Well-Posedness of an Electric Interface Model and its Finite Element Approximation

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Abstract

This work aims at providing a mathematical and numerical framework for the analysis on the effects of pulsed electric fields on the physical media that have a heterogeneous permittivity and a heterogeneous conductivity. Well-posedness of the model interface problem and the regularity of its solutions are established. A fully discrete finite element scheme is proposed for the numerical approximation of the potential distribution as a function of time and space simultaneously for an arbitrary shaped pulse, and it is demonstrated to enjoy the optimal convergence order in both space and time. The new results and numerical scheme have potential applications in the fields of electromagnetism, medicine, food sciences, and biotechnology.

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Key Words: pulsed electric field, heterogeneous medium, well-posedness, numerical schemes, finite element convergence.

1 Introduction

This work is concerned with a mathematical and numerical framework to help understand and analyse the effects of pulsed electric fields on the physical media that have a heterogeneous permittivity and a heterogeneous conductivity. The governing system involves an electric interface model, which may find wide applications in electromagnetism, medicine, food sciences, and biotechnology. A direct example is that the electrical properties of biological tissues and cell suspensions determine the pathways of current flow through the medium [4, 19, 21], and a biological tissue is described as having a permittivity and a conductivity [20]. The conductivity can be regarded as a measure of the ability of its charge to be transported throughout its volume by an applied electric field while the permittivity is a measure of the ability of its dipoles to rotate or its charge to be stored by an applied external field. At low frequencies, biological tissues behave like a conductor but capacitive effects become important at higher frequencies due to the membranous structures [25, 26].

Our goal is to study the electric behavior of a physical medium under the influence of a pulsed electric field. It is of great importance to understand the effects of the pulse shape on the potential distribution in the medium. We provide a numerical scheme for computing the potential distribution as a function of time and space simultaneously for an arbitrary shaped pulse. Our results are expected to have important applications in neural activation during deep brain simulations [6, 13], debacterization of liquids, food processing [27], and biofouling prevention [24]. Our numerical scheme can be also used for selective spectroscopic imaging of the electrical properties of biological media [3]. It is challenging to specify the pulse shape in order to give rise to selective imaging of cell suspensions [14, 16].

The paper is organized as follows. In section 2 we introduce the model equation and some notations and preliminary results. We recall the method of continuity and the notions of weak and strong

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solutions. Section 3 is devoted to existence, uniqueness, and regularity for the solutions to the model interface problem. We first derive an a priori energy estimate, then prove existence and uniqueness of the weak solution. Finally, we investigate the interface problem where the conductivity and permittivity distributions may be discontinuous or have large jumps across the medium interface, which is a common feature of biological media. It is shown in section 3 that the solution to the interface problem has a higher regularity in each individual region than in the entire domain. This regularity result is critical for our further numerical analysis. In section 4 we investigate the numerical approximation of the solution to the interface problem. Assuming that the domain is a convex polygon, we present a semi-discrete scheme and show its error estimates in both H^1- and L^2-norms. With these estimates at hand, we then process to propose a fully-discrete scheme and establish its error estimates in both H^1- and L^2-norms. It is worth mentioning that both semi-discrete and fully discrete scheme achieve optimal convergence order in both H^1- and L^2-norms, provided that the interface is numerically resolved.

Let us end this section with some notation used in this paper. For a domain $U \subset \mathbb{R}^n$ ($n = 2, 3$), each integer $k \geq 0$ and real p with $1 \leq p \leq \infty$, $W^{k,p}(U)$ denotes the standard Sobolev space of functions with their weak derivatives of order up to k in the Lebesgue space $L^p(U)$. When $p = 2$, we write $H^k(U)$ for $W^{k,2}(U)$. The scalar product of $L^2(\Omega)$ is denoted by $(\cdot, \cdot)$. If X is a Banach space with norm $\|\cdot\|_X$ and $J \subset \mathbb{R}$ is an interval, then $L^2(J; X)$ represents the Banach space consisting of all square integrable functions $f : J \to X$ (in Bôcher sense) with norm: $\|f(t)\|_{L^2(J;X)} := \left(\int_J \|f(t)\|_X^2 dt\right)^{1/2}$. We denote by $H^1(J; X)$ the space of all functions $u \in L^2(J; X)$ such that u', the weak derivative of u with respect to time variable, exists and belongs to $L^2(J; X)$, endowed with the norm $\|u\|_{H^1(J;X)} = (\|u\|^2_{L^2(J;X)} + \|u'\|^2_{L^2(J;X)})^{1/2}$. For $1 \leq i, j \leq n$, we write $D_i u = \partial u/\partial x_i$ and $D_{i,j} u = \partial^2 u/\partial x_i \partial x_j$. For $u \in H^1(U)$ and $f \in H^1(J; H^1(U))$, we also set the semi-norms $|u|_{H^1(U)} := \|\nabla u\|_{L^2(U)}$ and $|f|_{L^2(J;H^1(U))} := (\int_J |f(t)|^2_{H^1(U)} dt)^{\frac{1}{2}}$. For ease of notation, we do not always distinguish between the notation of u, $u(t)$, $u(t, x)$ and $u(t, \cdot)$. Sometimes, the notation is not changed when a function defined on Ω restricted to a subset. For the sake of brevity, we systematically use the expression $A \lesssim B$ to indicate that $A \leq CB$ for constant C that is independent of A and B. In some special cases, we may specify more the constants involved.

2 Preliminaries

Let Ω be a bounded domain in $\mathbb{R}^n$ ($n = 2, 3$) with a Lipschitz boundary, and σ and ε the conductivity and permittivity distributions inside Ω. We assume that σ and ε belong to $L^\infty(\Omega)$. Biological tissues induce capacitive effects due to their cell membrane structures [20]. When they are exposed to electric pulses, the voltage potential u is a solution to the following time-dependent equation [15, 22]

$$\begin{cases} -\nabla \cdot (\sigma(x)\nabla u(t,x) + \varepsilon(x)\nabla u'(t,x)) &= f(t,x), \quad (t,x) \in (0,T) \times \Omega, \\ u &= 0, \quad (t,x) \in (0,T) \times \partial\Omega, \\ u(0,x) &= u_0, \quad x \in \Omega, \end{cases} \tag{2.1}$$

where u_0 is the initial voltage, T is the finite terminal observation time and $f \in L^2(]0, T[; H^{-1}(\Omega))$ is the electric pulse.

The goal of this work is to establish the well-posedness of the model interface system (2.1) and derive a fully discrete finite element scheme for the numerical solution of the system. Of our special interest is the case when the physical coefficients are discontinuous in Ω, namely they may have large jumps across the interface between two different media, which is a common feature in applications, and the conductivity distribution $\sigma(x)$ does not need to be bounded below strictly positively. As far as we know, this is the first mathematical and numerical work on pulsed electric interface fields in capacitive media. The main difficulty comes from the fact that (2.1) does not belong to the well-studied classes of time-dependent equations. Our results in this paper have potential applications in cell electrofusion and electroporation using eletric pulses [22] and in electrosensing [1].

In this section, we first introduce some notions and preliminary results. For the sake of brevity, we write $I =]0, T[$, $H = L^2(\Omega)$, $V = H_0^1(\Omega)$ with its dual space $V' = H^{-1}(\Omega)$ and $\mathcal{X} = H_0^1(\Omega) \cap H^2(\Omega)$. Clearly, $V \subset H \subset V'$ is a triple of spaces (cf. [28, Chapter 1]), i.e.,

(1) the embeddings $V \subset H \subset V'$ are dense and continuous;

(2) $\{V', V\}$ forms an adjoint pair with duality product $\langle \cdot, \cdot \rangle_{V' \times V}$;

(3) the duality product $\langle \cdot, \cdot \rangle_{V' \times V}$ satisfies

$$\langle u, v \rangle_{V' \times V} = (u, v), \quad \forall\ u \in H,\ v \in V.$$

We also introduce two bilinear forms $a_1(u, v)$ and $a_2(u, v)$ on V as follows:

$$a_1(u, v) = \int_\Omega \sigma(x) \nabla u(x) \cdot \nabla v(x) dx, \quad a_2(u, v) = \int_\Omega \varepsilon(x) \nabla u(x) \cdot \nabla v(x) dx, \quad u, v \in V. \tag{2.2}$$

Next, we define the weak and strong solutions to the equation (2.1). We adapt the widely used notions of weak and strong solutions of parabolic equations (see, for instance, [23]).

Definition 2.1. *Let $u_0 \in V$ and $f \in L^2(I; V')$. A function $u \in H^1(I; V)$ is called a week solution of (2.1) if $u(0) = u_0$ and it satisfies the following weak formulation:*

$$a_1(u(t, \cdot), v) + a_2(u'(t, \cdot), v) = \langle f(t, \cdot), v \rangle_{V' \times V} \tag{2.3}$$

for all $v \in H_0^1(\Omega)$ and a.e. $t \in I$.

Definition 2.2. *Let $f \in L^2(I; H)$ and $u_0 \in \mathcal{X}$. Then, a function $u \in H^1(I; \mathcal{X})$ is called a strong solution of (2.1) if $u(0) = u_0$ and the relation*

$$-\nabla \cdot (\sigma(x) \nabla u(t, x) + \varepsilon(x) \nabla u'(t, x)) = f(t, x) \tag{2.4}$$

holds for a.e. $t \in I$ and a.e. $x \in \Omega$.

Remark 2.3. *Let X be a Banach space. From [23, Proposition 7.1] we know that $H^1(I; X) \Subset C(\overline{I}; X)$ continuously and*

$$\sup_{t \in \overline{I}} \|u(t)\| \lesssim \|u\|_{H^1(I;X)}. \tag{2.5}$$

In particular, we have that $u \in C(\overline{I}; V)$ for $u \in H^1(I; V)$.

