Convergence of an Adaptive Finite Element Method for Distributed Flux Reconstruction

Yifeng Xu[1]   and   Jun Zou[2]

Abstract

We shall establish the convergence of an adaptive conforming finite element method for the reconstruction of the distributed flux in a diffusion system. The adaptive method is based on a posteriori error estimators for the distributed flux, state and costate variables. The sequence of discrete solutions produced by the adaptive algorithm is proved to converge to the true triplet satisfying the optimality conditions in the energy norm and the corresponding error estimator converges to zero asymptotically.

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1   Introduction

The heat flux distributions are of significant practical interest in thermal and heat transfer problems, e.g., the real-time monitoring in steel industry [2] and the visualization by liquid crystal thermography [17]. Considering its accurate distribution is rather difficult to obtain in some inaccessible part of the physical domain, such as the interior boundary of nuclear reactors and steel furnaces, engineers attempt to recover the heat flux from some measured data, which leads naturally to the inverse problem of reconstructing the distributed heat flux from the measurements on the accessible part of the boundary or the Cauchy problem for an elliptic/parabolic equation. Several numerical methods have been proposed for this classical ill-posed problem, among which the least-squares formulation [39] [41] [42] has received intensive investigations and has been implemented by means of the boundary integral method [42] and the finite element method [39].

However, the story is far from complete from the viewpoint of numerical simulations. One main challenge is to detect local features of unknown fluxes accurately and efficiently, particularly in the presence of non-smooth boundaries and discontinuity or singularity in fluxes. Compared with the finite element reconstruction over meshes generated by a uniform refinement, which often requires formidable computational costs to achieve a high resolution, adaptive finite element methods (AFEM) are clearly a preferable candidate to remedy the situation as it is able to retrieve the same result with much fewer degrees of freedom.

A standard adaptive finite element method consists of successive loops of the form:

$$\text{SOLVE} \rightarrow \text{ESTIMATE} \rightarrow \text{MARK} \rightarrow \text{REFINE}. \tag{1.1}$$

That is, one first solves the discrete problem for the finite element solution on the current mesh, computes the related a posteriori error estimator, marks elements to be subdivided, and then refines the current mesh to generate a new finer one.

A major force to drive the process (1.1) is the module ESTIMATE, which relies on some computable quantities (often called a posteriori estimation), formed by the discrete solution on the current mesh and given data. Since the pioneer work [3], a posteriori error estimations have been extensively investigated for finite element approximations of direct partial differential equations and the theory has reached a mature level for elliptic systems; see the monographs [1] [6] [40] and the references therein. As far as PDE-based inverse problems are concerned, there are also some important developments, e.g., [5] [8] [9] [10] [11] [12] [19] [21] [24]. But a vast amount of literature is available on PDE-constrained optimal control problems; see [7] [22] [23] [25] [26] and references therein, although inverse problems are quite different in nature due to the severe instability by data noise.

On the other hand, the study of AFEMs itself is also a research topic of great interest and has made a substantial progress in the past decade. Specifically, the convergence and the computational complexity of an AFEM have been analyzed in depth for the numerical solution of second order boundary value problems; see [13] [14] [16] [18] [31] [32] [33] [35] [36]. But there are still no developments available for inverse problems. To our knowledge, the only related work is the one in [20] and it studied the asymptotic error reduction property of an adaptive finite element approximation for the distributed control problems with control constraints, where the adaptive algorithm requires one extra step for some oscillation terms in the module MARK and the interior node property in the module REFINE.

In this work, we shall fill in the gap and establish a first convergence result for an adaptive finite element method for inverse problems, namely, we shall demonstrate that both the finite element error (in some appropriate norm) and the estimator converge to zero when the AFEM is applied to reconstruct the distributed flux on some inaccessible part of the boundary from partial measurements on an accessible boundary part. Compared with [20] for an optimal control problem, the algorithm studied here is of the same framework as the standard one for (direct) elliptic problems (e.g. [14] [33]), particularly no more marking for oscillation terms as well as no interior node property is enforced in the module MARK and the module REFINE, therefore it is advantageous to practical computations. Our basic arguments follow some principles in [35] [32] for a class of linear direct boundary value problems. In this sense, the current work may be viewed as an extension of [32] [35] for the AFEM to inverse problems, but due to the nature of the inverse problem there are some essential technical differences as mentioned below.

- The direct problems of some linear partial differential equations were considered in [32] [35], while a nonlinear optimization problem for solving an inverse problem with the temperature field (state) and the flux (control) coupled in a diffusion equation is the focus of this work, which leads to a saddle-point system.

- In [32] [35] for linear direct problems, a key observation is the strong convergence of a sequence of discrete solutions generated by the adaptive process (1.1) to some limit, which is a direct consequence of the standard finite element convergence theory such as the Cea's lemma [15]. In contrast, achieving such a result for the inverse problem is highly nontrivial. We shall view the approximate fluxes generated by (1.1) as the minimizers to a discrete optimal system, and employ some techniques from the nonlinear optimizations to establish the strong convergence of the adaptive sequence to a minimizer of some limiting optimal system.

- The convergence was established in [35] by first demonstrating the weak vanishing limit of a sequence of residuals associated with the adaptive solutions, then proving the strong limit of the sequence of adaptive solutions is the exact solution. But this approach does not apply to our current problem as the exact state and the limiting state depend on the exact flux and the limiting flux respectively. As a remedy, we shall introduce an auxiliary state depending on the limiting flux to help us realize the desired convergence.

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Our convergence theory are basically established in three steps. In the first step, we shall show the sequence of discrete triplets (the approximate state, costate and flux) produced by the adaptive algorithm converges strongly to some limiting triplet. Unlike for the direct problem of differential equations, we need to deal with a nonlinear optimization system with PDE constraints; see section 4. In the second step, we will prove the limiting triplet is the exact one. To do so, we have to consider and study the limiting behaviors of the residuals associated with the approximate state and costate and introduce an auxiliary problem to resolve a technical difficulty; see section 5.2. Finally in the last step, we will demonstrate that the error estimator has a vanishing limit. This will be the consequence of the previous steps and the efficiency of the error estimator; see the proof of Theorem 5.2.

The rest of this paper is organized as follows. In section 2, we give a description of the flux reconstruction problem and its finite element method. A standard adaptive algorithm based on an existing residual-type a posteriori error estimator is stated in section 3. In section 4, we prove the sequence of discrete triplets converges to some limiting triplet. The main results are presented in section 5 and finally the paper is ended with some concluding remarks in section 6.

Throughout the paper we adopt the standard notation for the Lebesgue space $L^\infty(G)$ and L^2-based Sobolev spaces $H^m(G)$ for integer $m \geq 0$ on an open bounded domain $G \subset \mathbb{R}^d$. Related norms and semi-norms of $H^m(G)$ as well as the norm of $L^\infty(G)$ are denoted by $\|\cdot\|_{m,G}$, $|\cdot|_{m,G}$ and $\|\cdot\|_{\infty,G}$ respectively. We use $(\cdot,\cdot)_G$ to denote the L^2 scalar product on G, and the subscript is omitted when no confusion is caused. Moreover, we shall use C, with or without subscript, for a generic constant independent of the mesh size and it may take a different value at each occurrence.

2 Mathematical formulations

Let $\Omega \subset \mathbb{R}^d$ ($d = 2, 3$) be an open and bounded polyhedral domain. The boundary Γ of Ω is made up of two disjoint parts Γ_a and Γ_i such that $\Gamma = \Gamma_a \cup \Gamma_i$, where Γ_a and Γ_i are the accessible and inaccessible parts respectively. The governing diffusion system of our interest is of the form

$$-\boldsymbol{\nabla} \cdot (\alpha \boldsymbol{\nabla} u) = f \quad \text{in } \Omega, \tag{2.1}$$

$$\alpha \frac{\partial u}{\partial n} + \gamma u = \gamma u_a \quad \text{on } \Gamma_a; \quad \alpha \frac{\partial u}{\partial n} = -q \quad \text{on } \Gamma_i, \tag{2.2}$$

where $\boldsymbol{n}$ is the unit outward normal on Γ and the given data include the source $f \in L^2(\Omega)$, the ambient temperature $u_a \in L^2(\Gamma_a)$, the heat transfer coefficient $\gamma > 0$ and the diffusivity coefficient $\alpha > 0$. For simplicity γ and α are both assumed to be constants, but it is straightforward to extend all our analyses and results to the case when both are variable functions. The inverse problem is to recover the distributed flux q, when the partial measurement data z of temperature u is available on Γ_a. We note this problem is highly ill-posed since the Cauchy data z imposed on Γ_a is inevitably contaminated with observation errors in practice [39]. To overcome this difficulty, we often formulate it as a constrained minimization problem with the Tikhonov regularization:

$$\min_{q \in L^2(\Gamma_i)} \mathcal{J}(q) = \frac{1}{2} \|u(q) - z\|_{0,\Gamma_a}^2 + \frac{\beta}{2} \|q\|_{0,\Gamma_i}^2, \tag{2.3}$$

where $u := u(q) \in H^1(\Omega)$ satisfies the variational formulation of (2.1)-(2.2):

$$a(u, \phi) = (f, \phi) + (\gamma u_a, \phi)_{\Gamma_a} - (q, \phi)_{\Gamma_i} \quad \forall\, \phi \in H^1(\Omega) \tag{2.4}$$

and the constant $\beta > 0$ is the regularization parameter. Here $a(\cdot, \cdot) := (\alpha \boldsymbol{\nabla}\cdot, \boldsymbol{\nabla}\cdot) + (\gamma\cdot, \cdot)_{\Gamma_a}$ is a weighted inner product over $H^1(\Omega)$ and its induced norm $\|\cdot\|_a$ is equivalent to the usual H^1-norm due to the Poincaré inequality. There exists a unique minimizer to the system (2.3)-(2.4) [39].

