A Convergent Adaptive Edge Element Method for an Optimal Control Problem in Magnetostatics

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Abstract

This work is concerned with an adaptive edge element solution of an optimal control problem associated with a magnetostatic saddle-point Maxwell's system. An a posteriori error estimator of the residue type is derived for the lowest-order edge element approximation of the problem and proved to be both reliable and efficient. With the estimator and a general marking strategy, we propose an adaptive edge element method, which is demonstrated to generate a sequence of discrete solutions converging strongly to the exact solution satisfying the resulting optimality conditions and guarantee a vanishing limit of the error estimator.

Keywords: optimal control, magnetostatic Maxwell equation, a posteriori error estimate, edge element, adaptive convergence.

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1 Introduction

We are concerned in this work with the following stationary saddle-point system [5] [6] [16]

$$\left\{\begin{array}{rcll} \boldsymbol{\nabla}\times(\nu\boldsymbol{\nabla}\times\boldsymbol{y}) &=& \chi_c\boldsymbol{u} & \text{in} \quad \Omega, \\ \boldsymbol{\nabla}\cdot\boldsymbol{y} &=& 0 & \text{in} \quad \Omega, \\ \boldsymbol{y}\times\boldsymbol{n} &=& \boldsymbol{0} & \text{on} \quad \partial\Omega, \end{array}\right. \tag{1.1}$$

which is often encountered in magnetostatic simulations. Here $\Omega\subset\mathbb{R}^3$ is an open bounded polyhedral domain with a connected boundary $\partial\Omega$, $\boldsymbol{n}$ is the unit outward normal on $\partial\Omega$. The three-dimensional vector $\boldsymbol{u}\in\boldsymbol{L}^2(\Omega_c)$ represents an exciting current density in a Lipschitz polyhedral subdomain Ω_c satisfying $\overline{\Omega}_c\subset\Omega$ and χ_c is the characteristic function of Ω_c. The coefficient $\nu(\boldsymbol{x})$ is the inverse of the magnetic permeability and is assumed to be piecewisely $W^{1,\infty}(\Omega)$ such that $0<\nu_1\leq\nu(\boldsymbol{x})\leq\nu_2$ a.e. in Ω for two positive constants ν_1 and ν_2.

Edge elements are very popular in numerical solutions of the saddle-point system (1.1), resulting in some symmetric positive definite systems, which arise from the first equation of (1.1) with an extra zeroth order term [6] [8].

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The main interest of this work is to propose an adaptive edge element method for an optimal control problem related to the system (1.1) with the applied current density $\boldsymbol{u}$ and the potential $\boldsymbol{y}$ as the control and the state respectively. Mathematically, it is formulated as a constrained minimization problem [28]:

$$\min_{\boldsymbol{u}\in\boldsymbol{U}} \mathcal{J}(\boldsymbol{u}) = \frac{1}{2}\|\boldsymbol{\nabla}\times\boldsymbol{y}(\boldsymbol{u}) - \boldsymbol{\nabla}\times\boldsymbol{y}_d\|_0^2 + \frac{\gamma}{2}\|\boldsymbol{u}\|_{0,\Omega_c}^2, \tag{1.2}$$

where $\boldsymbol{y}(\boldsymbol{u})$ solves the system (1.1), the desired field $\boldsymbol{y}_d \in \boldsymbol{H}_0(\mathbf{curl};\Omega)$ and $\boldsymbol{\nabla}\times\boldsymbol{\nabla}\times\boldsymbol{y}_d \in \boldsymbol{L}^2(\Omega)$ (all the subsequent results can be easily extended to the case when $\boldsymbol{\nabla}\times\boldsymbol{y}_d$ in (1.2) is replaced by a target field $\tilde{\boldsymbol{y}}_d \in \boldsymbol{L}^2(\Omega)$ with $\boldsymbol{\nabla}\times\tilde{\boldsymbol{y}}_d \in \boldsymbol{L}^2(\Omega)$). The constant γ is a stabilisation parameter and the admissible space is defined as

$$\boldsymbol{U} := \{\boldsymbol{u}\in\boldsymbol{L}^2(\Omega_c) \mid (\boldsymbol{u},\boldsymbol{\nabla}q)_{\Omega_c} = 0 \quad \forall\, q\in H^1(\Omega_c)\}. \tag{1.3}$$