To prove the existence below, we will use the so-called "method of continuity", whose key tool is the following lemma (e.g., [11]).

Lemma 2.4. *Let X be a Banach space, Y a normed linear space, and L_0, L_1 two bounded linear operators from X to Y. For each $\lambda \in [0, 1]$, set*

$$L_\lambda = (1 - \lambda) L_0 + \lambda L_1,$$

and suppose that there exists a constant C such that

$$\|x\|_X \le C \|L_\lambda x\|_Y, \quad \forall\ x \in X, \quad \lambda \in [0, 1].$$

Then L_1 maps X onto Y if and only if L_0 maps X onto Y.

Let u be a function in a domain $U \subset \mathbb{R}^n$, $W \Subset U$ and e_k the unit coordinate vector in the x_k direction. We define the difference quotient of u in the direction e_k by

$$D_k^h u(x) = \frac{u(x + h e_k) - u(x)}{h} \tag{2.6}$$

for $x \in W$ and $h \in \mathbb{R}$ with $0 < |h| < \operatorname{dist}(W, \partial U)$. We will use the following lemma in the proof of Theorem 3.5, concerning the difference quotient of functions in Sobolev spaces (cf. [11, Lemma 7.23]).

Lemma 2.5. *Suppose that $u \in H^1(U)$. Then for each $W \Subset U$,*

$$\|D_k^h u\|_{L^2(W)} \le \|D_k u\|_{L^2(U)}, \quad \forall\, h:\ 0 < |h| < \frac{1}{2}\mathrm{dist}(W, \partial U)\,.$$

We end up with an analogue of [11, Lemma 7.24], and provide an outline of its proof.

Lemma 2.6. *Let $u \in L^2(I; L^2(U))$, $W \Subset U$ and suppose that there exists a positive constant K such that $\|D_k^h u\|_{L^2(I;L^2(W))} \le K$ for all $0 < |h| < \frac{1}{2}\mathrm{dist}(W, \partial U)$. Then $\|D_k u\|_{L^2(I;L^2(W))} \le K$.*

Proof. Banach-Alaoglu theorem implies that there exists a sequence $\{h_m\}_{m=1}^{\infty}$ with $h_m \to 0$ and a function $v \in L^2(I; L^2(W))$ such that $\|v\|_{L^2(I;L^2(W))} \le K$, and for any $\varphi \in C_0^\infty(W)$ and $\alpha \in C_0^\infty(I)$,

$$\int_I \int_W \alpha(t)\varphi D_k^{h_m} u(t)dxdt \to \int_I \int_W \alpha(t)\varphi v(t)dxdt \quad \text{as} \ \ m \to \infty\,.$$

On the other hand, we have

$$\int_I \int_W \alpha(t)\varphi D_k^{h_m} u(t)dxdt = -\int_I \int_W \alpha(t)u(t)D_k^{-h_m}\varphi dxdt \to -\int_I \int_W \alpha(t)u(t)D_k\varphi dxdt,$$

as $m \to \infty$. Hence, we have

$$\int_I \int_W \alpha(t)\left(u(t)D_k\varphi + v(t)\varphi\right)dxdt = 0.$$

Using the arbitrariness of α and φ, we know for a.e. $t \in I$, $v(t) = D_k u(t)$ in weak sense, hence $v = D_k u$ in $L^2(I; L^2(W))$. □

Lemma 2.7. *Let $U \subset \mathbb{R}^n$ be a domain and $1 \le i \le n$. If $u \in H^1(I; L^2(U))$, $D_i u' \in L^2(I; L^2(U))$ and $D_i u(0) \in L^2(U)$, then $D_i u \in L^2(I; L^2(U))$ and*

$$\|D_i u\|_{L^2(I;L^2(U))} \lesssim \|D_i u'\|_{L^2(I;L^2(U))} + \|D_i u(0)\|_{L^2(U)}.$$

Proof. Since $u \in H^1(I; L^2(U))$, we have

$$u(t) = u(0) + \int_0^t u'(s)ds, \quad \forall\, t \in I.$$

By Fubini's theorem, we know that for any $\phi \in C_0^\infty(U)$,

$$\int_U u(t)D_i\phi dx = \int_U u(0)D_i\phi dx + \int_U \int_0^t u'(s)D_i\phi ds dx = -\int_U \left(D_i u(0) + \int_0^t D_i u'(s)ds\right)\phi dx,$$

which implies that

$$D_i u(t) = D_i u(0) + \int_0^t D_i u'(s)ds.$$

This completes the proof. □

3 Existence and regularity

We now introduce a basic assumption for the existence and uniqueness of weak solutions to (2.1).

(*A*1) σ, $\varepsilon \in L^\infty(\Omega)$, and there exist two positive constants m and M such that $0 \le \sigma(x) \le M$ and $m \le \varepsilon(x) \le M$ for a.e. $x \in \Omega$.

Let us recall that there exist two operators $\mathcal{A}_1, \mathcal{A}_2 : V \to V'$ associated with the bilinear forms $a_1(\cdot,\cdot)$ and $a_2(\cdot,\cdot)$, respectively, i.e.,

$$\langle \mathcal{A}_1 u, v\rangle_{V'\times V} = a_1(u,v), \quad \langle \mathcal{A}_2 u, v\rangle_{V'\times V} = a_2(u,v), \quad u, v \in V.$$

From [28, Theorem 1.24] we know that $\mathcal{A}_1$ is a bounded operator and satisfies the following estimate

$$\|\mathcal{A}_1 u\|_{V'} \le M\|u\|_V, \quad \forall u \in V, \tag{3.1}$$

and $\mathcal{A}_2$ is actually an isomorphism from V to V' and satisfies

$$m\|u\|_V \le \|\mathcal{A}_2 u\|_{V'} \le M\|u\|_V, \quad \forall u \in V. \tag{3.2}$$

3.1 Existence and uniqueness of weak solutions

In this subsection, we prove the existence and uniqueness of the weak solutions to (2.1). The first auxiliary result is the following a prior estimate, which lays the foundation for our subsequent existence and regularity results of weak solutions to (2.1).

Theorem 3.1. *Let $f \in L^2(I;V')$, $u_0 \in V$ and u be the weak solution to* (2.1)*. Under the assumption* (A1)*, we have*

$$\int_0^T \int_\Omega |\nabla u'|^2 dxdt + \sup_{t\in\overline{I}} \|u(t)\|_V^2 \lesssim \|f\|_{L^2(I;V')}^2 + \|u_0\|_V^2, \tag{3.3}$$

and

$$\|u\|_{H^1(I;V)} \lesssim \|f\|_{L^2(I;V')} + \|u_0\|_V. \tag{3.4}$$

Proof. Choosing $v = u'$ in (2.3) and integrating over $(0,T)$, we obtain

$$\int_0^T \int_\Omega \left(\sigma\nabla u(t)\cdot\nabla u'(t) + \varepsilon|\nabla u'(t)|^2\right) dxdt \le \int_0^T \|f(t)\|_{V'}\|u'(t)\|_V dt. \tag{3.5}$$

From this and the identity that

$$\int_0^T \int_\Omega \sigma\nabla u(t)\cdot\nabla u'(t) dxdt = \frac{1}{2}\left(\|\sqrt{\sigma}\nabla u(T)\|_H^2 - \|\sqrt{\sigma}\nabla u(0)\|_H^2\right),$$

it follows that

$$\int_0^T \int_\Omega \varepsilon|\nabla u'(t)|^2 dxdt \le \int_0^T \|f(t)\|_{V'}\|u'(t)\|_V dt + M\|u_0\|_V. \tag{3.6}$$

Using Young's inequality, we have

$$\|u'\|_{L^2(I;V)} \lesssim \|f\|_{L^2(I;V')} + \|u_0\|_V.$$

From Lemma 2.7 and Remark 3.2, the desired results follow immediately. □

With estimate (3.4) in hand, we can prove the first existence result of (2.1).

Theorem 3.2. *Let $f \in L^2(I;V')$ and $u_0 \in V$. Under the assumption* (A1)*, equation* (2.1) *admits a unique weak solution.*

Proof. We first establish the result for $u_0 = 0$. The uniqueness is nothing but a direct consequence of Theorem 3.1. We use Lemma 2.4 to prove the existence. First, we construct a linear operator $\mathcal{L} : H_0^1(I;V) \to L^2(I;V')$ by setting

$$(\mathcal{L}u)(t) := \mathcal{A}_1 u(t) + \mathcal{A}_2 u'(t), \quad \forall u \in H_0^1(I;V),$$

where $H_0^1(I;V)$ is defined by

$$H_0^1(I;V) = \{u \in H^1(I;V); u(0) = 0\}.$$

It is a closed subspace of the Banach space $H^1(I;V)$, since $H^1(I;V) \Subset C(\overline{I};V)$ continuously. From (3.1) and (3.2) it follows that

$$\|\mathcal{L}u\|_{L^2(I,V')} \le M\|u\|_{H^1(I;V)},$$

which implies that $\mathcal{L}$ is well-defined and continuous.

For each $\lambda \in [0,1]$, we introduce a linear operator $\mathcal{L}_\lambda : H_0^1(I;V) \to L^2(I;V')$ as follows:

$$\mathcal{L}_\lambda u := \lambda\mathcal{L}u + (1-\lambda)\mathcal{L}_0 u, \quad \forall u \in H_0^1(I;V),$$

where we set $\mathcal{L}_0 u = -\Delta u - \Delta u'$ for $u \in H_0^1(I;V)$. Here $(-\Delta)$ is seen as an operator from V to V' and it is actually an isomorphism (cf. [28, Theorem 2.2]).