Moreover, with a costate $p^* \in H^1(\Omega)$ involved, the minimizer $(q^*, u^*(q^*))$ is characterized by the following optimality conditions [24]:

$$a(u^*, \phi) = (f, \phi) + (\gamma u_a, \phi)_{\Gamma_a} - (q^*, \phi)_{\Gamma_i} \quad \forall \, \phi \in H^1(\Omega) \tag{2.5}$$

$$a(p^*, v) = (u^* - z, v)_{\Gamma_a} \quad \forall \, v \in H^1(\Omega), \tag{2.6}$$

$$(\beta q^* - p^*, w)_{\Gamma_i} = 0 \quad \forall \, w \in L^2(\Gamma_i). \tag{2.7}$$

Next we introduce a finite element method to approximate the continuous problem (2.3)-(2.4). Let $\mathcal{T}_h$ be a shape-regular conforming triangulation of $\bar{\Omega}$ into a set of closed simplices, with the diameter $h_T := |T|^{1/d}$ for each element $T \in \mathcal{T}_h$. Let V_h be the usual H^1-conforming linear element space over $\mathcal{T}_h$, and $V_{h,\Gamma_i} := V_h|_{\Gamma_i}$ be the feasible discrete space for q. Then the minimization (2.3)-(2.4) is approximated by

$$\min_{q_h \in V_{h,\Gamma_i}} \mathcal{J}_h(q_h) = \frac{1}{2} \|u_h(q_h) - z\|_{0,\Gamma_a}^2 + \frac{\beta}{2} \|q_h\|_{0,\Gamma_i}^2, \tag{2.8}$$

where $u_h := u_h(q_h) \in V_h$ solves the discrete problem

$$a(u_h, \phi_h) = (f, \phi_h) + (\gamma u_a, \phi_h)_{\Gamma_a} - (q_h, \phi_h)_{\Gamma_i} \quad \forall \, \phi_h \in V_h. \tag{2.9}$$

As in the continuous case, there exists a unique minimizer to (2.8)-(2.9), and the minimizer $q_h^* \in V_{h,\Gamma_i}$, the discrete state and costate $u_h^* \in V_h$ and $p_h^* \in V_h$ satisfy the optimality conditions:

$$a(u_h^*, \phi_h) = (f, \phi_h) + (\gamma u_a, \phi_h)_{\Gamma_a} - (q_h^*, \phi_h)_{\Gamma_i} \quad \forall \, \phi_h \in V_h \tag{2.10}$$

$$a(p_h^*, v_h) = (u_h^* - z, v_h)_{\Gamma_a} \quad \forall \, v_h \in V_h, \tag{2.11}$$

$$(\beta q_h^* - p_h^*, w_h)_{\Gamma_i} = 0 \quad \forall \, w_h \in V_{h,\Gamma_i}. \tag{2.12}$$

3 A posteriori error estimation and an adaptive algorithm

In this section we review a residual-type a posteriori error estimate and a related adaptive algorithm developed in [24]. For this purpose, some more notation and definitions are needed.

The collection of all faces (resp. all interior faces) in $\mathcal{T}_h$ is denoted by $\mathcal{F}_h$ (resp. $\mathcal{F}_h(\Omega)$) and its restriction on Γ_a and Γ_i by $\mathcal{F}_h(\Gamma_a)$ and $\mathcal{F}_h(\Gamma_i)$ respectively. The scalar $h_F := |F|^{1/(d-1)}$ stands for the diameter of $F \in \mathcal{F}_h$, which is associated with a fixed normal unit vector $\boldsymbol{n}_F$ in the interior of Ω and $\boldsymbol{n}_F = \boldsymbol{n}$ on the boundary Γ. We use D_T (resp. D_F) for the union of all elements in $\mathcal{T}_h$ with non-empty intersection with element $T \in \mathcal{T}_h$ (resp. $F \in \mathcal{F}_h$). Furthermore, for any $T \in \mathcal{T}_h$ we denote by ω_T the union of elements in $\mathcal{T}_h$ sharing a common face with T, while for any $F \in \mathcal{F}_h(\Omega)$ (resp. $F \in \mathcal{F}_h(\Gamma_a) \cup \mathcal{F}_h(\Gamma_i)$) we denote by ω_F the union of two elements in $\mathcal{T}_h$ sharing the common face F (resp. the element with F as an edge).

For any $(\phi_h, v_h, w_h) \in V_h \times V_h \times V_{h,\Gamma_i}$, we define two element residuals for each $T \in \mathcal{T}_h$ by

$$R_{T,1}(\phi_h) = f + \boldsymbol{\nabla} \cdot (\alpha \boldsymbol{\nabla} \phi_h) \quad \text{and} \quad R_{T,2}(v_h) = -\boldsymbol{\nabla} \cdot (\alpha \boldsymbol{\nabla} v_h) \,,$$

and two face residuals for each face $F \in \mathcal{F}_h$ by

$$J_{F,1}(\phi_h, w_h) = \begin{cases} [\alpha \boldsymbol{\nabla} \phi_h \cdot \boldsymbol{n}_F] & \text{for } F \in \mathcal{F}_h(\Omega), \\ \gamma u_a - \gamma \phi_h - \alpha \boldsymbol{\nabla} \phi_h \cdot \boldsymbol{n}_F & \text{for } F \in \mathcal{F}_h(\Gamma_a), \\ -w_h - \alpha \boldsymbol{\nabla} \phi_h \cdot \boldsymbol{n}_F & \text{for } F \in \mathcal{F}_h(\Gamma_i) \end{cases}$$

and

$$J_{F,2}(v_h, \phi_h) = \begin{cases} [\alpha \boldsymbol{\nabla} v_h \cdot \boldsymbol{n}_F] & \text{for } F \in \mathcal{F}_h(\Omega), \\ \phi_h - z - \gamma v_h - \alpha \boldsymbol{\nabla} v_h \cdot \boldsymbol{n}_F & \text{for } F \in \mathcal{F}_h(\Gamma_a), \\ -\alpha \boldsymbol{\nabla} v_h \cdot \boldsymbol{n}_F & \text{for } F \in \mathcal{F}_h(\Gamma_i), \end{cases}$$

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where $[\alpha\boldsymbol{\nabla}\phi_h \cdot \boldsymbol{n}_F]$ and $[\alpha\boldsymbol{\nabla}v_h \cdot \boldsymbol{n}_F]$ are the jumps across $F \in \mathcal{F}_h$. Then for any $\mathcal{M}_h \subseteq \mathcal{T}_h$, we introduce the error estimator

$$
\begin{aligned}
\eta_h^2(\phi_h, v_h, w_h, f, u_a, z, \mathcal{M}_h) &:= \sum_{T \in \mathcal{M}_h} \eta_{T,h}^2(\phi_h, v_h, w_h, f, u_a, z) \\
&:= \sum_{T \in \mathcal{M}_h} (\eta_{T,h,1}^2(\phi_h, w_h, f, u_a) + \eta_{T,h,2}^2(v_h, \phi_h, z))
\end{aligned}
$$

with

$$
\eta_{T,h,1}^2(\phi_h, w_h, f, u_a) := h_T^2 \|R_{T,1}(\phi_h)\|_{0,T}^2 + \sum_{F \subset \partial T} h_F \|J_{F,1}(\phi_h, w_h)\|_{0,F}^2
$$

and

$$
\eta_{T,h,2}^2(v_h, \phi_h, z) := h_T^2 \|R_{T,2}(v_h)\|_{0,T}^2 + \sum_{F \subset \partial T} h_F \|J_{F,2}(v_h, \phi_h)\|_{0,F}^2 \,,
$$

and the following oscillation errors that involve the given data and the related elementwise projections:

$$
\mathrm{osc}_h^2(f, \mathcal{M}_h) := \sum_{T \in \mathcal{M}_h} h_T^2 \|f - \bar{f}_T\|_{0,T}^2,
$$

$$
\mathrm{osc}_h^2(\phi_h, w_h, \mathcal{S}_h) := \sum_{F \in \mathcal{S}_h} h_F \|J_{F,1} - \bar{J}_{F,1}\|_{0,F}^2,
$$

$$
\mathrm{osc}_h^2(v_h, \phi_h, \mathcal{S}_h) := \sum_{F \in \mathcal{S}_h} h_F \|J_{F,2} - \bar{J}_{F,2}\|_{0,F}^2
$$

for some $\mathcal{M}_h \subseteq \mathcal{T}_h$ and $\mathcal{S}_h \subseteq \mathcal{F}_h$, where $\bar{f}_T$ (resp. $\bar{J}_{F,1}$ and $\bar{J}_{F,2}$) is the integral average of f (resp. $J_{F,1}$ and $J_{F,2}$) over T (resp. F). When $\mathcal{M}_h = \mathcal{T}_h$ or $\mathcal{S}_h = \mathcal{F}_h$, $\mathcal{M}_h$ or $\mathcal{S}_h$ will be dropped in the parameter list of the error estimator or the oscillation errors above.

