilde{u}$ and $\tilde{u} = Eu\Big|_{\Omega_{\rm N}^H}$. Then from Lemma 3 and Lemma 1, we obtain

$$\begin{aligned} ||u_0||_{1,\Omega} &= ||\mathscr{I}_h \tilde{\mathscr{R}}_H \tilde{u}||_{1,\Omega} \le C\,||\tilde{\mathscr{R}}_H \tilde{u}||_{1,\Omega^H} = C\,||\tilde{\mathscr{R}}_H \tilde{u}||_{1,\Omega_{\rm N}^H} \\ &\le C\,||\tilde{u}||_{1,\Omega_{\rm N}^H} \le C\,||\tilde{u}||_{1,\hat{\Omega}} \le C\,||u||_{1,\Omega}, \end{aligned} \tag{56}$$

which implies (54), and

$$\begin{aligned} ||u-u_0||_{0,\Omega} &\le ||u-\tilde{\mathscr{R}}_H \tilde{u}||_{0,\Omega} + ||\tilde{\mathscr{R}}_H \tilde{u} - \mathscr{I}_h \tilde{\mathscr{R}}_H \tilde{u}||_{0,\Omega} \\ &\le ||\tilde{u}-\tilde{\mathscr{R}}_H \tilde{u}||_{0,\Omega_{\rm N}^H} + C\,h\,|\tilde{\mathscr{R}}_H \tilde{u}|_{1,\Omega^H} \\ &\le CH|\tilde{u}|_{1,\Omega_{\rm N}^H} + C\,h|\tilde{u}|_{1,\Omega_{\rm N}^H} \le C\,H|\tilde{u}|_{1,\hat{\Omega}} \le C\,H|u|_{1,\Omega}. \end{aligned} \tag{57}$$

It is well-known, (see Dryja and Widlund [11] or Bramble et al. [3]), that there exists a partition $\{\theta_i\}_{i=1}^p$ of unity for Ω related to the subdomains $\{\Omega_i'\}$ such that $\sum_{i=1}^p \theta_i(x) = 1$ on Ω and for $i = 1, 2, \cdots, p$,

$$\operatorname{supp}\theta_i \subset \Omega_i' \cup \partial\Omega,\ 0 \le \theta_i \le 1 \text{ and } ||\nabla\theta_i||_{L^\infty(\Omega_i)} \le C\,\delta_i^{-1}. \tag{58}$$

Now for any $u \in V^h$, let $u_0 = \mathscr{I}_h \tilde{\mathscr{R}}_H\,\tilde{u} \in V^h$ be chosen as above, and let $u_i = \Pi_h \theta_i (u-u_0)$ with Π_h being the standard interpolation of V^h. Obviously, $u_i \in V_i^h$ and

$$u = u_0 + u_1 + \cdots + u_p. \tag{59}$$

Then (53) follows in the standard way; see Dryja and Widlund [11] and Smith [22]. We give a complete proof here so that one can see clearly that no quasi-uniformity assumption on $\mathscr{T}^h$ and the subdomains $\{\Omega_i\}$ is required in the present case. Let τ be any element belonging to Ω_k' with h_τ being its diameter and $\bar{\theta}_k$ the average of θ_k on element τ. Then from (58) and the fact that $u-u_0 \in V^h$, we get

$$\begin{aligned} |u_k|_{1,\tau}^2 &\le 2|\bar{\theta}_k \Pi_h (u-u_0)|_{1,\tau}^2 + 2|\Pi_h(\theta_k - \bar{\theta}_k)(u-u_0)|_{1,\tau}^2 \\ &\le 2|u-u_0|_{1,\tau}^2 + 2|\Pi_h(\theta_k-\bar{\theta}_k)(u-u_0)|_{1,\tau}^2. \end{aligned}$$

By using the local inverse inequality, which requires only the shape regularity of $\mathscr{T}^h$ (see Proposition 3.2 in Xu [26]), we obtain

$$
\begin{aligned}
|u_k|^2_{1,\tau} &\le 2|u-u_0|^2_{1,\tau} + C\,h_\tau^{-2}||\Pi_h(\theta_k-\bar{\theta}_k)(u-u_0)||^2_{0,\tau} \\
&\le 2|u-u_0|^2_{1,\tau} + C\,h_\tau^{-2}\frac{h_\tau^2}{\delta_k^2}||u-u_0||^2_{0,\tau} \\
&\le 2|u-u_0|^2_{1,\tau} + C\,\frac{1}{\delta_k^2}||u-u_0||^2_{0,\tau}.
\end{aligned}
$$