Let $\sigma_\lambda = \lambda\sigma + (1-\lambda)\chi_\Omega$ and $\varepsilon_\lambda = \lambda\varepsilon + (1-\lambda)\chi_\Omega$ for $\lambda \in [0,1]$. Then the functions σ_λ and ε_λ satisfy

$$m' := \min\{m,1\} \le \varepsilon_\lambda(x) \le M' := \max\{M,1\} \quad \text{for a.e. } x \in \Omega,$$

and

$$0 \le \sigma_\lambda(x) \le M' \quad \text{for a.e. } x \in \Omega.$$

Then, an application of Theorem 3.1 yields that there exists a positive constant C, depending only on m', M' and T, such that

$$\|u\|_{H^1(I;V)} \le C\|\mathcal{L}_\lambda u\|_{L^2(I,V')} \quad \forall u \in H_0^1(I;V).$$

In view of Lemma 2.4, it remains to prove that the mapping $\mathcal{L}_0 : H_0^1(I;V) \to L^2(I;V')$ is onto. To this end, for an arbitrary $f \in L^2(I;V')$, we construct a function $w(t) := \int_0^t e^{-t+s}h(s)ds$ for $t \in I$, where $h(s) = (-\Delta)^{-1}f(s)$ for $s \in I$. Since $(-\Delta)^{-1} : V' \to V$ is bounded, we have that $h \in L^2(I;V)$ and hence $w \in H^1(I;V)$. Moreover, a direct computation yields $w(0) = 0$ and $-\Delta w'(t) - \Delta w(t) = f(t)$ for $t \in I$, which ensures that $\mathcal{L}_0$ is onto. Therefore, we can conclude that the method of continuity applies and Theorem 3.2 holds for $u_0 = 0$.

For $u_0 \neq 0$, we choose $w \in H^1(I;V)$ such that $w(0) = u_0$ and write $f^* = \mathcal{A}_1 w + \mathcal{A}_2 w'$. Clearly, $f^* \in L^2(I;V')$. Then the proof above for $u_0 = 0$ confirms the existence of a unique function $v \in H^1(I;V)$ such that $v(0) = 0$ and

$$\mathcal{A}_1 v + \mathcal{A}_2 v' = f - f^*.$$

Therefore, the function $u := w + v$ is the desired weak solution. □

3.2 Regularity of the solutions to the interface problem

In this subsection we consider the regularity of the weak solution for (2.1), which is important not only for its theoretical interest but also for the subsequent numerical analysis. Of our prime concern in this paper is the case when the coefficients $\sigma(x)$ and $\varepsilon(x)$ are discontinuous, and the conductivity distribution $\sigma(x)$ in (2.1) is unnecessary to be bounded below strictly positively. This feature is common to biological applications. Due to the (possibly sharp) jumps of $\sigma(x)$ and $\varepsilon(x)$ across the medium interface, the solution to (2.1) does not expect a desired global regularity like $H^2(\Omega)$, but it is shown in this section that this H^2-regularity is true locally in each medium region Ω_i for $i = 1, 2$. And such local H^2-regularity is proved sufficient in the next section for us to establish the desired optimal convergence order for finite element approximations. As we are not aware of proofs of such local regularities for time-dependent PDEs with large jumps in coefficients in literature, even for standard parabolic equations, we will present a rigorous proof here for the non-standard time-dependent PDE (2.1). We start with the introduction of some standard assumptions.

(*A2*) Ω consists of two C^2-subdomains Ω_1 and Ω_2 with $\Omega_1 \Subset \Omega$, $\Omega_2 := \Omega\backslash\overline{\Omega}_1$;

(A3) $\varepsilon_i := \varepsilon|_{\Omega_i}$ and $\sigma_i := \sigma|_{\Omega_i}$ are continuously differentiable in $\overline{\Omega}_i$ $(i = 1, 2)$.

The interface problem (2.1) is often complemented with the following physical interface conditions:

$$[u(t)] = 0 \quad on\ I \times \Gamma, \quad [\sigma \frac{\partial u(t)}{\partial \nu} + \varepsilon \frac{\partial u'(t)}{\partial \nu}] = 0 \quad on\ I \times \Gamma, \tag{3.7}$$

where $\Gamma := \partial\Omega_1$ is the interface, and $[u(t)] := u_1|_\Gamma - u_2|_\Gamma$, $[\sigma \frac{\partial u(t)}{\partial \nu} + \varepsilon \frac{\partial u'(t)}{\partial \nu}] := \sigma_1 \frac{\partial u_1(t)}{\partial \nu_1} + \sigma_2 \frac{\partial u_2(t)}{\partial \nu_2} + \varepsilon_1 \frac{\partial u_1'(t)}{\partial \nu_1} + \varepsilon_2 \frac{\partial u_2'(t)}{\partial \nu_2}$ on Γ. Here u_i stands for the restrictions of u to Ω_i, and $\partial/\partial\nu_i$ denotes the outer normal derivative with respect to Ω_i, $i = 1, 2$. To deal with the interface problem, we introduce a Banach space

$$\mathcal{Y} = \{u \in V; u_i \in H^2(\Omega_i),\ i = 1, 2\}$$

with the norm

$$\|u\|_{\mathcal{Y}} = \|u\|_V + \|u_1\|_{H^2(\Omega_1)} + \|u_2\|_{H^2(\Omega_2)}, \quad \forall u \in \mathcal{Y}.$$

Definition 3.3. *Let $f \in L^2(I; H)$ and $u_0 \in \mathcal{Y}$. A function $u \in H^1(I; \mathcal{Y})$ is called a strong solution of* (2.1) *with the jump conditions* (3.7) *if $u(0) = u_0$ and the relation*

$$-\nabla \cdot (\sigma(x)\nabla u(t, x) + \varepsilon(x)\nabla u'(t, x)) = f(t, x) \tag{3.8}$$

holds for a.e. $t \in I$ and a.e. $x \in \Omega_i$ $(i = 1, 2)$.

Before proving the existence of a strong solution to the interface problem, we first establish the following result.

Lemma 3.4. *Let u be the weak solution of* (2.3). *Assume that $f \in L^2(I; H)$, $u_0 \in \mathcal{Y}$, $u \in H^1(I; \mathcal{Y})$, $\partial\Omega_1$ and $\partial\Omega_2$ are Lipschitz continuous. Then u is a strong solution for* (2.1) *and* (3.7).

Proof. We obtain, upon integration by parts, that for a.e. $t \in I$,

$$\int_{\Omega_i} (-\nabla \cdot (\sigma\nabla u + \varepsilon\nabla u')\, v - fv)\, dx = \int_{\Omega_i} (\sigma\nabla u \cdot \nabla v + \varepsilon\nabla u \cdot \nabla v - fv)\, dx, \quad \forall v \in H_0^1(\Omega_i), \tag{3.9}$$

which implies that

$$-\nabla \cdot (\sigma(x)\nabla u(t, x) + \varepsilon(x)\nabla u'(t, x)) = f(t, x)$$

holds for a.e. $t \in I$ and a.e. $x \in \Omega_i$ $(i = 1, 2)$. It remains to show that the weak solution also satisfies the jump conditions (3.7). By integration by parts we have for a.e. $t \in I$,

$$\begin{aligned} 0 &= \int_{\Omega_1 \cup \Omega_2} (-\nabla \cdot (\sigma\nabla u + \varepsilon\nabla u')\, v - fv)\, dx \\ &= \int_\Omega (\sigma\nabla u \cdot \nabla v + \varepsilon\nabla u \cdot \nabla v - fv)\, dx - \int_\Gamma [\sigma \frac{\partial u}{\partial \nu} + \varepsilon \frac{\partial u'}{\partial \nu}] v dx, \quad \forall v \in V. \end{aligned}$$

From this and the definition of weak solutions it follows that

$$\int_\Gamma [\sigma \frac{\partial u}{\partial \nu} + \varepsilon \frac{\partial u'}{\partial \nu}] v dx = 0, \quad \forall\, v \in V.$$

The arbitrariness of v shows that u satisfies the second jump condition in (3.7). The first condition in (3.7) is a direct consequence of the fact that $u \in H^1(I; V)$. This completes the proof. □

From the lemma above, we know that the key point is to get the regularity of the weak solutions, which is the main subject of the following theorem.

Theorem 3.5. *Let $f \in L^2(I; H)$ and $u_0 \in \mathcal{Y}$. Under the assumptions* (A1), (A2) *and* (A3), *the interface problem* (2.1) *and* (3.7) *admits a unique strong solution u, which satisfies*

$$\|u\|_{H^1(I;\mathcal{Y})} \lesssim \|f\|_{L^2(I;H)} + \|u_0\|_{\mathcal{Y}}.$$

Proof. From Corollary 3.2, there exists a weak solution $u \in H^1(I;V)$ to (2.1). In view of Lemma 3.4 and Theorem 3.1, it suffices to show that $u \in H^1(I;\mathcal{Y})$ and

$$\|u\|_{H^1(I;\mathcal{Y})} \lesssim \|u\|_{H^1(I;V)} + \|f\|_{L^2(I;H)} + \|u_0\|_{\mathcal{Y}}.$$

The proof is divided into two parts. We only show that $u|_{\Omega_1} \in H^2(\Omega_1)$, since the result that $u|_{\Omega_2} \in H^2(\Omega_2)$ can be proven in the same way. Henceforth we denote by C a generic constant that depends only on the cut-off functions, the final observation time T and the coefficients ε and σ, and is always independent of the size of the difference parameter h in (2.6).