With the above preparations, we are now ready to present the upper and lower bounds for the errors of the finite element solutions in terms of a residual-type estimator [24].

Theorem 3.1. *Let (u^*, p^*, q^*) and (u_h^*, p_h^*, q_h^*) be the solutions of (2.5)-(2.7) and (2.10)-(2.12) respectively, then there exists a constant C depending on the shape-regularity of $\mathcal{T}_h$ and the coefficients α and γ, such that*

$$
\|u^* - u_h^*\|_1^2 + \|p^* - p_h^*\|_1^2 + \|q^* - q_h^*\|_0^2 \le C\beta^{-2}\eta_h^2(u_h^*, p_h^*, q_h^*, f, u_a, z). \tag{3.1}
$$

Theorem 3.2. *There exists a constant C depending on the shape-regularity of $\mathcal{T}_h$ and the coefficients α and γ, such that for any $T \in \mathcal{T}_h$,*

$$
\begin{aligned}
\eta_{T,h}^2(u_h^*, p_h^*, q_h^*, f, u_a, z) \le C(&\|u^* - u_h^*\|_{0,\omega_T}^2 + \|p^* - p_h^*\|_{0,\omega_T}^2 + \|q^* - q_h^*\|_{0,\partial T \cap \Gamma_i}^2 \\
&\mathrm{osc}_h^2(f, \omega_T) + \mathrm{osc}_h^2(u_h^*, q_h^*, \partial T) + \mathrm{osc}_h^2(p_h^*, u_h^*, \partial T)).
\end{aligned} \tag{3.2}
$$

Based on the error estimators provided in Theorems 3.1 and 3.2 above, the following adaptive algorithm was proposed for the flux reconstruction in [24]. In what follows the dependence on the triangulations is indicated by the number k of the mesh refinements.

Algorithm 3.1. *Given a parameter $\theta \in [0, 1]$ and a conforming initial mesh $\mathcal{T}_0$. Set $k := 0$.*

1. (SOLVE) *Solve the discrete problems (2.10)-(2.12) on $\mathcal{T}_k$ for $(u_k^*, p_k^*, q_k^*) \in V_k \times V_k \times V_{k,\Gamma_i}$.*

2. (ESTIMATE) *Compute the error estimator $\eta_k(u_k^*, p_k^*, q_k^*, f, u_a, z)$.*

3. (MARK) *Mark a subset* $\mathcal{M}_k \subset \mathcal{T}_k$ *such that*

$$\forall\, T \in \mathcal{M}_k \quad \eta_{T,k}(u_k^*, p_k^*, q_k^*, f, u_a, z) \geq \theta \max_{T \in \mathcal{T}_k} \eta_{T,k}(u_k^*, p_k^*, q_k^*, f, u_a, z). \tag{3.3}$$

4. (REFINE) *Refine each triangle* $T \in \mathcal{M}_k$ *by the newest vertex bisection to get* $\mathcal{T}_{k+1}$.

5. *Set* $k := k + 1$ *and go to Step 1.*

A stopping criterion is normally included after step 2 to terminate the iteration, which is omitted here for the notational convenience. The maximum strategy [3], one of the most common marking criteria, is used in the module MARK and we will discuss more about other strategies in section 5.3. In addition, the newest vertex bisection in the module REFINE guarantees the uniform shape-regularity of $\{\mathcal{T}_k\}$ [28] [29] [30] [37] [38] [40]. In other words, all constants only depend on the initial mesh and the given data but not on any particular mesh in the sequel. We point out a practically important feature in our algorithm: the additional marking for oscillation errors and the interior node property for the refinement are both not required, which are needed in the adaptive algorithm for an optimal control problem in [20]. Finally, as the solution $(u_k^*, q_k^*) \in V_k \times V_{k,\Gamma_i}$ is also the minimizer to problem (2.8)-(2.9) with $h = k$, we shall view both of them as the same unless specified otherwise.

The adaptive Algorithm 3.1 was implemented and analysed in [24]. Several nontrivial numerical examples were tested there, with different types of singular fluxes, including fluxes with large jumps, shape-spike fluxes and dipole-like fluxes. From the numerical experiments we have observed that Algorithm 3.1 is able to locate the singularities of fluxes accurately, with the desired local mesh refinements around singularities. Moreover, all the examples in [24] have shown that Algorithm 3.1 ensures the convergence of the flux errors in L^2-norm, even with essentially fewer degrees of freedom than the uniform refinement. The aim of this work is to provide a rigorous mathematical justification of the convergence of the adaptive finite element Algorithm 3.1.

4 A limiting triplet

In this section, we demonstrate the convergence of the sequence $\{(u_k^*, p_k^*, q_k^*)\}$ generated by Algorithm 3.1. To this end, with $\{V_k\}$ and $\{V_{k,\Gamma_i}\}$ induced by Algorithm 3.1, we define two limiting spaces:

$$V_\infty := \overline{\bigcup_{k \geq 0} V_k} \ \ (\text{in } H^1\text{-norm}) \quad and \quad Q_\infty := \overline{\bigcup_{k \geq 0} V_{k,\Gamma_i}} \ \ (\text{in } L^2\text{-norm}).$$

We remark that V_∞ and Q_∞ are a closed subspace of $H^1(\Omega)$ and $L^2(\Gamma_i)$ respectively. Then we introduce a constrained minimization problem over Q_∞:

$$\min_{q \in Q_\infty} \mathcal{J}_\infty(q) = \frac{1}{2}\|u_\infty(q) - z\|_{0,\Gamma_a}^2 + \frac{\beta}{2}\|q\|_{0,\Gamma_i}^2, \tag{4.1}$$

where $u_\infty := u_\infty(q) \in V_\infty$ satisfies the variational problem:

$$a(u_\infty, \phi) = (f, \phi) + (\gamma u_a, \phi)_{\Gamma_a} - (q, \phi)_{\Gamma_i} \quad \forall\, \phi \in V_\infty. \tag{4.2}$$

Following the arguments of [39] for the system (2.3)-(2.4), we can show that there exists a unique minimizer to the optimization problem (4.1)-(4.2).

Next we present the first result of this section, namely the sequence q_k^* generated by Algorithm 3.1 converges strongly to the minimizer q_∞^* of problem (4.1)-(4.2). For the purpose we need some auxiliary results.

Lemma 4.1. *Let $\{V_k \times V_{k,\Gamma_i}\}$ be a sequence of discrete spaces generated by Algorithm 3.1. If the sequence $\{q_k\} \subset \bigcup_{k \geq 0} V_{k,\Gamma_i}$ weakly converges to some $q^* \in Q_\infty$ in $L^2(\Gamma_i)$, then there exists a subsequence $\{q_m\}$ with $m = k_n$, such that for the sequence $\{u_m(q_m)\} \subset \bigcup_{k \geq 0} V_k$ produced by (2.9) with h replaced by m and $u_\infty(q^*) \in V_\infty$ generated by (4.2) with $q = q^*$ there holds*

$$u_m(q_m) \to u_\infty(q^*) \quad in \ L^2(\Gamma). \tag{4.3}$$

Proof. Taking $\phi_k = u_k(q_k)$ in (2.9), we immediately know that $\|u_k(q_k)\|_1$ is uniformly bounded independently of k and hence there exist a subsequence, denoted by $\{u_m(q_m)\}$ with $m = k_n$, and some $u^* \in H^1(\Omega)$ such that

$$u_m(q_m) \to u^* \quad \text{weakly in } H^1(\Omega), \qquad u_m(q_m) \to u^* \quad \text{in } L^2(\Gamma). \tag{4.4}$$

We only need to show $u^* = u_\infty(q^*)$. As V_∞ is weakly closed, $u^* \in V_\infty$. For any positive integer l, when we choose $m \geq l$, we know from (2.9) that

$$(\alpha \nabla u_m(q_m), \nabla \phi_l) + (\gamma u_m(q_m), \phi_l)_{\Gamma_a} = (f, \phi_l) + (\gamma u_a, \phi_l)_{\Gamma_a} - (q_m, \phi_l)_{\Gamma_i} \quad \forall \ \phi_l \in V_l.$$