By taking the sum over $\tau \in \Omega'_k$, we have

$$
|u_k|^2_{1,\Omega'_k} \le 2|u-u_0|^2_{1,\Omega'_k} + C\,\frac{1}{\delta_k^2}||u-u_0||^2_{0,\Omega'_k}. \tag{60}
$$

Recall the assumption made previously that any point $x \in \Omega$ belongs only to a finite number of subdomains $\{\Omega'_i\}$, it follows from (56), (57), and (60) that

$$
\sum_{k=1}^{p}|u_k|^2_{1,\Omega'_k} \le C\left(|u-u_0|_1^2 + \frac{1}{\delta^2}||u-u_0||_0^2\right) \tag{61}
$$

$$
\le C\left(1+\frac{H}{\delta}\right)^2|u|_1^2. \tag{62}
$$

Analogously, we derive that

$$
\sum_{k=1}^{p}||u_k||^2_{0,\Omega'_k} \le C\left(1+\frac{h^2}{\delta^2}\right)||u||_0^2,
$$

which completes the proof of (53). □

In the rest of this section, we prove Theorem 1. We first state a general abstract lemma that is a natural extension of the one due to Lions [17], Nepomnyaschikh [20], Dryja and Widlund [11], Zhang [28], and Griebel and Oswald [14]. The proof is straightforward, similar to the one of Theorem A in [14].

Given a Hilbert space V and a symmetric, positive definite bilinear form $a(\cdot,\cdot)$ and a set of auxiliary spaces V_i for which the bilinear form is also defined, but which are not necessarily the subspaces of V. Suppose there exist "interpolation" operators $I_i : V_i \to V$ and define $T_i : V \to V_i$ by

$$
a(T_i u, v) = a(u, I_i v), \ \forall\, v \in V_i. \tag{63}
$$

Then $T = \sum_{i=0}^{p} I_i T_i$ satisfies the following lemma.

Lemma 5.

$$
a(T^{-1}u,u) = \min_{\substack{u_i \in V_i \\ u=\sum_{i=0}^{p} I_i u_i}} \sum_{i=0}^{p} a(u_i,u_i). \tag{64}
$$

Proof of Theorem 1. The estimate of $\kappa(M_2 A)$ is quite routine by using Lemma 4 and (27). To get the bound of $\kappa(M_1 A)$, it suffices to show that there exist two constants C_0 and C_1 independent of H, δ, h such that for any $u^h \in V^h$,

$$C_0\, a(\tilde{P}u^h, u^h) \le a(u^h, u^h) \le C_1 \left(1 + \frac{H}{\delta}\right)^2 a(\tilde{P}u, u). \tag{65}$$

We first provide the upper bound. From (16) we see that

$$a(P_H u^h, P_H u^h) = a(\mathscr{I}_h P_H u^h, u^h). \tag{66}$$

Thus, by Cauchy-Schwarz's inequality and the stability of $\mathscr{I}_h$,

$$||P_H u^h||^2_{a,\Omega^H} \le ||u^h||_a\, ||\mathscr{I}_h P_H u^h||_a \le C\, ||u^h||_a\, ||P_H u^h||_{a,\Omega^H}, \tag{67}$$

i.e., $||P_H u^h||_{a,\Omega^H} \le C\, ||u^h||_a$, which leads to

$$a(\tilde{P}_0 u^h, u^h) = a(P_H u^h, P_H u^h) \le C\, a(u^h, u^h). \tag{68}$$

From standard coloring arguments and the fact that the norm of a projection operator equals one, (see Zhang [28]),

$$\sum_{i=1}^{p} a(P_i u^h, u^h) \le C\, a(u^h, u^h). \tag{69}$$

Therefore we have proved the first inequality in (65). For the second inequality, we choose $I_0 = \mathscr{I}_h$, I_i for $i > 0$ to be the identity operator, and $V_0 = V^H$, then applying Lemma 5 and Lemma 4 gives our results.