We first establish the interior regularity of the solution and its desired estimate. Let $U \Subset \Omega_1$ and choose a domain W such that $U \Subset W \Subset \Omega_1$. We then select a cut-off function $\eta \in C_0^\infty(W)$ such that $\eta \equiv 1$ on U and vanishes outside of W. Now let $|h| > 0$ be small, and e_k be the unit coordinate vector in x_k direction for $k \in \{1, \cdots, n\}$, and define a function $v = -D_k^{-h}(\eta^2 D_k^h u')$ (see (2.6) for the definition of D_k^h). Clearly, we know $v(t) \in H_0^1(\Omega_1)$, hence also to V for $t \in I$. Now, letting $\sigma^h(x) = \sigma(x + he_k)$, $\varepsilon^h(x) = \varepsilon(x + he_k)$ for $x \in W$, substituting this v into the left-hand side of (2.3), and integrating it over I, we find that

$$\begin{aligned}
A &:= \int_I \left(a_1(u(t), v(t)) + a_2(u'(t), v(t))\right) dt \\
&= \int_I \int_\Omega \left(D_k^h\left(\sigma \nabla u(t)\right) \cdot \nabla\left(\eta^2 D_k^h u'(t)\right) + D_k^h\left(\varepsilon \nabla u'(t)\right) \cdot \nabla\left(\eta^2 D_k^h u'(t)\right)\right) dxdt \\
&= \int_I \int_W \left(\varepsilon^h \eta^2 \nabla D_k^h u'(t) \cdot \nabla D_k^h u'(t) + \sigma^h \eta^2 \nabla D_k^h u'(t) \cdot \nabla D_k^h u(t)\right) dxdt \\
&\quad + \int_I \int_W \left(2\eta D_k^h \varepsilon D_k^h u'(t) \nabla u'(t) \cdot \nabla \eta + \eta^2 D_k^h \varepsilon \nabla u'(t) \cdot \nabla D_k^h u'(t) + 2\varepsilon^h \eta D_k^h u'(t) \nabla D_k^h u'(t) \cdot \nabla \eta\right) dxdt \\
&\quad + \int_I \int_W \left(2\eta D_k^h \sigma D_k^h u'(t) \nabla u(t) \cdot \nabla \eta + \eta^2 D_k^h \sigma \nabla u(t) \cdot \nabla D_k^h u'(t) + 2\sigma^h \eta D_k^h u'(t) \nabla D_k^h u(t) \cdot \nabla \eta\right) dxdt \\
&=: (\mathbf{J})_{\mathbf{1}} + (\mathbf{J})_{\mathbf{2}} + (\mathbf{J})_{\mathbf{3}}\,.
\end{aligned}$$

We now estimate $(\mathbf{J})_{\mathbf{1}}$, $(\mathbf{J})_{\mathbf{2}}$, and $(\mathbf{J})_{\mathbf{3}}$ one by one. It is easy to see that

$$(\mathbf{J})_{\mathbf{1}} = \frac{1}{2}\int_W \sigma^h \eta^2 |\nabla D_k^h u(T)|^2 dx - \frac{1}{2}\int_W \sigma^h \eta^2 |\nabla D_k^h u(0)|^2 dx + \int_I \int_W \varepsilon^h \eta^2 |\nabla D_k^h u'(t)|^2 dxdt. \tag{3.10}$$

We note that there exists a constant $K > 0$ such that $|D_k^h \sigma(x)| \le K$ and $|D_k^h \varepsilon(x)| \le K$ for all $x \in W$ and $0 < |h| < \frac{1}{2}\mathrm{dist}(W, \partial\Omega_1)$. Using Young's inequality and Lemma 2.5, we obtain

$$|(\mathbf{J})_{\mathbf{2}}| \le \frac{m}{5}\int_I \int_W \eta^2 |\nabla D_k^h u'(t)|^2 dxdt + C\int_I \int_\Omega |\nabla u'(t)|^2 dxdt. \tag{3.11}$$

Similarly, we can derive

$$\begin{aligned}
|(\mathbf{J})_{\mathbf{3}}| \le \frac{m}{5}\int_I \int_W \eta^2 |\nabla D_k^h u'(t)|^2 dxdt + \delta \int_I \int_W \eta^2 |\nabla D_k^h u(t)|^2 dxdt \\
+ C\int_I \int_\Omega \left(|\nabla u(t)|^2 + |\nabla u'(t)|^2\right) dxdt,
\end{aligned} \tag{3.12}$$

where δ is a positive constant to be specified later. An interplay of Lemmas 2.5 and 2.7 implies that

$$\int_W \eta |\nabla D_k^h u(t)|^2 dx \le C'\left(\int_\Omega |\nabla D_k u(0)|^2 dx + \int_I \int_\Omega |\nabla D_k u'(s)|^2 dxds\right), \quad \forall\, t \in I,$$

with some constant $C' > 0$, whence (3.12) ensures that

$$|(\mathbf{J})_{\mathbf{3}}| \le \frac{2m}{5}\int_I \int_W \eta^2 |\nabla D_k^h u'(t)|^2 dxdt + C\int_I \int_\Omega \left(|\nabla u(t)|^2 + |\nabla u'(t)|^2\right) dxdt, \tag{3.13}$$

if δ is chosen small enough, say $\delta = m/(5TC')$.

On the other hand, using Young's inequality and Lemma 2.5 again, we deduce

$$\begin{aligned} B &:= \int_I \int_\Omega f(t)v(t)dxdt \\ &\leq \frac{m}{5}\int_I\int_W \eta^2|\nabla D_k^h u'(t)|^2 dxdt + C\left(\int_I\int_\Omega \left(|f(t)|^2 + |\nabla u'(t)|^2\right)dxdt + \|u_0\|_{\mathcal{Y}}^2\right). \end{aligned}$$

Since $A = B$, we combine (3.10) with (3.14) to get

$$\begin{aligned} &\frac{2m}{5}\int_I\int_W \eta^2|\nabla D_k^h u'(t)|^2 dxdt \\ \leq\ &\frac{m}{5}\int_I\int_W \eta^2|\nabla D_k^h u'(t)|^2 dxdt + C(\int_I\int_\Omega \left(|\nabla u(t)|^2 + |\nabla u'(t)|^2 + |f(t)|^2\right)dxdt + \|u_0\|_{\mathcal{Y}}^2), \end{aligned} \tag{3.14}$$

which implies

$$\sum_{i=1}^n \int_I \|D_k^h D_i u'(t)\|_{L^2(U)}^2 dt \lesssim \int_I \left(\|u'(t)\|_V^2 + \|u(t)\|_V^2 + \|f(t)\|_H^2\right)dt + \|u_0\|_{\mathcal{Y}}^2$$

for all $k = 1, 2, \cdots, n$ and sufficiently small $|h| \neq 0$. By applying Lemmas 2.6 and 2.7, we come to

$$\|u\|_{H^1(I;H^2(U))} \lesssim \|f\|_{L^2(I;H)} + \|u\|_{H^1(I;V)} + \|u_0\|_{\mathcal{Y}}. \tag{3.15}$$

Next, we establish the boundary regularity and the desired estimate. We first use the standard argument to straighten out the boundary, i.e., flattening out the boundary by changing the coordinates near a boundary point (cf. [11, Chap. 6.2]). Given $x_0 \in \partial\Omega_1$, there exists a ball $B = B_r(x_0)$ with radius r and a C^2-diffeormorphism $\Psi : B \to \Psi(B) \subset \mathbb{R}^n$ such that $\det|\nabla\Psi| = 1$, $U' := \Psi(B)$ is an open set, $\Psi(B\cap\Omega_1) \subset \mathbb{R}^n_+$ and $\Psi(B\cap\partial\Omega_1) \subset \partial\mathbb{R}^n_+$, where $\mathbb{R}^n_+$ is the half-space in the new coordinates. Henceforth we write $y = \Psi(x) = (\Psi_1(x), \cdots, \Psi_n(x))$ for $x \in B$. Then we have $\{y_n > 0; y \in U'\} = \Psi(B\cap\Omega_1)$. Let $\Phi = \Psi^{-1}$, $B^+ = B_{\frac{r}{2}}(x_0)\cap\Omega_1$, $G = \Psi(B_{\frac{r}{2}}(x_0))$ and $G^+ = \Psi(B^+)$, then we can see $G \Subset U'$ and $G^+ \subset G$. We shall write $D_i w = \partial w/\partial y_i$ for $i = 1, \cdots, n$, and $w(y) = u(\Phi(y))$, $\hat{f}(y) = f(\Phi(y))$ for $y \in U'$. Now using the transformation function Ψ, the original equation on $I \times B$ can be transformed into an equation of the same form on $I \times U'$, i.e., for a.e. $t \in I$,

$$\int_{U'} \big(\sum_{i,j=1}^n \hat{\sigma}_{ij} D_i w(t) D_j v + \sum_{i,j=1}^n \hat{\varepsilon}_{ij} D_i w'(t) D_j v\big)dy = \int_{U'} \hat{f}(t)v dy, \quad \forall v \in H_0^1(U'), \tag{3.16}$$

where the coefficients $\hat{\sigma}_{ij}(y)$ and $\hat{\varepsilon}_{ij}(y)$ are given by

$$\hat{\sigma}_{ij}(y) := \sum_{r=1}^n \sigma(\Phi(y))\frac{\partial\Psi_i}{\partial x_r}(\Phi(y))\frac{\partial\Psi_j}{\partial x_r}(\Phi(y)),\ \hat{\varepsilon}_{ij}(y) := \sum_{r=1}^n \varepsilon(\Phi(y))\frac{\partial\Psi_i}{\partial x_r}(\Phi(y))\frac{\partial\Psi_j}{\partial x_r}(\Phi(y)) \tag{3.17}$$

for $1 \leq i, j \leq n$ and $y \in U'$. It is not difficult to see that

$$\sum_{i,j=1}^n \hat{\varepsilon}_{ij}(y)\xi_i\xi_j \geq m|\xi|^2, \qquad \sum_{i,j=1}^n \hat{\sigma}_{ij}(y)\xi_i\xi_j \geq 0, \quad \forall\ (y,\xi) \in U' \times \mathbb{R}^n, \tag{3.18}$$