Letting m go to infinity and noting the convergence results in (4.4) as well as the weak convergence of $\{q_k\}$, we find

$$(\alpha \nabla u^*, \nabla \phi_l) + (\gamma u^*, \phi_l)_{\Gamma_a} = (f, \phi_l) + (\gamma u_a, \phi_l)_{\Gamma_a} - (q^*, \phi_l)_{\Gamma_i} \quad \forall \ \phi_l \in V_l.$$

As l and $\phi_l \in V_l$ are arbitrary, we further obtain

$$(\alpha \nabla u^*, \nabla \phi) + (\gamma u^*, \phi)_{\Gamma_a} = (f, \phi) + (\gamma u_a, \phi)_{\Gamma_a} - (q^*, \phi)_{\Gamma_i} \quad \forall \ \phi \in V_\infty,$$

which leads to the desired conclusion. □

Lemma 4.2. *Let $\{V_k \times V_{k,\Gamma_i}\}$ be a sequence of discrete spaces generated by Algorithm 3.1. If the sequence $\{q_k\} \subset \bigcup_{k \geq 0} V_{k,\Gamma_i}$ strongly converges to some $q^* \in Q_\infty$ in $L^2(\Gamma_i)$, then for the sequence $\{u_k(q_k)\} \subset \bigcup_{k \geq 0} V_k$ given by (2.9) with h replaced by k and $u_\infty(q^*) \in V_\infty$ given by (4.2) with $q = q^*$ there holds*

$$u_k(q_k) \to u_\infty(q^*) \quad in \ H^1(\Omega). \tag{4.5}$$

Proof. We begin with an auxiliary discrete problem: Find $u_k(q^*) \in V_k$ such that

$$a(u_k(q^*), \phi) = (f, \phi) + (\gamma u_a, \phi)_{\Gamma_a} - (q^*, \phi)_{\Gamma_i} \quad \forall \ \phi \in V_k. \tag{4.6}$$

Subtracting (4.6) from (2.9) with $\phi = u_k(q_k) - u_k(q^*)$ and using the trace theorem as well as the norm equivalence we come to the estimate

$$\|u_k(q^*) - u_k(q_k)\|_1 \leq C \|q^* - q_k\|_{0,\Gamma_i}.$$

On the other hand, we note that (4.6) is a finite element approximation of (4.2) with $q = q^* \in Q_\infty$, so the Cea's lemma admits an optimal approximation property

$$\|u_\infty(q^*) - u_k(q^*)\|_1 \leq C \inf_{v \in V_k} \|u_\infty(q^*) - v\|_1.$$

Finally, the desired convergence (4.5) is the consequence of the above two estimates and the density of $\bigcup_{k \geq 0} V_k$ in V_∞. □

Now we are in a position to show the first main result of this section.

Theorem 4.1. *Let $\{V_k \times V_{k,\Gamma_i}\}$ be a sequence of discrete spaces generated by Algorithm 3.1, and $\{q_k^*\}$ be the corresponding sequence of minimizers to the discrete problem (2.8)-(2.9) when finite element spaces V_h and V_{h,Γ_i} are replaced by V_k and V_{k,Γ_i} (and functional $\mathcal{J}_h$ will be denoted by $\mathcal{J}_k$ accordingly). Then the whole sequence $\{q_k^*\}$ converges strongly in $L^2(\Gamma_i)$ to the unique minimizer q_∞^* of the problem (4.1)-(4.2).*

Proof. The fact that $\|q_k^*\|_{0,\Gamma_i}$ is uniformly bounded implies there exist a subsequence, also denoted by $\{q_k^*\}$ and some $q^* \in Q_\infty$ such that

$$q_k^* \rightarrow q^* \quad \text{weakly in } L^2(\Gamma_i). \tag{4.7}$$

Then from Lemma 4.1, we know by extracting a subsequence with $m = k_n$ that

$$u_m(q_m^*) \rightarrow u_\infty(q^*) \quad \text{in } L^2(\Gamma_a). \tag{4.8}$$

On the other hand, for any $q \in Q_\infty$ there exists a sequence $\{q_l\} \subset \bigcup_{l \geq 0} V_{l,\Gamma_i}$ such that

$$\lim_{l \to \infty} \|q_l - q\|_{0,\Gamma_i} = 0, \tag{4.9}$$

which, by Lemma 4.2 and the trace theorem, implies

$$\lim_{l \to \infty} \|u_l(q_l) - z\|_{0,\Gamma_a}^2 = \|u_\infty(q) - z\|_{0,\Gamma_a}^2. \tag{4.10}$$

Choosing $k \geq l$ for sufficiently large l and noting the whole sequence $\{q_k^*\}$ are minimizers of $\mathcal{J}_k$ over $\{V_{k,\Gamma_i}\}$, we can derive

$$\mathcal{J}_k(q_k^*) \leq \mathcal{J}_l(q_l) = \frac{1}{2}\|u_l(q_l) - z\|_{0,\Gamma_a}^2 + \frac{\beta}{2}\|q_l\|_{0,\Gamma_i}^2.$$

Then a collection of (4.7)-(4.10) gives

$$\mathcal{J}_\infty(q^*) = \frac{1}{2}\|u_\infty(q^*) - z\|_{0,\Gamma_a}^2 + \frac{\beta}{2}\|q^*\|_{0,\Gamma_i}^2$$

$$\leq \lim_{m \to \infty} \frac{1}{2}\|u_m(q_m^*) - z\|_{0,\Gamma_a}^2 + \liminf_{m \to \infty} \frac{\beta}{2}\|q_m^*\|_{0,\Gamma_i}^2$$

$$\leq \liminf_{m \to \infty} \mathcal{J}_m(q_m^*) \leq \limsup_{m \to \infty} \mathcal{J}_m(q_m^*) \leq \limsup_{k \to \infty} \mathcal{J}_k(q_k^*) \leq \limsup_{l \to \infty} \mathcal{J}_l(q_l) = \mathcal{J}_\infty(q) \quad \forall\, q \in Q_\infty,$$

which indicates that $q^* = q_\infty^*$ is the unique minimizer of the problem (4.1)-(4.2). Then the whole sequence $\{q_k^*\}$ converges weakly to q_∞^*. Moreover the choice $q = q^*$ in the above estimate yields equality $\lim_{m \to \infty} \mathcal{J}_m(q_m^*) = \mathcal{J}_\infty(q^*) = \inf \mathcal{J}_\infty(Q_\infty)$ and it follows that $\lim_{k \to \infty} \mathcal{J}_k(q_k^*) = \inf \mathcal{J}_\infty(Q_\infty)$ for the whole sequence $\{q_k^*\}$. Similarly, the strong convergence in (4.8) also holds true for the whole sequence $\{u_k^*(q_k^*)\}$. These two facts guarantee that $\lim_{k \to \infty} \|q_k^*\|_{0,\Gamma_i}^2 = \|q_\infty^*\|_{0,\Gamma_i}^2$, which, along with the weak convergence, implies the strong convergence. □

Like the continuous case, after we introduce a Lagrangian multiplier $p_\infty \in V_\infty$ to relax the constraint (4.2), the minimization problem (4.1) is recast as a saddle-point problem of the following Lagrangian functional over $V_\infty \times V_\infty \times Q_\infty$:

$$\mathcal{L}(u_\infty, p_\infty, q) = \frac{1}{2}\|u_\infty - z\|_{0,\Gamma_a}^2 + \frac{\beta}{2}\|q\|_{0,\Gamma_i}^2 - a(u_\infty, p_\infty) + (f, p_\infty) + (\gamma u_a, p_\infty)_{\Gamma_a} - (q, p_\infty)_{\Gamma_i}.$$

The minimizer q_∞^* of (4.1) and the related state u_∞^* are determined by the following system:

$$a(u_\infty^*, \phi) = (f, \phi) + (\gamma u_a, \phi)_{\Gamma_a} - (q_\infty^*, \phi)_{\Gamma_i} \quad \forall\, \phi \in V_\infty \tag{4.11}$$

$$a(p_\infty^*, v) = (u_\infty^* - z, v)_{\Gamma_a} \quad \forall\, v \in V_\infty, \tag{4.12}$$

$$(\beta q_\infty^* - p_\infty^*, w)_{\Gamma_i} = 0 \quad \forall\, w \in Q_\infty. \tag{4.13}$$

Finally for the above system, we have the second main result of this section.