The mathematical theory, including sensitivity analysis of control-to-state mapping, optimality conditions and regularity of the optimal control, of the problem (1.2) has been investigated under some reasonable assumptions on the nonlinear reluctivity $\nu(\boldsymbol{x},|\boldsymbol{B}|)$ in [28]. Moreover, relevant a priori error finite element analysis is also conducted when the control and the state are both discretized by the lowest order edge elements of Nédélec's first family. We also mention [23] [27] for latest results on optimal control problems in electromagnetism. However, the existing studies have still not focused on numerical treatments of the practically important situations when the solution to the problem (1.1)-(1.2) encounters local singularities or internal interface layers due to reentrant corners on $\partial\Omega$ or jumps of the coefficient ν across interfaces of different media, which affects numerical performance and accuracy greatly on uniformly refined meshes. Adaptive finite elements are an popular and effective strategy to improve local accuracies of numerical solutions.

An adaptive finite element method (AFEM) typically takes the successive loops of the form:

$$\mathsf{SOLVE} \to \mathsf{ESTIMATE} \to \mathsf{MARK} \to \mathsf{REFINE}. \tag{1.4}$$

That is, one first solves the discrete problem for the finite element solution on the current mesh, computes the relevant a posteriori error estimator, marks elements to be subdivided, and then refines the current mesh to generate a new finer one.

The most crucial ingredient of the above process is the module ESTIMATE, which measures the error in terms of some computable quantities formed by the discrete solution, the mesh size and the given data, i.e., a posteriori error estimation. This topic has been examined in depth for partial differential equations; see two systematic reviews [1] [24] and the references therein. In the past decade many important progresses have been made on a posteriori error analyses for PDE-constrained optimal control problems, see, e.g., [11] [15]. As far as adaptive finite elements for Maxwell' s equations are concerned, the theory has already reached a mature level; see [2] [5] [7] [19]. But the development of adaptive methods for optimal control problems of Maxwell's equations is still at an early stage. A residual-type a posteriori error estimator is obtained in [13] for the lowest order edge element approximation of an $\boldsymbol{H}(\mathbf{curl})$-elliptic distributed control problem with a pointwise control constraint.

On the other hand, the convergence and computational complexity of AFEM have also been investigated extensively in the past decade. The issue has been well understood for second order linear elliptic problems; see [4] [17] [18] [21]. A very recent survey in [3] provides an abstract framework for quasi-optimal convergence rates of various adaptive schemes based on conforming, nonconforming and mixed methods for second order linear and nonlinear problems. The theory of AFEM has also been generalized to adaptive edge element methods for the Maxwell's equations; see [5] [12] [21] [29]. But as far as optimal control problems are concerned, we are only aware of the work [10] on an asymptotic error reduction property of an adaptive finite element approximation for the distributed control problems. The authors have recently studied the AFEM for the PDE-based inverse problems, and established the convergence of AFEM for the problem of flux reconstruction in [26].

The aim of this work is two fold. First, we shall derive an a posteriori error estimator for the optimality system of the constrained minimization problem (1.1)-(1.2) when the lowest order edge element of the first family is used for approximation. Then an adaptive algorithm of the form (1.4) will be proposed and its convergence will be established: the sequence of adaptively-generated minimizers to discrete problems converges strongly to the minimizer of the problem (1.1)-(1.2) and the relevant estimator is a null sequence. We note that the algorithm under consideration is of the same framework as the standard one for elliptic problems (e.g. [4] [17] [18]). Thus it is of great convenience from the point of view of implementation. In the course of analysis, we shall make full use of some effective arguments and analysis tools in literature for linear elliptic and electrogmagnetic problems and their related optimal controls, in addition to several new techniques introduced here to handle some essential difficulties and major differences due to the constraint system (1.1) and the control constraint (cf. (1.3)).

An adaptive FEM was investigated in [13] for the optimal control problem associated with the Maxwell system, where only the $\boldsymbol{H}(\mathbf{curl})$-elliptic system is considered, namely the first equation in (1.1) has an extra zeroth order term. Our current interest is the more technical case where the zeroth order term is absent, so the divergence constraint must be enforced for the uniqueness, resulting in the saddle-point system (1.1), for which a Lagrangian multiplier will be introduced in the state equation.

In the finite element analysis of [28], in parallel with the continuous case the discrete control set is required to be the orthogonal complement of the edge element space to the gradient of the continuous linear element space. Then the L^2-norm convergence of the discrete minimizers follows from the discrete compactness property of the edge elements [16]. However, it remains still open whether this property is true for a family of triangulations that are generated by adaptive local refinements, where the mesh sizes of the triangulations may converge to zero only over part of the domain Ω. So for proving a similar convergence result over adaptively refined meshes, we shall resort to the weak convergence and a simple yet crucial observation: the sequence of minima to discrete objective functionals converges to the minimum of a functional with respect to some limiting optimization problem.