Choosing a domain W' such that $G \Subset W' \Subset U'$, we then select a cut-off function, which is still denoted by η, such that $\eta \equiv 1$ on G and vanishes outside W'. Now let $|h| > 0$ be small, and $\hat{e}_k$ be the unit coordinate vector in the y_k direction for $k \in \{1, \cdots, n-1\}$. In the sequel, D_k^h stands for the difference quotient in the direction $\hat{e}_k$. We observe that there exists a constant $K' > 0$ such that $|D_k^h \hat{\sigma}_{i,j}(y)| \leq K'$

and $|D_k^h \hat{\varepsilon}_{i,j}(y)| \le K'$ for a.e. $y \in W'$, all $0 < |h| < \frac{1}{2}\mathrm{dist}(W', \partial U')$ and $1 \le i,j \le n$. Then, a natural variant of the reasoning leading to (3.14) shows that

$$\begin{aligned} & \frac{2m}{5}\int_I \int_{W'} \left(\sum_{i=1}^n \eta^2 (D_k^h D_i w'(t))^2\right) dydt \\ \le\ & \frac{m}{5}\int_I \int_{W'} \left(\sum_{i=1}^n \eta^2 (D_k^h D_i w'(t))^2\right) dydt + C\int_I (\|w(t)\|^2_{H^1(U')} + \|w'(t)\|^2_{H^1(U')} + \|\hat{f}(t)\|_{L^2(U')})dt \\ & + C\|w(0)\|_{H^2(U'_- \cup U'_+)}, \end{aligned}$$

where $\|w(0)\|_{H^2(U'_- \cup U'_+)} := \|w(0)\|_{H^2(U'_-)} + \|w(0)\|_{H^2(U'_+)}$ with $U'_+ = U' \cap \mathbb{R}^n_+$ and $U'_- = U' \setminus \overline{U'_+}$. We can derive from the resulting inequality that

$$\begin{aligned} & \sum_{i=1}^n \int_I \|D_k^h D_i w'(t)\|^2_{L^2(G^+)} dt \\ \lesssim\ & \int_I \left(\|w'(t)\|^2_{H^1(U')} + \|w(t)\|^2_{H^1(U')} + \|\hat{f}(t)\|^2_{L^2(U')}\right) dt + \|w(0)\|_{H^2(U'_- \cup U'_+)} \end{aligned}$$

for $k = 1, \cdots, n-1$ and all sufficiently small $|h| \neq 0$, where we have also used the fact $\eta = 1$ on G^+. Using Lemma 2.6, we have

$$\sum_{1 \le i,j < 2n} \|D_{i,j} w\|_{H^1(I;L^2(G^+))} \lesssim \|\hat{f}\|_{L^2(I;L^2(U'))} + \|w\|_{H^1(I;H^1(U'))} + \|w(0)\|_{H^2(U'_- \cup U'_+)}, \tag{3.19}$$

where $D_{i,j} w = D_i D_j w$. From (3.16) we obtain upon integration by parts that for a.e. $t \in I$,

$$\begin{aligned} & \int_{G^+} \hat{\sigma}_{nn} D_n w(t) D_n \varphi + \hat{\varepsilon}_{nn} D_n w'(t) D_n \varphi dy \\ & = \int_{G^+} \left(\hat{f}(t) + \sum_{1 \le i,j < 2n} D_i(\hat{\varepsilon}_{ij} D_j w'(t)) + \sum_{1 \le i,j < 2n} D_i(\hat{\sigma}_{ij} D_j w(t))\right) \varphi dy \end{aligned}$$

for any $\varphi \in C_0^\infty(G^+)$. Noting that $\hat{\sigma}_{ij}$ and $\hat{\varepsilon}_{ij}$ are both continuously differentiable in $\overline{G}^+$ and the estimate (3.19), the right-hand side of the equation above is well-defined and so we find that for a.e. $t \in I$, the weak derivative of $\hat{\sigma}_{nn} D_n w(t) + \hat{\varepsilon}_{nn} D_n w'(t)$ with respect to y_n exists and it satisfies

$$-D_n\left(\hat{\sigma}_{nn} D_n w(t) + \hat{\varepsilon}_{nn} D_n w'(t)\right) = \hat{f}(t) + \sum_{1 \le i,j < 2n} D_i(\hat{\varepsilon}_{ij} D_j w'(t)) + \sum_{1 \le i,j < 2n} D_i(\hat{\sigma}_{ij} D_j w(t)). \tag{3.20}$$

For the sake of brevity, we write $g := \hat{\varepsilon}_{nn} H D_n w$, where

$$H(t,y) := \exp\left(\frac{\hat{\sigma}_{nn}(y)}{\hat{\varepsilon}_{nn}(y)} t\right) \quad (t,y) \in I \times G^+.$$

It follows readily that H is strictly positive and $H \in C^1(I \times G^+)$. (3.20) ensures that for a.e. $t \in I$,

$$-D_n\left(\frac{g'(t)}{H}\right) = \hat{f}(t) + \sum_{1 \le i,j < 2n} D_i(\hat{\varepsilon}_{ij} D_j w'(t)) + \sum_{1 \le i,j < 2n} D_i(\hat{\sigma}_{ij} D_j w(t)). \tag{3.21}$$

A direct computation yields for a.e. $t \in I$,

$$D_n g'(t) = \frac{D_n H(t)}{H(t)} g'(t) - H(t)\Big(\hat{f}(t) + \sum_{1 \le i,j < 2n} D_i(\hat{\varepsilon}_{ij} D_j w'(t)) + \sum_{1 \le i,j < 2n} D_i(\hat{\sigma}_{ij} D_j w(t))\Big). \tag{3.22}$$

Since $\|g'\|_{L^2(I;L^2(G^+))} \lesssim \|w\|_{H^1(I;H^1(G^+))}$, we infer from (3.19) and (3.22) that

$$\|D_n g'\|_{L^2(I;L^2(G^+))} \lesssim \|\hat{f}\|_{L^2(I;L^2(U'))} + \|w\|_{H^1(I;H^1(U'))} + \|w(0)\|_{H^2(U'_-\cup U'_+)}. \tag{3.23}$$

As $\|D_n g(0)\| \lesssim \|w(0)\|_{H^2(U'_+)}$, an application of Lemma 2.7 yields

$$\|D_n g\|_{L^2(I;L^2(G^+))} \lesssim \|\hat{f}\|_{L^2(I;L^2(U'))} + \|w\|_{H^1(I;H^1(U'))} + \|w(0)\|_{H^2(U'_-\cup U'_+)}. \tag{3.24}$$

We can then conclude from (3.23) and (3.24) that

$$\|D_{n,n} w\|_{H^1(I;L^2(G^+))} \lesssim \|\hat{f}\|_{L^2(I;L^2(U'))} + \|w\|_{H^1(I;H^1(U'))} + \|w(0)\|_{H^2(U'_-\cup U'_+)}.$$

Combining this with estimate (3.19), and transforming w back to u in the resulting inequality, we find

$$\|u\|_{H^1(I;H^2(B^+))} \lesssim \|f\|_{L^2(I;H)} + \|u\|_{H^1(I;V)} + \|u_0\|_{\mathcal{Y}}. \tag{3.25}$$

By choosing a finite set of balls $\{B_{r_i/2}(x_i)\}_{i=1}^N$ such that it covers the boundary and then adding the estimates over these balls, we obtain the desired result. □

Using the standard arguments (cf. [12, Theorem 3.2.1.2]) with some natural modifications and the estimates above, we can prove the following regularity result in a general convex domain.

Theorem 3.6. *Let $f \in L^2(I;H)$ and $u_0 \in \mathcal{Y}$. Assume that Ω is a bounded and convex domain, $\Omega_1 \Subset \Omega$ a C^2-subdomain, and that (A1) and (A3) hold. Then, the interface problem (2.1) and (3.7) admits a unique strong solution, which satisfies*

$$\|u\|_{H^1(I;\mathcal{Y})} \lesssim \|f\|_{L^2(I;H)} + \|u_0\|_{\mathcal{Y}}. \tag{3.26}$$

3.3 Existence of a strong solution for smooth coefficients

For the case with smooth coefficients, if we use

(*A*4) $\partial\Omega$ is C^2 and $\sigma, \varepsilon \in C^1(\overline{\Omega})$,

instead of (*A*2) and (*A*3), then we can obtain a better regularity result as follows, using the same argument as in the poof of Theorem 3.5:

Theorem 3.7. *Let $f \in L^2(I;H)$ and $u_0 \in \mathcal{X}$. Under assumptions (A1) and (A4), the equation (2.1) admits a unique strong solution u, which satisfies the following estimate:*

$$\|u\|_{H^1(I;\mathcal{X})} \lesssim \|f\|_{L^2(I;H)} + \|u_0\|_{\mathcal{X}}.$$

Remark 3.8. *By the standard semigroup theory (cf. [28]), we may achieve a better estimate, i.e., under the assumptions of Theorem 3.7, we have $u \in C^1([0,T];\mathcal{X})$. That is, u is a classical solution.*

4 Finite element approximation and error estimates

In this section we propose a fully discrete finite element scheme to approximate the solution of the interface problem (2.1) and (3.7), and establish its optimal convergence under the minimum regularity assumptions on the given data. To do so, we first consider an auxiliary semi-discrete finite element scheme for the concerned interface problem and develop its optimal convergence, which will lead to the optimal convergence of the fully discrete scheme.