Theorem 4.2. *Let $\{V_k \times V_{k,\Gamma_i}\}$ be a sequence of discrete spaces generated by Algorithm 3.1, then the sequence $\{(u_k^*, p_k^*, q_k^*)\}$ of discrete solutions converges to $(u_\infty^*, p_\infty^*, q_\infty^*)$, the solution of the problem (4.11)-(4.13), in the following sense:*

$$\|u_k^* - u_\infty^*\|_1 \to 0, \quad \|p_k^* - p_\infty^*\|_1 \to 0, \quad \|q_k^* - q_\infty^*\|_{0,\Gamma_i} \to 0 \quad as \ k \to \infty. \tag{4.14}$$

Proof. The third convergence follows directly from Theorem 4.1. Then by Lemma 4.2 we obtain the first result. It remains to show the second one. We introduce an auxiliary problem: Find $\widetilde{p}_k \in V_k$ such that

$$(\alpha \boldsymbol{\nabla} \widetilde{p}_k, \boldsymbol{\nabla} v) + (\gamma \widetilde{p}_k, v)_{\Gamma_a} = (u_\infty^* - z, v)_{\Gamma_a} \quad \forall \ v \in V_k. \tag{4.15}$$

Combining (2.11) and (4.15) with $v = \widetilde{p}_k - p_k^*$ and using the trace theorem as well as the norm equivalence we obtain

$$\|\widetilde{p}_k - p_k^*\|_1 \le C \|u_k^* - u_\infty^*\|_1. \tag{4.16}$$

On the other hand, it is not difficult to find that the problem (4.15) is a discrete version of (4.12). Hence the Cea's lemma gives

$$\|p_\infty^* - \widetilde{p}_k\|_1 \le C \inf_{v \in V_k} \|p_\infty^* - v\|_1. \tag{4.17}$$

The desired result comes readily from (4.16), (4.17), the first convergence in (4.14) and the construction of V_∞. $\qquad\square$

Remark 4.1. As a matter of fact, the convergence results in Theorem 4.2 have no connections with any particular strategy adopted in the module MARK as in the case of linear elliptic problems [32] [33] [35]. So other marking strategies work also here; see section 5.3 for details.

5 Convergence

In this section, we shall establish the convergence of Algorithm 3.1 in the following senses: (1) the discrete solutions $\{(u_k^*, p_k^*, q_k^*)\}$ converge strongly to the true solution of the problem (2.5)-(2.7); (2) the error estimator η_k converges to zero. With some properties of adaptively generated triangulations and the error estimator stated in section 5.1, the proof of our main results is presented in section 5.2. We will discuss the generalizations of the current arguments to other marking strategies in section 5.3.

5.1 Preliminaries

We first introduce a convenient classification of all elements generated during an adaptive algorithm. For each mesh $\mathcal{T}_k$, we define [35]:

$$\mathcal{T}_k^+ := \bigcap_{l \ge k} \mathcal{T}_l \quad \text{and} \quad \mathcal{T}_k^0 := \mathcal{T}_k \setminus \mathcal{T}_k^+.$$

So $\mathcal{T}_k^+$ consists of all elements not refined after the k-th iteration and the sequence $\{\mathcal{T}_k^+\}$ satisfies $\mathcal{T}_l^+ \subset \mathcal{T}_k^+$ for all $k > l$. On the other hand, all elements in $\mathcal{T}_k^0$ are refined at least once after the k-th iteration, that is to say for any $T \in \mathcal{T}_k^0$, there exists $l \ge k$ such that $T \in \mathcal{T}_l$ but $T \notin \mathcal{T}_{l+1}$. Correspondingly, the domain Ω is split into two parts covered by $\mathcal{T}_k^+$ and $\mathcal{T}_k^0$ respectively, i.e.

$$\bar{\Omega} = \Omega(\mathcal{T}_k^+) \cup \Omega(\mathcal{T}_k^0) =: \Omega_k^+ \cup \Omega_k^0.$$

We also define a mesh-size function $h_k : \bar{\Omega} \to \mathbb{R}^+$ almost everywhere by $h_k(x) = h_T$ for x in the interior of an element $T \in \mathcal{T}_k$ and $h_k(x) = h_F$ for x in the relative interior of a face $F \in \mathcal{F}_k$. It is clear that the sequence $\{h_k\}$ given by Algorithm 3.1 strictly decreases on the region refined by the newest vertex bisection. In fact, we have the following observations (Corollary 3.3, [35]).

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Lemma 5.1. *Let χ_k^0 be the characteristic function of Ω_k^0, then the definition of $\mathcal{T}_k^0$ implies that*

$$\lim_{k\to\infty} \|h_k \chi_k^0\|_\infty = \lim_{k\to\infty} \|h_k\|_{\infty,\Omega_k^0} = 0. \tag{5.1}$$

In the subsequent convergence analysis, the sum of $\eta_{T,k,1}$ and $\eta_{T,k,2}$ over $\mathcal{T}_k$ will be split by $\mathcal{T}_k^0$ and $\mathcal{T}_k^+$, and with the help of Lemma 5.1 and the local approximation properties of a classic nodal interpolation operator [15], we are able to control the relevant residual in Ω_k^0 (see the proof of Lemma 5.3 below). For the remaining part, we have to resort to the marking strategy (3.3), which implies that the maximal error indicator in $\mathcal{T}_k \setminus \mathcal{M}_k$ is dominated by the maximal error indicator in $\mathcal{M}_k$. Therefore it is necessary to study only the convergence behavior of the latter.

Lemma 5.2. *Let $\{\mathcal{T}_k, V_k \times V_{k,\Gamma_i}, (u_k^*, p_k^*, q_k^*)\}$ be the sequence of meshes, finite element spaces and discrete solutions produced by Algorithm 3.1 and $\mathcal{M}_k$ the set of marked elements given by (3.3). Then the following convergence holds*

$$\lim_{k\to\infty} \max_{T\in\mathcal{M}_k} \eta_{T,k}(u_k^*, p_k^*, q_k^*, f, u_a, z) = 0. \tag{5.2}$$

Proof. Let $\widetilde{T}$ be the element where the error indicator attains the maximum among $\mathcal{M}_k$. As $\widetilde{T} \in \mathcal{M}_k \subset \mathcal{T}_k^0$, the local quasi-uniformity of $\mathcal{T}_k$ and Lemma 5.1 tell that

$$|D_{\widetilde{T}}| \le C|\widetilde{T}| \le C\|h_k^d\|_{\infty,\Omega_k^0} \to 0 \quad \text{as } k\to\infty. \tag{5.3}$$

By means of a trace theorem, the inverse estimate and the triangle inequality, we can estimate the error indicator $\eta_{\widetilde{T},k} := (\eta_{\widetilde{T},k,1}^2 + \eta_{\widetilde{T},k,2}^2)^{1/2}$ as follows

$$
\begin{aligned}
\eta_{\widetilde{T},k,1}^2(u_k^*, q_k^*, f, u_a) &\le C(\|u_k^*\|_{1,D_{\widetilde{T}}}^2 + h_{\widetilde{T}}\|q_k^*\|_{0,\partial\widetilde{T}\cap\Gamma_i}^2 + h_{\widetilde{T}}^2\|f\|_{0,\widetilde{T}}^2 + h_{\widetilde{T}}\|u_a\|_{0,\partial\widetilde{T}\cap\Gamma_a}^2) \\
&\le C(\|u_k^* - u_\infty^*\|_1^2 + \|u_\infty^*\|_{1,D_{\widetilde{T}}}^2 + \|q_k^* - q_\infty^*\|_{0,\Gamma_i}^2 + h_{\widetilde{T}}\|q_\infty^*\|_{0,\partial\widetilde{T}\cap\Gamma_i}^2 \\
&\qquad + h_{\widetilde{T}}^2\|f\|_{0,\widetilde{T}}^2 + h_{\widetilde{T}}\|u_a\|_{0,\partial\widetilde{T}\cap\Gamma_a}^2), \\
\eta_{\widetilde{T},k,2}^2(p_k^*, u_k^*, z) &\le C(\|p_k^*\|_{1,D_{\widetilde{T}}}^2 + \|u_k^*\|_{1,D_{\widetilde{T}}}^2 + h_{\widetilde{T}}\|z\|_{0,\partial\widetilde{T}\cap\Gamma_a}^2) \\
&\le C(\|p_k^* - p_\infty^*\|_1^2 + \|p_\infty^*\|_{1,D_{\widetilde{T}}}^2 + \|u_k^* - u_\infty^*\|_1^2 + \|u_\infty^*\|_{1,D_{\widetilde{T}}}^2 + h_{\widetilde{T}}\|z\|_{0,\partial\widetilde{T}\cap\Gamma_a}^2).
\end{aligned}
$$

Now the result follows from (4.14), (5.3) and the absolute continuity of $\|\cdot\|_1$ and $\|\cdot\|_{0,\Gamma}$ with respect to the Lebesgue measure. $\qquad\square$

Remark 5.1. By inverse estimates we can deduce the following stability estimates for any $T \in \mathcal{T}_k$:

$$
\begin{aligned}
\eta_{T,k,1}(u_k^*, q_k^*, f, u_a) &\le C(\|u_k^*\|_{1,D_T} + \|q_k^*\|_{0,\partial T\cap\Gamma_i} + \|f\|_{0,T} + \|u_a\|_{0,\partial T\cap\Gamma_a}), \tag{5.4} \\
\eta_{T,k,2}(p_k^*, u_k^*, z) &\le C(\|p_k^*\|_{1,D_T} + \|u_k^*\|_{1,D_T} + \|z\|_{0,\partial T\cap\Gamma_a}). \tag{5.5}
\end{aligned}
$$

Remark 5.2. From the proof of Lemma 5.2, we know that the maximum strategy (3.3) in the module MARK is not utilized. Therefore this lemma is valid also for other markings.