Next, we give a brief description of our subsequent arguments. Thanks to the special structure of the system (1.1), we shall adopt an equivalent energy norm, the inf-sup condition, a regular decomposition and two quasi-interpolation operators in deriving the estimator for errors of the state, the costate and the control in the a posteriori error estimation; see section 3. Important in the course of convergence analysis is an auxiliary limit of discrete minimizers/discrete triplets (the approximate state, costate and control) given by the adaptive process (1.4). By applying techniques from nonlinear optimization, we first achieve the convergence of discrete cost functionals (the proof of Theorem 5.1). Then the weak limit of discrete controls can be upgraded to a strong one (Theorem 5.2), by which, a norm convergence of the discrete state and costate variables is further guaranteed (Theorem 5.3). The first desired convergence result (Theorem 6.1) follows after we verify that the limiting triplet also satisfies the optimality conditions for the problem (1.1)-(1.2), and the second result (Theorem 6.2) is established by the help of the efficiency; see section 6.

The rest of this paper is organized as follows. In section 2, we present the optimality system of the problem (1.1)-(1.2) and the corresponding edge element method. Section 3 is devoted to the reliability and the efficiency of an residual-type error estimator, which allows us to design an adaptive algorithm in section 4. We discuss the convergence of discrete solutions to some limiting triplet in section 5 before main results are given in section 6. The paper is ended with some concluding remarks in section 7.

Throughout the paper we adopt the standard notation for the Lebesgue space $L^\infty(G)$ and Sobolev spaces $W^{m,p}(G)$ for integer $m \geq 0$ on an open bounded domain $G \subset \mathbb{R}^3$. Related norms and seminorms of $H^m(G)$ $(p=2)$ as well as the norm of $L^\infty(G)$ are denoted by $\|\cdot\|_{m,G}$, $|\cdot|_{m,G}$ and $\|\cdot\|_{\infty,G}$ respectively. We use $(\cdot,\cdot)_G$ to denote the L^2 scalar product G, and the subscript is omitted when $G=\Omega$. Moreover, we shall use C, with or without subscript, for a generic constant independent of the mesh size and it may take a different value at each occurrence.

2 Variational formulation and edge element approximation

For numerical treatments by edge elements, we need to reformulate the system (1.1) as a variational problem. For this purpose, we need the following Sobolev spaces

$$\begin{aligned}
&\boldsymbol{H}_0(\mathbf{curl};\Omega) = \{\boldsymbol{v} \in \boldsymbol{L}^2(\Omega) \mid \boldsymbol{\nabla}\times\boldsymbol{v} \in \boldsymbol{L}^2(\Omega),\ \boldsymbol{v}\times\boldsymbol{n} = \mathbf{0} \text{ on } \partial\Omega\},\\
&\boldsymbol{X} = \{\boldsymbol{v} \in \boldsymbol{H}_0(\mathbf{curl};\Omega) \mid (\boldsymbol{v},\boldsymbol{\nabla} q) = 0\ \forall\, q \in H_0^1(\Omega)\},\\
&\boldsymbol{H}(\mathbf{curl};\Omega_c) = \{\boldsymbol{v} \in \boldsymbol{L}^2(\Omega_c) \mid \boldsymbol{\nabla}\times\boldsymbol{v} \in \boldsymbol{L}^2(\Omega_c)\},\\
&\boldsymbol{X}^c = \{\boldsymbol{v} \in \boldsymbol{H}(\mathbf{curl};\Omega_c) \mid (\boldsymbol{v},\boldsymbol{\nabla} q) = 0\ \forall\, q \in H^1(\Omega_c)\},
\end{aligned}$$

all of which are equipped with graph norms $\|\cdot\|_{\boldsymbol{H}(\mathbf{curl})}$. With a Lagrange multiplier $\phi \in H_0^1(\Omega)$ introduced to relax the divergence condition in (1.1), integration by parts yields the following saddle-point problem: find $(\boldsymbol{y}(\boldsymbol{u}),\phi(\boldsymbol{u})) \in \boldsymbol{H}_0(\mathbf{curl};\Omega)\times H_0^1(\Omega)$ such that

$$\begin{cases} (\nu\boldsymbol{\nabla}\times\boldsymbol{y},\boldsymbol{\nabla}\times\boldsymbol{v}) + (\boldsymbol{v},\boldsymbol{\nabla}\phi) = (\boldsymbol{u},\boldsymbol{v})_{\Omega_c} \quad \forall\, \boldsymbol{v} \in \boldsymbol{H}_0(\mathbf{curl};\Omega),\\ (\boldsymbol{y},\boldsymbol{\nabla} q) = 0 \quad \forall\, q \in H_0^1(\Omega). \end{cases} \tag{2.1}$$