Unless otherwise notified, we assume below that $f \in L^2(I;H)$ and $u_0 \in \mathcal{Y}$. For the sake of exposition, we further make the following assumptions:

(*A*5) Ω is a convex polygon or polyhedron in $\mathbb{R}^n$ with $n = 2$ or 3, and $\Omega_1 \Subset \Omega$ is a domain with C^2-boundary;

(*A*6) The coefficients ε and σ are constants in each domain, namely, $\varepsilon = \varepsilon_i$ and $\sigma = \sigma_i$ in Ω_i, $i = 1, 2$, where ε_i and σ_i are two positive constants.

Clearly, assumption (*A*6) implies (*A*1). It follows from Theorem 3.6 that there exists a strong solution to the interface problem (2.1) and (3.7).

Remark 4.1. *For the sake of exposition, we assume that Ω is a convex polygon (if $n = 2$) or a convex polyhedral domain (if $n = 3$). The actual curved boundary can be treated in the same manner as we handle the interface Γ in our subsequent analysis of this section.*

We now introduce a triangulation of the domain Ω. First we triangulate Ω_1 using a quasi-uniform mesh $\mathcal{T}_h^1$ with simplicial elements of size h, which form a polyhedral domain $\Omega_{1,h}$. By a quasi-uniform mesh, we mean [18]

$$h \le h_K \lesssim h \quad K \in \mathcal{T}_h^1, \tag{4.1}$$

where $h_K := 1/2\,\mathrm{diam}K$ and $h := \min_{K\in\mathcal{T}_h^1} \rho_K$ with ρ_K being the maximal radius such that $B_{\rho_K}(b_K) \subset K$. Here $B_{\rho_K}(b_K)$ denotes the closed ball of radius ρ_K centered at the barycenter of the element K. We also require that the triangulation $\mathcal{T}_h^1$ is done such that all the boundary vertices of $\Omega_{1,h}$ lie on the boundary of Ω_1. Then we triangulate Ω_2 using a quasi-uniform mesh $\mathcal{T}_h^2$ with simplicial elements of size h, which form a polyhedral domain $\Omega_{2,h}$. The triangulation $\mathcal{T}_h^2$ is done such that all the vertices of the outer polyhedral boundary $\partial\Omega$ are also the vertices of $\Omega_{2,h}$, while all the vertices on the inner boundary of $\Omega_{2,h}$ match the boundary vertices of $\Omega_{1,h}$. More precisely, the triangulation $\mathcal{T}_h := \mathcal{T}_h^1 \cup \mathcal{T}_h^2$ satisfies the following conditions:

(*T*1) $\overline{\Omega} = \cup_{K\in\mathcal{T}_h} K$;

(*T*2) if $K_1, K_2 \in \mathcal{T}_h$ with $K_1 \neq K_2$, then either $K_1 \cap K_2 = \emptyset$ or $K_1 \cap K_2$ is a common vertex, an edge or a face;

(*T*3) for each K, all its vertices are completely contained in either $\overline{\Omega}_1$ or $\overline{\Omega}_2$.

Now we define V_h to be the continuous piecewise linear finite element space on the triangulation $\mathcal{T}_h$ and V_h^0 the closed subspace of V_h with its functions vanishing on the boundary $\partial\Omega$. Then we study the approximation of piecewise smooth functions by finite element functions in V_h. Clearly, the accuracy of this approximation depends on how well the mesh $\mathcal{T}_h$ resolve the interface Γ. Following the notation used in [18], we define, for $\lambda > 0$ with $\lambda < \min\{\mathrm{dist}(\Gamma, \partial\Omega), h/2\}$, a tubular neighborhood S_λ of Γ by

$$S_\lambda := \{x \in \Omega;\ \mathrm{dist}(x, \partial\Gamma) < \lambda\}\,;$$

see Fig. 1. Then we decompose $\mathcal{T}_h$ into three disjoint subsets $\mathcal{T}_h = \mathring{\mathcal{T}}_h^1 \cup \mathring{\mathcal{T}}_h^2 \cup \mathcal{T}_*$, where

$$\mathring{\mathcal{T}}_h^i = \{K \in \mathcal{T}_h; K \subset \overline{\Omega_i}\backslash S_\lambda\}, \quad i = 1, 2,$$

and $\mathcal{T}_* := \mathcal{T}_h\backslash(\mathring{\mathcal{T}}_h^1 \cup \mathring{\mathcal{T}}_h^2)$. Furthermore, we write $\mathcal{T}_*^i = \{K \in \mathcal{T}_*; K \subset \Omega_i \cup S_\lambda\}$. And we know $\mathcal{T}_*^1 \cap \mathcal{T}_*^2 = \emptyset$. To see this, for any element $K \in \mathcal{T}_*^1 \cap \mathcal{T}_*^2$, it satisfies that $K \subset S_\lambda$ by noting the fact that $\Omega_1 \cap \Omega_2 = \emptyset$, thus $B_{\rho_K}(b_K) \subset S_\lambda$, which, together with (4.1), implies $h < \lambda$. This contradicts to the condition that $2\lambda < h$, leading to the conclusion that $\mathcal{T}_*^1 \cap \mathcal{T}_*^2 = \emptyset$. Furthermore, since Γ is of class C^2, we know from [18] that there exists $\lambda > 0$ such that

$$\lambda = O(h^2), \tag{4.2}$$

and $\mathcal{T}_* = \mathcal{T}_*^1 \cup \mathcal{T}_*^2$, provided that h is appropriately small. We may also refer to [7, page 1890], [8, pp.180-181], and [10] for more detail. Figure 1 illustrates an example of such decomposition.

For this triangulation, we have an important observation that $\overline{\Omega}_{i,h} = \cup\{K; K \in \mathring{\mathcal{T}}_h^i \cup \mathcal{T}_*^i\}$, $i = 1, 2$, i.e., $\mathcal{T}_h^i = \mathring{\mathcal{T}}_h^i \cup \mathcal{T}_*^i$. The notation S_λ not only quantifies how well the mesh $\mathcal{T}_h$ resolves the interface, but

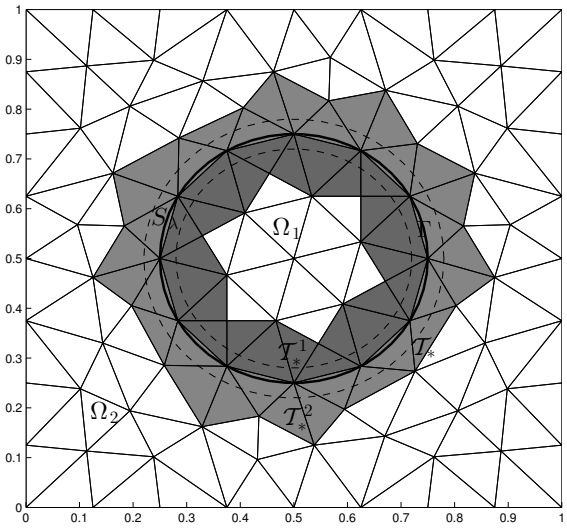

Figure 1: An illustrative example of the finite element triangulation

it also allows us to use the important auxiliary results in Lemmas 4.3-4.5, which were established in [18], in the subsequent analysis.

We note that the evaluation of the entries of the stiffness matrix involving interface elements is not trivial in the three-dimensional case if the mesh is not aligned with the interface. So we shall adopt the following more convenient approximation bilinear forms $a_{i,h}(\cdot,\cdot) : V \times V \to \mathbb{R}$:

$$a_{1,h}(u,v) := \sum_{i=1}^{2} \int_{\Omega_{i,h}} \sigma_i \nabla u \cdot \nabla v dx \quad \text{and} \quad a_{2,h}(u,v) := \sum_{i=1}^{2} \int_{\Omega_{i,h}} \varepsilon_i \nabla u \cdot \nabla v dx.$$

To approximate the problem in space optimally, we introduce the projection operator $Q_h : \mathcal{Y} \cap V \to V_h^0$. For each $u \in \mathcal{Y}$, let $f^* = -\varepsilon_i \Delta u_i$ in Ω_i, $i = 1, 2$, and $g^* = [\varepsilon \frac{\partial u}{\partial \nu}]$. Clearly, $f^* \in H$ and $g^* \in L^2(\Gamma)$. Then, we can define $Q_h : \mathcal{Y} \cap V \to V_h^0$ by

$$a_{2,h}(Q_h u, v_h) = (f^*, v_h) + \langle g^*, v_h \rangle, \quad \forall v \in V_h^0,$$

where $\langle \cdot, \cdot \rangle$ denotes the scalar product in $L^2(\Gamma)$. We note that the right-hand side $L(\cdot) := (f^*, \cdot) + \langle g^*, \cdot \rangle$ is independent of h. Thus, we can follow the proof of [18, Theorems 4.1 and 4.8], which mainly focuses on the case when $g^* = 0$, to obtain the following result.

Lemma 4.2. *We have*

$$a_2(u, v_h) = a_{2,h}(Q_h u, v_h), \quad \forall v_h \in V_h^0. \tag{4.3}$$

Moreover, for any $u \in \mathcal{Y}$, the following error estimate holds:

$$\|u - Q_h u\|_H + h\|u - Q_h u\|_V \lesssim h^2 \|u\|_{\mathcal{Y}}.$$

Now, we present some auxiliary results. For the difference between the bilinear form $a_i(\cdot,\cdot)$ and its approximated bilinear form $a_{i,h}(\cdot,\cdot)$, we have the following estimate (cf. [18, p. 27]).

Lemma 4.3. *Both $a_{1,h}(\cdot,\cdot)$ and $a_{2,h}(\cdot,\cdot)$ are bounded, and $a_{2,h}(\cdot,\cdot)$ is coercive. Moreover, the form $a_{i,h}^{\Delta}(u,v) := a_i(u,v) - a_{i,h}(u,v)$ $(i = 1, 2)$ satisfies*

$$|a_i^{\Delta}(u,v)| \lesssim |u|_{H^1(S_\lambda)} |v|_{H^1(S_\lambda)}.$$

To estimate the energy-norm and the L^2-norm of a function over S_λ, we will frequently use the following estimates (cf. [18, Lemma 2.1 and Remark 4.2]).