5.2 Main results

Now we turn our attention to the main results of this work. It is not difficult to know that once we can prove the solution triplet $(u_\infty^*, p_\infty^*, q_\infty^*)$ to the system (4.11)-(4.13) is the exact solution triplet (u^*, p^*, q^*) to the system (2.5)-(2.7) in some appropriate norm, then our expected first convergence result, namely the sequence of discrete solutions $\{(u_k^*, p_k^*, q_k^*)\}$ generated by Algorithm 3.1 converges strongly to the true solution of the problem (2.5)-(2.7), will follow immediately from Theorem 4.2.

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To do so, we shall first show the two residuals with respect to u_k^* as well as p_k^* have weak vanishing limits for u_∞^* and p_∞^* (see Lemmas 5.3 and 5.4). It is worth noting that compared with the case of the direct boundary value problems, the inverse problem under consideration involves a major difficulty, i.e., u^* and u_∞^* are determined by different fluxes q^* and q_∞^* respectively. To overcome the difficulty, we define an auxiliary pair $(u(q_\infty^*), p(q_\infty^*))$ through (2.5)-(2.6) with q^* replaced by q_∞^*. Then we will show that the pair $(u(q_\infty^*), p(q_\infty^*))$ is the same as the limiting pair (u_∞^*, p_∞^*).

As stated above, the first two residuals with respect to $u_k^*(q_k^*)$ and $p_k^*(q_k^*)$ are defined by

$$\langle \mathcal{R}(u_k^*), \phi \rangle := (f, \phi) + (\gamma u_a, \phi)_{\Gamma_a} - (q_k^*, \phi)_{\Gamma_i} - a(u_k^*, \phi) \quad \forall\, \phi \in H^1(\Omega),$$

$$\langle \mathcal{R}(p_k^*), v \rangle := (u_k^* - z, v)_{\Gamma_a} - a(p_k^*, v) \quad \forall\, v \in H^1(\Omega).$$

Since $\{q_k^*\}$ is a converging sequence of minimizers by Theorem 4.1, it is uniformly bounded in $L^2(\Gamma_i)$, so are $\{u_k^*\}$ and $\{p_k^*\}$ in $H^1(\Omega)$ by means of (2.10) and (2.11). Thus, we know $\{\mathcal{R}(u_k^*)\}$ and $\{\mathcal{R}(p_k^*)\}$ are two sequences of uniformly bounded linear functionals in $H^1(\Omega)'$, namely there exist two constants independent of k such that

$$\|\mathcal{R}(u_k^*)\|_{H^1(\Omega)'} \leq C_{un1}, \quad \|\mathcal{R}(p_k^*)\|_{H^1(\Omega)'} \leq C_{un2}. \tag{5.6}$$

In addition, we can easily observe from (2.10) and (2.11) that

$$\langle \mathcal{R}(u_k^*), v \rangle = 0 \quad \text{and} \quad \langle \mathcal{R}(p_k^*), v \rangle = 0 \quad \forall\, v \in V_k. \tag{5.7}$$

Using these relations, we can establish the following weak convergence.

Lemma 5.3. *The sequence $\{(u_k^*, p_k^*, q_k^*)\}$ produced by Algorithm 3.1 satisfies*

$$\lim_{k \to \infty} \langle \mathcal{R}(u_k^*), \phi \rangle = 0, \quad \lim_{k \to \infty} \langle \mathcal{R}(p_k^*), \phi \rangle = 0 \quad \forall\, \phi \in H^1(\Omega). \tag{5.8}$$

Proof. We prove only the first result by borrowing some techniques from [35], as the second convergence can be done in the same manner. We easily see that $\mathcal{T}_l^+ \subset \mathcal{T}_k^+ \subset \mathcal{T}_k$ for $k > l$. This implies $\Omega_l^0 = \Omega(\mathcal{T}_k \setminus \mathcal{T}_l^+) := \bigcup \{T \in \mathcal{T}_k, T \notin \mathcal{T}_l^+\}$ and any refinement of $\mathcal{T}_k$ does not affect any element in $\mathcal{T}_l^+$. Now we set $\Omega_l^* := \bigcup \{T \in \mathcal{T}_k, T \cap \Omega_l^0 \neq \emptyset\}$ and $\Omega_k^\times := \bigcup \{T \in \mathcal{T}_k, T \cap \Omega_l^+ \neq \emptyset\}$, and write I_k and I_k^{sz} for the Lagrange and Scott-Zhang interpolations respectively associated with V_k [15] [34]. Then for any $\psi \in C^\infty(\bar{\Omega})$, we can derive for $w = \psi - I_k \psi \in H^1(\Omega)$ by using the orthogonality (5.7) and elementwise integration by parts that

$$
\begin{aligned}
&|\langle \mathcal{R}(u_k^*), \psi \rangle| = |\langle \mathcal{R}(u_k^*), \psi - I_k \psi \rangle| = |\langle \mathcal{R}(u_k^*), w - I_k^{sz} w \rangle| \\
&\leq C \sum_{T \in \mathcal{T}_k} \eta_{T,k,1}(u_k^*, q_k^*, f, u_a) \|\psi - I_k \psi\|_{1,D_T} \\
&= C \Big(\sum_{T \in \mathcal{T}_k \setminus \mathcal{T}_l^+} \eta_{T,k,1}(u_k^*, q_k^*, f, u_a) \|\psi - I_k \psi\|_{1,D_T} + \sum_{T \in \mathcal{T}_l^+} \eta_{T,k,1}(u_k^*, q_k^*, f, u_a) \|\psi - I_k \psi\|_{1,D_T} \Big).
\end{aligned}
\tag{5.9}
$$

Using (5.4) and the uniform boundedness of $\|u_k^*\|_1$ and $\|q_k^*\|_{0,\Gamma_i}$, we have

$$\Big(\sum_{T \in \mathcal{T}_k \setminus \mathcal{T}_l^+} \eta_{T,k,1}^2(u_k^*, q_k^*, f, u_a) \Big)^{1/2} \leq C(\|u_k^*\|_1 + \|q_k^*\|_{0,\Gamma_i} + \|f\|_0 + \|u_a\|_{\Gamma_a}) \leq \widetilde{C} \tag{5.10}$$

where $\widetilde{C}$ is independent of k. Furthermore, we can apply the local interpolation error estimate for I_k [15] and the monotonicity of the mesh-size function h_k to obtain

$$\|w - I_k \psi\|_{1,\Omega_l^*} \leq C\|h_l\|_{\infty,\Omega_l^*}\|\psi\|_2, \quad \|\psi - I_k \psi\|_{1,\Omega_k^\times} \leq C\|h_l\|_{\infty,\Omega_k^\times}\|\psi\|_2 \leq C\|\psi\|_2. \tag{5.11}$$

Now it follows readily from (5.9)-(5.11) and the local quasi-uniformity of $\mathcal{T}_l$ that for any $k > l$,

$$|\langle \mathcal{R}(u_k), \psi \rangle| \le C_1 \|\psi\|_2 \|h_l\|_{\infty, \Omega_l^0} + C_2 \|\psi\|_2 \big(\sum_{T \in \mathcal{T}_l^+} \eta_{T,k,1}^2(u_k^*, q_k^*, f, u_a) \big)^{1/2}. \tag{5.12}$$

To proceed our estimation, we can choose for any given $\varepsilon > 0$ some sufficiently large l by using Lemma 5.1 such that

$$\|h_l\|_{\infty, \Omega_l^0} \le \frac{\varepsilon}{2C_1 \|\psi\|_2}. \tag{5.13}$$

In addition, the marking strategy (3.3) and Lemma 5.2 ensure that

$$\lim_{k \to \infty} \max_{T \in \mathcal{T}_k \backslash \mathcal{M}_k} \eta_{T,k}(u_k^*, p_k^*, q_k^*, f, u_a, z) \le \lim_{k \to \infty} \max_{T \in \mathcal{M}_k} \eta_{T,k}(u_k^*, p_k^*, q_k^*, f, u_a, z) = 0,$$

which, together with $\mathcal{T}_l^+ \cap \mathcal{M}_k = \emptyset$, implies

$$\lim_{k \to \infty} \max_{T \in \mathcal{T}_l^+} \eta_{T,k}(u_k^*, p_k^*, q_k^*, f, u_a, z) = 0.$$

Therefore, we can choose $K > l$ for some fixed l such that when $k \ge K$,

$$\max_{T \in \mathcal{T}_l^+} \eta_{T,k,1}(u_k^*, q_k^*, f, u_a) \le \max_{T \in \mathcal{T}_l^+} \eta_{T,k}(u_k^*, p_k^*, q_k^*, f, u_a, z) \le \frac{\varepsilon}{2C_2 \|\psi\|_2} |\mathcal{T}_l^+|^{-\frac{1}{2}}. \tag{5.14}$$

Then we can see from (5.12)-(5.14) that $\langle \mathcal{R}(u_k), \psi \rangle$ is controlled by ε for any $k \ge K$ and $\psi \in C^\infty(\bar{\Omega})$, i.e.,

$$\lim_{k \to \infty} \langle \mathcal{R}(u_k^*), \psi \rangle = 0 \quad \forall \, \psi \in C^\infty(\bar{\Omega}). \tag{5.15}$$

This gives the first convergence in (5.8) by the density of $C^\infty(\bar{\Omega})$ in $H^1(\Omega)$. $\qquad \square$

Remark 5.3. One may see from the second estimate in (5.14) that for a fixed l,

$$\lim_{k \to \infty} \eta_k(u_k^*, p_k^*, q_k^*, f, u_a, z, \mathcal{T}_l^+) = 0. \tag{5.16}$$

This observation will be used in the subsequent proof of Theorem 5.2.