As the control-to-state map $\boldsymbol{y}(\boldsymbol{u})$ is linear, standard arguments from optimization implies a unique solution $(\boldsymbol{u}^*,\boldsymbol{y}^*,\phi^*) \in \boldsymbol{U}\times\boldsymbol{H}_0(\mathbf{curl};\Omega)\times H_0^1(\Omega)$ to the problem (1.2) and (2.1) [28]. For our later use, we collect two important results here, i.e., the Poincaré-type inequality and the inf-sup condition (cf. [6] [16] [28]):

$$\|\boldsymbol{v}\|_0 \leq C\|\boldsymbol{\nabla}\times\boldsymbol{v}\|_0 \quad \forall\, \boldsymbol{v} \in \boldsymbol{X}\,, \tag{2.2}$$

$$\sup_{\mathbf{0}\neq\boldsymbol{v}\in\boldsymbol{H}_0(\mathbf{curl};\Omega)} \frac{(\boldsymbol{v},\boldsymbol{\nabla} q)}{\|\boldsymbol{v}\|_{\boldsymbol{H}(\mathbf{curl})}} \geq C\|q\|_1 \quad \forall\, q \in H_0^1(\Omega) \tag{2.3}$$

where C depends only on Ω. A direct consequence of (2.2) is that $\|\boldsymbol{\nabla}\times\cdot\|_0$ is equivalent to the graph norm on $\boldsymbol{X}$. Noting that $\boldsymbol{X}$ and $\boldsymbol{\nabla} H_0^1(\Omega)$ are L^2-orthogonal and $\boldsymbol{H}_0(\mathbf{curl};\Omega) = \boldsymbol{X}\oplus\boldsymbol{\nabla} H_0^1(\Omega)$ [16], we may define an alternative norm equivalent to the graph one on $\boldsymbol{H}_0(\mathbf{curl};\Omega)$: $(\|\boldsymbol{\nabla}\times\boldsymbol{v}\|_0^2 + \|\boldsymbol{v}^0\|_0^2)^{1/2}$, with $\boldsymbol{v}^0$ being the L^2-projection of $\boldsymbol{v}$ on $\boldsymbol{\nabla} H_0^1(\Omega)$.

With a costate $\boldsymbol{p}^* \in \boldsymbol{H}_0(\mathbf{curl};\Omega)$ and a corresponding Lagrangian multiplier $\psi^* \in H_0^1(\Omega)$ involved, the solution $(\boldsymbol{u}^*,\boldsymbol{y}^*,\phi^*) \in \boldsymbol{U}\times\boldsymbol{H}_0(\mathbf{curl};\Omega)\times H_0^1(\Omega)$ to the problem (1.2) and (2.1) is characterized by the following optimality conditions [28]:

$$\begin{cases} (\nu\boldsymbol{\nabla}\times\boldsymbol{y}^*,\boldsymbol{\nabla}\times\boldsymbol{v}) + (\boldsymbol{v},\boldsymbol{\nabla}\phi^*) = (\boldsymbol{u}^*,\boldsymbol{v})_{\Omega_c} \quad \forall\, \boldsymbol{v} \in \boldsymbol{H}_0(\mathbf{curl};\Omega),\\ (\boldsymbol{y}^*,\boldsymbol{\nabla} q) = 0 \quad \forall\, q \in H_0^1(\Omega), \end{cases} \tag{2.4}$$

$$\begin{cases} (\nu\boldsymbol{\nabla}\times\boldsymbol{p}^*,\boldsymbol{\nabla}\times\boldsymbol{v}) + (\boldsymbol{v},\boldsymbol{\nabla}\psi^*) = (\boldsymbol{\nabla}\times\boldsymbol{y}^* - \boldsymbol{\nabla}\times\boldsymbol{y}_d,\boldsymbol{\nabla}\times\boldsymbol{v}) \quad \forall\, \boldsymbol{v} \in \boldsymbol{H}_0(\mathbf{curl};\Omega),\\ (\boldsymbol{p}^*,\boldsymbol{\nabla} q) = 0 \quad \forall\, q \in H_0^1(\Omega), \end{cases} \tag{2.5}$$

$$(\boldsymbol{u}^* + \gamma^{-1}\boldsymbol{p}^*,\boldsymbol{u})_{\Omega_c} = 0 \quad \forall\, \boldsymbol{u} \in \boldsymbol{U}. \tag{2.6}$$