Lemma 4.4. *For any $u \in V$, we have*

$$\|u\|^2_{L^2(S_\lambda)} \lesssim \lambda \|u\|^2_V. \tag{4.4}$$

Moreover, for any $u \in \mathcal{Y}$,

$$|u|^2_{H^1(S_\lambda)} \lesssim \lambda \|u\|^2_{\mathcal{Y}} \tag{4.5}$$

where $|\cdot|_{H^1(S_\lambda)}$ is the H^1-semi norm.

The following estimate is critical to the establishment of our main results (cf. [18, Lemma 4.5]).

Lemma 4.5. *There exists a positive constant μ independent of h such that*

$$\|w_h\|_{H^1(S_\lambda)} \lesssim \sqrt{\frac{\lambda}{h}} \|w_h\|_{H^1(S_{\mu h})} \quad \forall w_h \in V_h.$$

4.1 Semi-discrete finite element approximation and error estimates

We now consider an auxiliary semi-discrete finite element scheme for our considered interface problem (2.1) and (3.7) and develop its optimal convergence, which will lead directly to the optimal convergence of the fully discrete scheme in section 4.2.

Problem ($\mathbf{P}_h$). Let $u_h(0) = Q_h u_0$. Find $u_h \in H^1(I; V_h^0)$ such that for a.e. $t \in I$,

$$a_{1,h}(u_h(t), v_h) + a_{2,h}(u_h'(t), v_h) = \langle f(t), v_h \rangle_{V' \times V}, \quad \forall v_h \in V_h^0. \tag{4.6}$$

We first establish the following stability estimate, which will be used in the subsequent analysis.

Lemma 4.6. *If $f \in L^2(I; V')$ and u_h is the solution to Problem $(\mathbf{P_h})$, then we have*

$$\|u_h\|_{H^1(I;V)} \lesssim \|f\|_{L^2(I;V')} + \|Q_h u_0\|_V.$$

The proof of this lemma is similar to that of Theorem 3.1, so it is omitted. We are now in a position to establish the optimal error estimate of the scheme $(\mathbf{P}_h)$ in the energy norm.

Theorem 4.7. *Let u be the solution to the interface problem (2.1) and (3.7) and u_h the solution to Problem $(\mathbf{P}_h)$, then the following error estimate holds under the assumptions (A5) and (A6):*

$$\|u - u_h\|_{H^1(I;V)} \lesssim h \left(\|f\|_{L^2(I;H)} + \|u_0\|_{\mathcal{Y}} \right).$$

Proof. We first have the following decomposition:

$$\begin{aligned}
&\left(\int_0^T \left(\|u(t) - u_h(t)\|_V^2 + \|u'(t) - u_h'(t)\|_V^2 \right) dt \right)^{\frac{1}{2}} \\
&\leq \left(\int_0^T \left(\|u(t) - Q_h u(t)\|_V^2 + \|u'(t) - Q_h u'(t)\|_V^2 \right) dt \right)^{\frac{1}{2}} \\
&\quad + \left(\int_0^T \left(\|Q_h u(t) - u_h(t)\|_V^2 + \|Q_h u'(t) - u_h'(t)\|_V^2 \right) dt \right)^{\frac{1}{2}} \\
&=: (\mathrm{I})_1 + (\mathrm{I})_2.
\end{aligned} \tag{4.7}$$

Using Lemma 4.2 and Theorem 3.6, we obtain

$$(\mathrm{I})_1 \lesssim h \|u\|_{H^1(I;\mathcal{Y})} \lesssim h \left(\|f\|_{L^2(I;H)} + \|u_0\|_{\mathcal{Y}} \right).$$

It remains to establish a desired estimate for $(\mathrm{I})_2$. To this end, we first notice that the function $w := u_h - Q_h u$ belongs to $H^1(I; V_h^0)$. In addition, using the identity that $(Q_h u)'(t) = Q_h u'(t)$ for a.e. $t \in I$ and the definition of u and u_h, we find for a.e. $t \in I$,

$$a_{1,h}(w(t), v_h) + a_{2,h}(w'(t), v_h) = \langle F(t), v_h \rangle_{V' \times V}, \quad \forall v_h \in V_h^0,$$

where $F(t) \in V'$ for $t \in I$, defined by

$$\langle F(t), v \rangle_{V' \times V} := a_1(u - Q_h u, v) + a_2(u' - Q_h u', v) + a_1^{\Delta}(Q_h u, v) + a_2^{\Delta}(Q_h u', v), \quad \forall v \in V.$$

Analogously to Lemma 4.6, we derive

$$(\mathrm{I})_2 = \|w\|_{H^1(I;V)} \lesssim \|F\|_{L^2(I;V')}. \tag{4.8}$$

Thus, it remains to estimate $\|F\|_{L^2(I;V')}$. For $t \in I$ and any $v \in V$, we use Lemma 4.3 to obtain

$$\begin{aligned} &|\langle F(t), v \rangle_{V' \times V}| \\ &\lesssim \left(\|u(t) - Q_h u(t)\|_V + |Q_h u(t)|_{H^1(S_\lambda)} + \|u'(t) - Q_h u'(t)\|_V + |Q_h u'(t)|_{H^1(S_\lambda)} \right) \|v\|_V, \end{aligned}$$

which, together with the estimates

$$\begin{aligned} |Q_h u(t)|_{H^1(S_\lambda)} &\le |u(t)|_{H^1(S_\lambda)} + |u(t) - Q_h u(t)|_{H^1(S_\lambda)}, \\ |Q_h u'(t)|_{H^1(S_\lambda)} &\le |u'(t)|_{H^1(S_\lambda)} + |u'(t) - Q_h u'(t)|_{H^1(S_\lambda)}, \end{aligned}$$

implies that

$$\|F\|_{L^2(I;V')} \lesssim \|Q_h u - u\|_{H^1(I,V)} + \|u\|_{H^1(I;H^1(S_\lambda))}.$$

Now Lemmas 4.2 and 4.4, together with Theorem 3.6, yield

$$\|F\|_{L^2(I;V')} \lesssim (h + \sqrt{\lambda}) \left(\|f\|_{L^2(I;H)} + \|u_0\|_{\mathcal{Y}} \right).$$

From this, (4.2) and (4.8), the desired estimate for $(\mathrm{I})_2$ follows readily. □

Now, we shall derive the optimal L^2-norm error estimate for the scheme $\mathbf{(P}_h\mathbf{)}$.

Theorem 4.8. *Under the assumptions of Theorem 4.7, we have the following estimate in L^2-norm:*

$$\|u - u_h\|_{L^2(I;H)} \lesssim h^2 \left(\|f\|_{L^2(I;H)} + \|u_0\|_{\mathcal{Y}} \right).$$

Proof. For the duality argument, we define $w \in H^1(I; V)$ and $w_h \in H^1(I; V_h^0)$ such that for a.e. $t \in I$,

$$\begin{aligned} a_1(w(t), v) - a_2(w'(t), v) &= (u(t) - u_h(t), v), \quad \forall v \in V, \\ a_1(w_h(t), v) - a_2(w_h'(t), v) &= (u(t) - u_h(t), v), \quad \forall v \in V_h^0, \end{aligned}$$

which satisfies $w(T) = w_h(T) = 0$. That is, $w^*(t) := w(T - t)$ is the weak solution of (2.1) with initial value $w^*(0) = 0$ and f replaced by $u - u_h$. Then Theorem 3.6 implies that

$$\|w\|_{H^1(I;\mathcal{Y})} \lesssim \|u - u_h\|_{L^2(I;H)}. \tag{4.9}$$

Using the same argument employed in Theorem 4.7 with a natural modification, we find that

$$\|w - w_h\|_{H^1(I;V))} \lesssim h \|u - u_h\|_{L^2(I;H)}. \tag{4.10}$$

By integration by parts with respect to the time variable, identity (4.3) and taking advantage of the Galerkin orthogonality for $w - w_h$ and $e := u - u_h$, we know that

$$\begin{aligned} &\int_0^T \big(a_1(e, w_h) - a_2(e, w_h') \big) dt \\ &= \int_0^T \big(a_1(e, w_h) + a_2(e', w_h) \big) dt + a_2(u(0) - Q_h(0), w_h(0)) \\ &= \int_0^T \left(-a_1^{\Delta}(u_h, w_h) - a_2^{\Delta}(u_h', w_h) \right) dt + a_2^{\Delta}(Q_h u(0), w_h(0)), \end{aligned} \tag{4.11}$$

and for a.e. $t \in I$,

$$a_1(w(t) - w_h(t), v) - a_2(w'(t) - w_h'(t), v) = 0, \quad \forall v \in V_h^0. \tag{4.12}$$

Applying (4.11) and (4.12) and integrating by parts with respect to time variable, we obtain

$$\begin{aligned} \|e\|^2_{L^2(I;H)} &= \int_0^T \big(a_1(u - Q_h u, w - w_h) + a_2(u' - Q_h u', w - w_h)\big)dt \\ &\quad - \int_0^T \left(a_1^\Delta(u_h, w_h) + a_2^\Delta(u_h', w_h)\right) dt \\ &\quad + a_2^\Delta(Q_h u(0), w_h(0)) + a_2(u(0) - Q_h u(0), w(0) - w_h(0)) \\ &=: (\mathrm{II})_1 + (\mathrm{II})_2 + (\mathrm{II})_3. \end{aligned}$$