Lemma 5.3 yields a important direct consequence. Indeed, we know from (4.14) that for any $\phi \in H^1(\Omega)$ and $v \in H^1(\Omega)$,

$$\begin{aligned}
\langle \mathcal{R}(u_\infty^*), \phi \rangle &:= (f, \phi) + (\gamma u_a, \phi)_{\Gamma_a} - (q_\infty^*, \phi)_{\Gamma_i} - a(u_\infty^*, \phi) = \lim_{k \to \infty} \langle \mathcal{R}(u_k^*), \phi \rangle, \\
\langle \mathcal{R}(p_\infty^*), v \rangle &:= (u_\infty^* - z, v)_{\Gamma_a} - a(p_\infty^*, v) = \lim_{k \to \infty} \langle \mathcal{R}(p_k^*), v \rangle.
\end{aligned}$$

Then the application of Lemma 5.3 leads readily to the following results about the vanishing residuals associated with $u_\infty^*(q_\infty^*)$ and $p_\infty^*(q_\infty^*)$.

Lemma 5.4. *The solution of the problem* (4.11)-(4.13) *satisfies*

$$\langle \mathcal{R}(u_\infty^*), \phi \rangle = 0 \quad and \quad \langle \mathcal{R}(p_\infty^*), \phi \rangle = 0 \quad \forall \, \phi \in H^1(\Omega). \tag{5.17}$$

To continue our analysis, we now introduce two auxiliary continuous problems:
Find $u(q_\infty^*) \in H^1(\Omega)$ and $p(q_\infty^*) \in H^1(\Omega)$ such that

$$\begin{aligned}
a(u(q_\infty^*), \phi) &= (f, \phi) + (\gamma u_a, \phi)_{\Gamma_a} - (q_\infty^*, \phi)_{\Gamma_i} \quad \forall \, \phi \in H^1(\Omega), &\tag{5.18} \\
a(p(q_\infty^*), v) &= (u(q_\infty^*) - z, v)_{\Gamma_a} \quad \forall \, v \in H^1(\Omega). &\tag{5.19}
\end{aligned}$$

Lemma 5.5. *For the solution* $(u_\infty^*, p_\infty^*, q_\infty^*)$ *of the problem* (4.11)-(4.13) *and the solutions* $u(q_\infty^*)$, $p(q_\infty^*)$ *of the problems* (5.18) *and* (5.19)*, there hold that*

$$u_\infty^* = u(q_\infty^*) \quad and \quad p_\infty^* = p(q_\infty^*) \quad in \ H^1(\Omega). \tag{5.20}$$

Proof. The Poincaré inequality, (5.18) and Lemma 5.4 yield that

$$C\|u(q_\infty^*) - u_\infty^*\|_1 \leq \sup_{\|\phi\|_1=1} a(u(q_\infty^*) - u_\infty^*, \phi) = \sup_{\|\phi\|_1=1} \langle \mathcal{R}(u_\infty^*), \phi \rangle = 0,$$

so the first equality is proved. Then the second equality in (5.20) follows from the first result, (5.19) and the following estimates:

$$C\|p(q_\infty^*) - p_\infty^*(q_\infty^*)\|_1 \leq \sup_{\|v\|_1=1} a(p(q_\infty^*) - p_\infty^*, v) = \sup_{\|v\|_1=1} \left\{ (u(q_\infty^*) - z, v)_{\Gamma_a} - a(p_\infty^*, v) \right\}$$

$$= \sup_{\|v\|_1=1} \left\{ (u_\infty^*(q_\infty^*) - z, v)_{\Gamma_a} - a(p_\infty^*, v) \right\} = \sup_{\|v\|_1=1} \langle \mathcal{R}(p_\infty^*), v \rangle = 0.$$

$\square$

Now we are ready to present the first main result in this paper.

Theorem 5.1. *Let* (u^*, p^*, q^*) *be the solution of the problem* (2.5)-(2.7)*. Then Algorithm 3.1 produces a sequence of discrete solutions* (u_k^*, p_k^*, q_k^*) *which converge to* (u^*, p^*, q^*) *in the following sense*

$$\lim_{k \to \infty} \|u^* - u_k^*\|_1 = 0, \quad \lim_{k \to \infty} \|p^* - p_k^*\|_1 = 0, \quad \lim_{k \to \infty} \|q^* - q_k^*\|_{0,\Gamma_i} = 0. \tag{5.21}$$

Proof. We first show $q^* = q_\infty^*$, which, together with Theorem 4.2, leads to the third convergence. By means of the definition of Q_∞ in section 4, the trace theorem and the density of $\bigcup_{k \geq 0} V_k$ in V_∞, it is not difficult to get $p_\infty^*|_{\Gamma_i} \in Q_\infty$. Then there exists a sequence $\{p_k\} \subset \bigcup_{k \geq 0} V_k$ such that $p_k \to p_\infty^*$ in $H^1(\Omega)$, which, together with the trace theorem, allows

$$p_k|_{\Gamma_i} \to p_\infty^*|_{\Gamma_i} \quad in \ L^2(\Gamma_i).$$

Thus we have from (2.7) and (4.13) that

$$\beta q^* = p^*, \quad \beta q_\infty^* = p_\infty^* \quad on \ \Gamma_i. \tag{5.22}$$

On the other hand, we deduce from (2.5)-(2.6) and (5.18)-(5.19) that

$$a(u(q_\infty^*) - u^*, \phi) = (q^* - q_\infty^*, \phi)_{\Gamma_i} \quad \forall \ \phi \in H^1(\Omega), \tag{5.23}$$

$$a(p(q_\infty^*) - p^*, v) = (u(q_\infty^*) - u^*, v)_{\Gamma_a} \quad \forall \ v \in H^1(\Omega). \tag{5.24}$$

By taking $\phi = p(q_\infty^*) - p^*$ and $v = u(q_\infty^*) - u^*$ respectively in (5.23) and (5.24), we derive

$$\|u(q_\infty^*) - u^*\|_{0,\Gamma_a}^2 = (q^* - q_\infty^*, p(q_\infty^*) - p^*)_{\Gamma_i}.$$

With (5.22), we are further led to

$$\beta\|q^* - q_\infty^*\|_{0,\Gamma_i}^2 + \|u(q_\infty^*) - u^*\|_{0,\Gamma_a}^2 = (q^* - q_\infty^*, \beta q^* - \beta q_\infty^* + p(q_\infty^*) - p^*)_{\Gamma_i}$$

$$= (q^* - q_\infty^*, p(q_\infty^*) - p_\infty^*)_{\Gamma_i} \leq \|q^* - q_\infty^*\|_{0,\Gamma_i} \|p(q_\infty^*) - p_\infty^*\|_{0,\Gamma_i}, \tag{5.25}$$

which, together with the second equality in (5.20), implies

$$\|q^* - q_\infty^*\|_{0,\Gamma_i} \leq \beta^{-1}\|p(q_\infty^*) - p_\infty^*\|_{0,\Gamma_i} \leq C\beta^{-1}\|p(q_\infty^*) - p_\infty^*\|_1 = 0.$$

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So the last convergence in (5.21) holds thanks to Theorem 4.2. Moreover, it follows directly from (5.23) that

$$u(q_\infty^*) = u^* \quad \text{in } H^1(\Omega). \tag{5.26}$$

Now the first convergence in (4.14) and the first equality in (5.20) yield the first result in (5.21), i.e., $u_k^* \to u_\infty^* = u^*$ in $H^1(\Omega)$ as $k \to \infty$. Similarly, we can show using (5.24) and (5.26) that $p(q_\infty^*) = p^*$ in $H^1(\Omega)$, then the desired second convergence in (5.20) follows from Theorem 4.2 and Lemma 5.5. $\quad\square$

With the help of Theorem 5.1 and the local efficiency (3.2), we are ready to establish the second main result of this paper.