Remark 5. We can improve the bound of Theorem 1 by replacing $(1 + H/\delta)^2$ by $(1 + H_{\max}/\delta)$ if the subdomains $\{\Omega_i\}_{i=1}^p$ form a quasi-uniform triangulation of Ω and $H \le \beta H_{\max}$ for some fixed constant β. Here $H_{\max}$ is the maximum of the diameters of the subdomains. This can be done by using a result by Dryja and Widlund [13]; cf. Chan and Zou [6].

6. Numerical experiments

In this section, we give the results of two numerical experiments for the case $\mathscr{I}_h = \Pi_h$. In our first numerical experiment, we demonstrate that the assumption (A1) is necessary in practice, i.e., it is very important to cover the Neumann boundary. When the coarse grid does not completely cover the fine grid Neumann boundary, one obtains rather poor convergence.

We consider the Poisson problem on the unit square with either pure homogeneous Dirichlet or mixed boundary conditions. In the case of mixed boundary conditions, we prescribe homogeneous Dirichlet boundary condition for $x \le 0.2$ and homogeneous Neumann boundary condition for $x > 0.2$. A uniform triangulation using linear finite elements is used. The coarse grid is defined on the square $[0, 1 + \beta] \times [0, 1]$. If β is less than zero, we are not covering the right

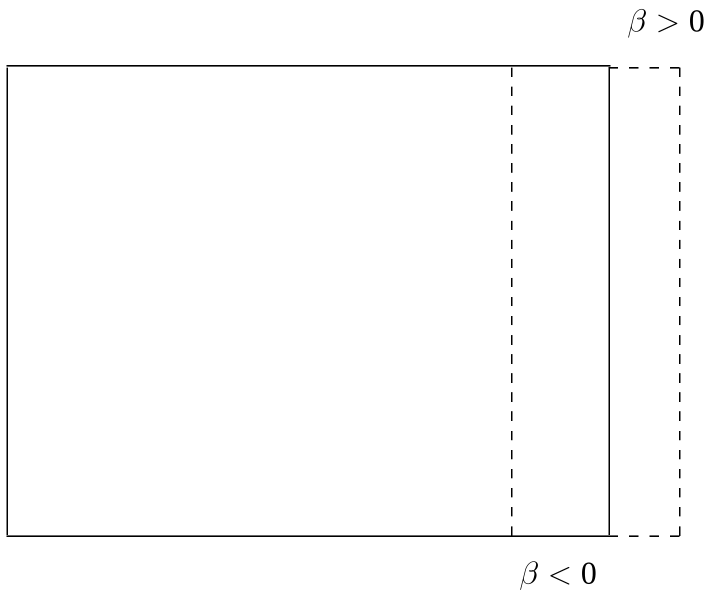

Fig. 2. Overlapping region

edge of the fine grid, see Fig. 2. Note that only when $\beta = 0$ do we have a nested coarse

grid space. In all of the experiments we use an extension by zero during the interpolation from the coarse grid for all fine grid nodes not covered by the coarse grid.

We ran with four size grids: 20 by 20, 40 by 40, 80 by 80 and 160 by 160. With each refinement the number of subdomains was increased by a factor of four from 16 to 64 to 256 to 1024. A constant overlap of one element was used. The coarse grid was refined from 5 by 5 to 10 by 10 to 20 by 20, to 40 by 40. The value of the "missing" overlap, $|\beta|$, was changed from 0.1 to 0.05 to 0.025 to 0.0125. Note that we keep $|\beta| \asymp H$. For all our calculations, we always choose the initial iterative guess of zero and stop the iteration when a relative decrease in the discrete norm of residual of 10^{-5} is obtained.

As one can see in Tables 1 and 2, the number of iterations was essentially unaffected by the "missing" overlap for the Dirichlet boundary conditions. However, for the case of Neumann boundary conditions, the number of iterations required to achieve the same tolerance increased greatly. This result agrees very well with our theory.