The rest of the proof can be done in a similar argument to that used for the standard elliptic interface problem (see, e.g., [18, Theorem 4.6]). First by applying the Cauchy-Schwarz inequality, Lemma 4.2, the regularity estimate (4.9) and Theorem 4.7, we can establish the desired boundedness of $(\mathrm{II})_1$. Then we can derive some necessary bounds of u_h, w and w_h in $H^1(I; H^1(S_\lambda))$ by means of Lemmas 4.2-4.5, the regularity estimates (4.9), (4.10) and the condition $2\lambda \le h$, and further estimate $(\mathrm{II})_2$ and $(\mathrm{II})_3$, where the embedding result (2.5) is also needed. The detailed proof can be found in [2]. □

4.2 Fully discrete finite element scheme and error estimates

In this subsection, we are going to formulate a fully discrete finite element scheme to approximate the solution to the interface problem (2.1) and (3.7). We shall use the backward Euler scheme for the time discretization. Let us start with dividing the time interval I into N equally spaced subintervals and using the following nodal points:

$$0 = t^0 < t^1 < \cdots < t^N = T,$$

where $t^n = n\tau$ for $n = 0, 1, \cdots, N$ and $\tau = T/N$. For any given discrete time sequence $\{u^n\}_{n=0}^N$ in V and a function $g(x,t)$ which is continuous with respect to t, we can define

$$\partial_\tau w^n = \frac{w^n - w^{n-1}}{\tau}, \quad \overline{g}^n = \frac{1}{\tau}\int_{t^{n-1}}^{t^n} g(\cdot, s)ds, \quad \widehat{g}^n(\cdot) = g(\cdot, t^n), \quad n = 1, \cdots, N.$$

Now, we propose a fully discrete finite element scheme to approximate the solution to the interface problem (2.1) and (3.7).

Problem ($\mathbf{P_{h,\tau}}$). Let $u_h^0 = Q_h u_0$. For each $n = 1, 2, \cdots, N$, find $u_h^n \in V_h^0$ such that

$$a_{1,h}(u_h^n, v_h) + a_{2,h}(\partial_\tau u_h^n, v_h) = (\widehat{f}^n, v_h), \quad \forall v_h \in V_h^0. \tag{4.13}$$

For a discrete sequence $\{u_h^n\}_{n=1}^N$ defined in Problem **($\mathbf{P_{h,\tau}}$)**, we can introduce a piecewise constant function in time by

$$u_{h,\tau}(\cdot, t) = u_h^n(\cdot) \quad \forall t \in (t^{n-1}, t^n], \quad n = 1, 2, \cdots, N. \tag{4.14}$$

Then, we say that $u_{h,\tau}$ is a solution of Problem **($\mathbf{P_{h,\tau}}$)**, which is a fully discrete approximation of the solution to the interface problem (2.1) and (3.7). In order to compute the error between $u_{h,\tau}$ and u, it suffices to establish the error between $u_{h,\tau}$ and u_h, i.e. the solution of the semi-discrete scheme (4.6), since the error between u_h and u has been studied in Section 4.1. To this end, we need the following auxiliary result, which can be obtained by applying the Lax-Milgram theorem, taking $v_h = 2\tau\partial w_h^n$ in (4.15) and then using some standard arguments (see [2]).

Lemma 4.9. *Let $\{F_n\}_{n=1}^N$ be a time discrete sequence lying in V' and $w_h^0 = 0$. There exists a unique sequence $\{w_h^n\}_{n=1}^N$ such that for $n = 1, 2, \cdots, N$,*

$$a_{1,h}(w_h^n, v_h) + a_{2,h}(\partial_\tau w_h^n, v_h) = \langle F_n, v\rangle_{V' \times V}, \quad \forall v_h \in V_h^0. \tag{4.15}$$

Moreover, the sequence $\{w_h^n\}_{n=1}^N$ *has the following stability estimate:*

$$\max_{1\le n\le N}\|w_h^n\|_V^2 \lesssim \tau\sum_{n=1}^{N}\|F_n\|_{V'}^2. \tag{4.16}$$

From the lemma above we notice that Problem $\mathbf{(P_{h,\tau})}$ always admits a unique solution.

Lemma 4.10. *Let* $u_{h,\tau}$ *and* u_h *be the solution of Problem* $\mathbf{(P_{h,\tau})}$ *and Problem* $\mathbf{(P_h)}$*, respectively, then under the assumptions* $(A5)$ *and* $(A6)$ *and the condition that* $f\in H^1(I;H)$*, the following estimate holds:*

$$\|u_h-u_{\tau,h}\|_{L^2(I;V)}\lesssim \tau\left(\|f'\|_{L^2(I;H)}+\|f\|_{L^2(I;H)}+\|u_0\|_{\mathcal{Y}}\right).$$

Proof. We first define a piecewise constant function in time such that $u_{h,\tau}^*(0)=Q_hu_0$ and

$$u_{h,\tau}^*(\cdot,t)=\widehat{u}_h^n(\cdot),\quad \forall t\in(t^{n-1},t^n],\quad n=1,2,\cdots,N.$$

Using Lemmas 4.2 and 4.6, it follows readily that

$$\|u_h-u_{h,\tau}^*\|_{L^2(I;V)}\lesssim\tau\|u_h\|_{H^1(I;V)}\lesssim\tau(\|f\|_{L^2(I;H)}+\|u_0\|_{\mathcal{Y}}). \tag{4.17}$$

Integrating (4.6) over (t^{n-1},t^n) and dividing both sides by τ, we have for $n=1,2,\cdots,N$,

$$a_{1,h}(\overline{u}_h^n,v_h)+a_{2,h}(\partial_\tau\widehat{u}_h^n,v_h)=(\overline{f}^n,v_h),\quad\forall v_h\in V_h^0. \tag{4.18}$$

Subtracting both sides of (4.18) above from those of (4.13), we can rewrite the resulting equation as

$$a_{1,h}(u_h^n-\widehat{u}_h^n,v_h)+a_{2,h}(\partial_\tau(u_h^n-\widehat{u}_h^n),v_h)=(\widehat{f}^n-\overline{f}^n,v_h)+a_{1,h}(\overline{u}_h^n-\widehat{u}_h,v_h),\quad\forall v_h\in V_h^0.$$

The right-hand side of the equation above defines a functional on V for each $n=1,2,\cdots,N$. Then we can apply Lemma 4.9, Theorem 4.7 and the triangle inequality to obtain the desired result; see [2] for more detail. □

From Lemma 4.10 and Theorems 4.7 and 4.8, the following result follows immediately.

Theorem 4.11. *Let* u *be the solution to the interface problem* (2.1) *and* (3.7) *and* $u_{h,\tau}$ *the solution to Problem* $\mathbf{(P_{h,\tau})}$*, then under the assumptions* $(A5)$ *and* $(A6)$ *and the condition that* $f\in H^1(I;H)$*, the following estimates hold:*

$$\|u-u_{h,\tau}\|_{L^2(I;V)}\lesssim(\tau+h)\left(\|f\|_{L^2(I;H)}+\|f'\|_{L^2(I;H)}+\|u_0\|_{\mathcal{Y}}\right),$$

$$\|u-u_{h,\tau}\|_{L^2(I;H)}\lesssim\left(\tau+h^2\right)\left(\|f\|_{L^2(I;H)}+\|f'\|_{L^2(I;H)}+\|u_0\|_{\mathcal{Y}}\right).$$

5 Concluding remarks

This work performs a systematic mathematical and finite element analysis on the effects of pulsed electric fields on heterogeneous physical media. Under rather general conditions on the given data, we have established the well-posedness of the governing time-dependent electric interface problem and the regularity of its solutions. Then we have proposed a fully discrete finite element scheme for the numerical approximation of the solutions to the interface problem, and derived the optimal error estimates of the finite element solution in both energy-norm and L^2-norm.

Time-dependent interface problems are frequently encountered in scientific computing and many applied sciences. The typical mathematical models are the parabolic- or wave-type interface equations with discontinuous coefficients, which arise when the physical processes involve two or more materials or media with different properties, such as the bulk modulus in acoustic propagation or the conductivity in heat diffusion; see [5, 7, 8] and references therein. We believe the new results and numerical scheme

here for a non-standard time-dependent interface problem may motivate some further research directions, and find more potential applications in the areas like electromagnetism, biomedicine, food sciences, and biotechnology.

For the sake of exposition and clarity, we have chosen to avoid a few rather technical but practically important issues in the current work. But the theory of this work can be generalized in principle in two directions. First of all, general high-order finite elements can be studied by combining the theory in this work with the analysis in [18]. Second, the model system in this work can be extended to the following time-dependent interface problem of more general form:

$$-\Big(\sum_{i,j=1}^{n} D_i(\sigma_{i,j}(x)D_j u(t,x)) + \sum_{i,j=1}^{n} D_i(\varepsilon_{i,j}(x)D_j u'(t,x))\Big) = f(t,x)\,, \tag{5.1}$$

complemented with the interface conditions:

$$[u] = 0\,, \quad [\frac{du}{dN_1} + \frac{du'}{dN_2}] = g \quad \text{on the interface } \Gamma,$$

where $g(t,x)$ is a known function representing the interface source strength, and two derivatives du/dN_1 and du/dN_2 are given by $du/dN_1 = \sum_{i,j=1}^{n} \sigma_{i,j} \frac{\partial u}{\partial x_j} \cos(\nu, x_i)$ and $du/dN_2 = \sum_{i,j=1}^{n} \varepsilon_{i,j} \frac{\partial u}{\partial x_j} \cos(\nu, x_i)$. Naturally, the coefficients $\{\sigma_{i,j}(x)\}$ and $\{\varepsilon_{i,j}(x)\}$ in (5.1) are assumed to satisfy the similar conditions to (3.18).

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