Theorem 5.2. *The sequence $\{\eta_k(u_k^*, p_k^*, q_k^*, f, u_a, z)\}$ of the estimators generated by Algorithm 3.1 converges to zero.*

Proof. We split the estimator for $k \geq l$ as in the proof of Lemma 5.3 that

$$\eta_k^2(u_k^*, p_k^*, q_k^*, f, u_a, z) = \eta_k^2(u_k^*, p_k^*, q_k^*, f, u_a, z, \mathcal{T}_k \setminus \mathcal{T}_l^+) + \eta_k^2(u_k^*, p_k^*, q_k^*, f, u_a, z, \mathcal{T}_l^+). \tag{5.27}$$

It follows from (2.10)-(2.11) and the strong convergence of $\{q_k^*\}$ that $\|u_k^*\|_1$, $\|p_k^*\|_1$ and $\|q_k^*\|_{0,\Gamma_i}$ are all uniformly bounded above by a constant C_{stab}. Summing up the lower bound (3.2) over all elements in $\mathcal{T}_k \setminus \mathcal{T}_l^+$, we obtain

$$
\begin{aligned}
\eta_k^2(u_k^*, p_k^*, q_k^*, f, u_a, z, \mathcal{T}_k \setminus \mathcal{T}_l^+) &\leq C \sum_{T \in \mathcal{T}_k \setminus \mathcal{T}_l^+} \big(\|u^* - u_k^*\|_{0,\omega_T}^2 + \|p^* - p_k^*\|_{0,\omega_T}^2 + \|q^* - q_k^*\|_{0,\partial T \cap \Gamma_i}^2 \\
&\qquad\qquad + \mathrm{osc}_k^2(f, \omega_T) + \mathrm{osc}_k^2(u_k^*, q_k^*, \partial T) + \mathrm{osc}_h^2(p_k^*, u_k^*, \partial T) \big) \\
&\leq C \big(\|u^* - u_k^*\|_1^2 + \|p^* - p_h^*\|_0^2 + \|q^* - q_h^*\|_{0,\Gamma_i}^2 \\
&\qquad + \max_{T \in \mathcal{T}_k \setminus \mathcal{T}_l^+} h_T(\|f\|_0^2 + \|u_a\|_{0,\Gamma_i}^2 + \|z\|_{0,\Gamma_a}^2 + \|u_k^*\|_1^2 + \|p_k^*\|_1^2 + \|q_k^*\|_{0,\Gamma_i}^2) \big), \\
&\leq C \big(\|u^* - u_k^*\|_1^2 + \|p^* - p_h^*\|_0^2 + \|q^* - q_h^*\|_{0,\Gamma_i}^2 \\
&\qquad + \max_{T \in \mathcal{T}_k \setminus \mathcal{T}_l^+} h_T(\|f\|_0^2 + \|u_a\|_{0,\Gamma_i}^2 + \|z\|_{0,\Gamma_a}^2 + C_{stab}^2) \big),
\end{aligned}
$$

where we used the facts that $\bar{f}_T$, $\bar{J}_{F,1}$ and $\bar{J}_{F,2}$ are the best L^2-projections onto constant spaces and $h_F \leq Ch_T$ for any $F \in \partial T \cap \mathcal{F}_h(\Gamma)$. To complete the proof, we recall that $\max_{T \in \mathcal{T}_k \setminus \mathcal{T}_l^+} h_T \leq \|h_l\|_{\infty,\Omega_l^0} \to 0$ as $l \to \infty$ by Lemma 5.1 and the monotonicity of h_k, and the convergences in (5.21) and (5.16), hence we can require two terms in (5.27) to be smaller than any given positive number once we fix a large l and choose k sufficiently large. $\quad\square$

5.3 Generalizations to other marking strategies

In this section we shall extend the convergence results of Algorithm 3.1 established in the previous section 5.2 to the cases when the marking criterion (3.3) in Algorithm 3.1 is replaced by three other popular marking strategies, i.e., the equidistribution strategy, the modified equidistribution strategy and the practical Dörfler strategy.

By carefully reviewing the previous analysis, it is not difficult to discover that Theorems 4.1-4.2 and Lemmas 5.1-5.2 are all independent of any specific marking strategy, and the maximum strategy (3.3) is only used in the proof of Lemma 5.3 for the condition

$$\max_{T \in \mathcal{T}_k \setminus \mathcal{M}_k} \eta_{T,k}(u_k^*, p_k^*, q_k^*, f, u_a, z) \leq \max_{T \in \mathcal{M}_k} \eta_{T,k}(u_k^*, p_k^*, q_k^*, f, u_a, z) \tag{5.28}$$

to hold. Therefore, it suffices for us to check whether this condition (5.28) is satisfied also by the aforementioned three strategies.

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The equidistribution strategy. Given a parameter $\theta \in [0,1]$ and a tolerance TOL, this strategy selects a subset $\mathcal{M}_k$ of all such elements $T \in \mathcal{T}_k$ to mark, which satisfies

$$\eta_{T,k}(u_k^*, p_k^*, q_k^*, f, u_a, z) \geq \theta \, \mathsf{TOL}/\sqrt{|\mathcal{T}_k|}. \tag{5.29}$$

In practice, if $\eta_k(u_k^*, p_k^*, q_k^*, f, u_a, z) \leq \mathsf{TOL}$, the adaptive algorithm is terminated. It is easy to verify that whenever $\eta_k(u_k^*, p_k^*, q_k^*, f, u_a, z) > \mathsf{TOL}$ the element with the maximal error indicator is always included in $\mathcal{M}_k$ according to (5.29). Hence, (5.28) holds for the equidistribution strategy. Then arguing as in Theorem 5.2 for the case of the maximum strategy, we have the following similar conclusion.

Theorem 5.3. *Let (u^*, p^*, q^*) be the solution of the problem (2.5)-(2.7) and $\{(u_k^*, p_k^*, q_k^*)\}$ be a sequence of discrete solutions produced by Algorithm 3.1 with (5.29) in place of (3.3) in the module* MARK*. Then for a given tolerance* TOL*, the following inequality holds after a finite number of iterations:*

$$\eta_k(u_k^*, p_k^*, q_k^*, f, u_a, z) \leq \mathsf{TOL}. \tag{5.30}$$

The modified equidistribution strategy. Given a parameter $\theta \in [0,1]$, this strategy selects a subset $\mathcal{M}_k$ of all such elements $T \in \mathcal{T}_k$ to mark, which satisfies

$$\eta_{T,k}(u_k^*, p_k^*, q_k^*, f, u_a, z) \geq \theta \eta_k(u_k^*, p_k^*, q_k^*, f, u_a, z)/\sqrt{|\mathcal{T}_k|}. \tag{5.31}$$

With this marking strategy, the convergence results (5.21) and Theorem 5.2 still hold true for Algorithm 3.1 since we may easily observe that the modified equidistribution strategy satisfies (5.28).

The practical Dörfler strategy. Given a parameter $\theta \in (0,1]$, this strategy marks a subset $\mathcal{M}_k$ of elements in $\mathcal{T}_k$ that satisfy

$$\eta_k(u_k^*, p_k^*, q_k^*, f, u_a, z, \mathcal{M}_k) \geq \theta \eta_k(u_k^*, p_k^*, q_k^*, f, u_a, z), \tag{5.32}$$

$$\min_{T \in \mathcal{M}_k} \eta_{T,k}(u_k^*, p_k^*, q_k^*, f, u_a, z) \geq \max_{T \in \mathcal{T}_k \backslash \mathcal{M}_k} \eta_{T,k}(u_k^*, p_k^*, q_k^*, f, u_a, z). \tag{5.33}$$

We can easily verify that (5.33) ensures the condition (5.28), so the convergence results (5.21) and Theorem 5.2 still follow.

Concluding remarks

We have investigated a new adaptive finite element method for distributed flux reconstruction proposed recently in [24]. It has been demonstrated that as the algorithm proceeds the adaptive sequence of the discrete triplets generated by the algorithm converges to the true flux in L^2-norm, the true state and costate variables in H^1-norm and the relevant sequence of estimators also has a vanishing limit. The latter guarantees that the adaptive algorithm may stop within any given tolerance after a finite number of iterations. For the sake of convenience, convergence results are established in the case of the maximum strategy in the module MARK and then extended to other more practical marking strategies.

In the course of the convergence analysis, we have employed some techniques from nonlinear optimizations to derive an important auxiliary result: the sequence of adaptive triplets generated by the algorithm converges strongly to some limiting triplet. We believe there exist similar results for other inverse problems in terms of output least-squares formulations with PDE constraints, so may follow the same line to study their related AFEMs.

The convergence theory developed here may be extended to some nonlinear inverse problems such as the reconstruction of the Robin coefficient on an inaccessible part of the boundary from some accessible boundary measurement data on the basis of an adaptive finite element method.

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