In our second experiment, we solve a mildly varying coefficient problem:

$$\frac{\partial}{\partial x}\left((1 + xy)\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left((1 + \sin(4x + 4y))\frac{\partial u}{\partial y}\right) = x^2 \sin(3y)$$

discretized by a standard piecewise linear finite element method on the unstructured airfoil grid shown in Fig. 3. The airfoil is embedded in the unit square.

Table 1. Convergence for multiplicative Schwarz

Boundary Conditions	β	Fine Mesh			
		20x20	40x40	80x80	160x160
Dirichlet	0	10	10	9	9
	+	10	10	11	12
	-	9	10	10	10
Mixed	0	10	10	10	10
	+	10	10	10	10
	-	15	22	30	43

Table 2. Convergence for additive Schwarz

Boundary Conditions	β	Fine Mesh			
		20x20	40x40	80x80	160x160
Dirichlet	0	30	28	26	25
	+	29	30	28	30
	-	27	28	28	29
Mixed	0	23	28	29	29
	+	23	28	29	28
	-	33	50	77	110

Table 3. Multiplicative DD iterations for the *Airfoil* mesh. 32 subdomains

Overlap (no. elements)	Coarse Grid	
0	None	55
0	G_2	30
0	G_1	20
1	None	31
1	G_2	17
1	G_1	11
2	None	24
2	G_2	13
2	G_1	9

We use nonhomogeneous Dirichlet boundary conditions for $x \leq 0.2$ and homogeneous Neumann boundary conditions for $x > 0.2$. Note that since the present software used for the calculations can generate only coarse grids that are interior to the fine grid, we do violate Assumption (A1) here. This explains why the iteration counts are slightly higher then one would expect for, for example, a Dirichlet boundary value problem. The subdomains are shown in Fig. 3 and two sets of coarse grids are given in Fig. 4. Since the theoretical convergence behavior of additive and multiplicative overlapping Schwarz is very similar, we have chosen to only include the results for the multiplicative case, see Table 3. Other numerical studies may be found in Chan and Smith [5].

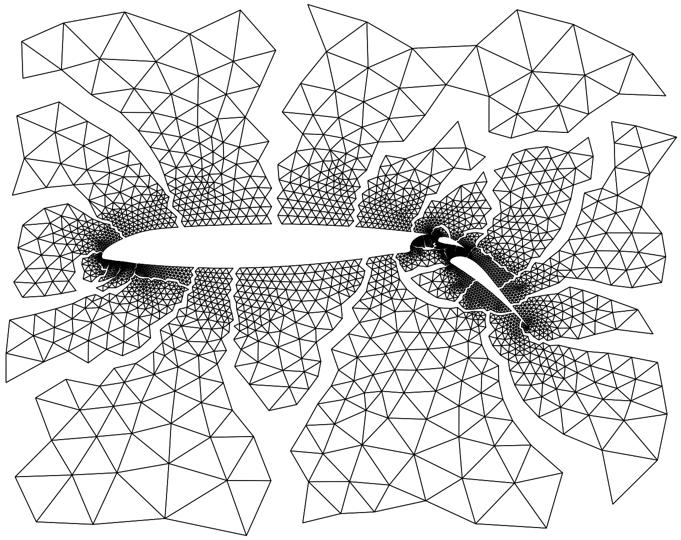

Fig. 3. Airfoil grid partitioned into 32 subdomains

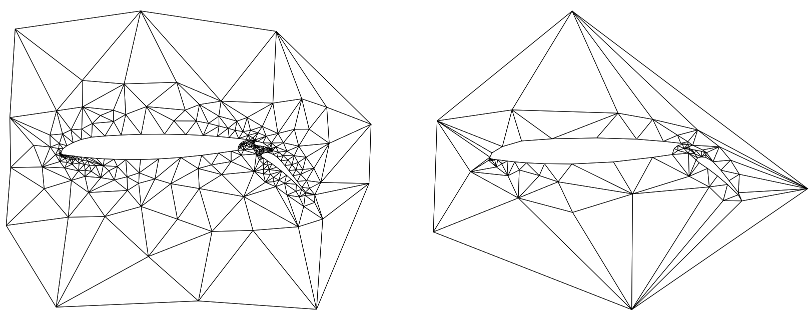

Fig. 4. Two coarse grids: G_1 (left), G_2 (right)

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