Noting that $\boldsymbol{u}^* \in \boldsymbol{U}$, we can easily see the Lagrange multiplier $\phi^* = 0$ by taking $\boldsymbol{v} = \boldsymbol{\nabla}\phi^*$ in the first equation of (2.4). Similarly, we have $\psi^* = 0$ by taking $\boldsymbol{v} = \boldsymbol{\nabla}\psi^*$ in the first equation of (2.5).

Next we introduce a finite element method to approximate the constrained minimization problem (1.2) and (2.1). Let $\mathcal{T}$ be a conforming and shape-regular triangulation of $\overline{\Omega}$ into a set of closed tetrahedra such that the local meshsize $h_T := |T|^{1/3}$ and the coefficient function ν is piecewise $W^{1,\infty}$ over $\mathcal{T}$. When restricted on the control region Ω_c, $\mathcal{T}$ induces a subset $\mathcal{T}^c$ satisfying $\overline{\Omega}_c = \bigcup_{T\in\mathcal{T}^c} T$. Then the lowest order edge element space of the first family is defined by [16]

$$\boldsymbol{V}_{\mathcal{T}} = \{\boldsymbol{v} \in \boldsymbol{H}_0(\mathbf{curl};\Omega) \mid \boldsymbol{v}|_T = \boldsymbol{a}_T + \boldsymbol{b}_T \times \boldsymbol{x} \quad \boldsymbol{a}_T, \boldsymbol{b}_T \in \mathbb{R}^3,\ \forall\, T \in \mathcal{T}\}.$$

For the numerical treatment of the Lagrange multiplier, we also need the standard $H_0^1(\Omega)$-conforming piecewise linear finite element space $S_{\mathcal{T}}$ [9], for which we know the following inclusion relation [16]

$$\boldsymbol{\nabla} S_{\mathcal{T}} \subset \boldsymbol{V}_{\mathcal{T}}. \tag{2.7}$$

Now we take $\boldsymbol{V}_{\mathcal{T}}^c := \boldsymbol{V}_{\mathcal{T}}|_{\Omega_c}$ and $S_{\mathcal{T}}^c$ to be the $H^1(\Omega_c)$-conforming piecewise linear finite element space, then introduce the following discrete admissible space for controls:

$$\boldsymbol{U}_{\mathcal{T}} = \{\boldsymbol{v} \in \boldsymbol{V}_{\mathcal{T}}^c \mid (\boldsymbol{v}, \boldsymbol{\nabla} q)_{\Omega_c} = 0 \ \forall\, q \in S_{\mathcal{T}}^c\}. \tag{2.8}$$

Now we approximate the optimal control problem (1.2) and (2.1) by the following discrete system

$$\min_{\boldsymbol{u}_{\mathcal{T}} \in \boldsymbol{U}_{\mathcal{T}}} \mathcal{J}_{\mathcal{T}}(\boldsymbol{u}_{\mathcal{T}}) = \frac{1}{2}\|\boldsymbol{\nabla}\times\boldsymbol{y}_{\mathcal{T}}(\boldsymbol{u}_{\mathcal{T}}) - \boldsymbol{\nabla}\times\boldsymbol{y}_d\|_0^2 + \frac{\gamma}{2}\|\boldsymbol{u}_{\mathcal{T}}\|_{0,\Omega_c}^2, \tag{2.9}$$

where $\boldsymbol{y}_{\mathcal{T}} := \boldsymbol{y}_{\mathcal{T}}(\boldsymbol{u}_{\mathcal{T}}) \in \boldsymbol{V}_{\mathcal{T}}$ and $\phi_{\mathcal{T}} := \phi_{\mathcal{T}}(\boldsymbol{u}_{\mathcal{T}}) \in S_{\mathcal{T}}$ satisfy the discrete problem:

$$\begin{cases} (\nu\boldsymbol{\nabla}\times\boldsymbol{y}_{\mathcal{T}}, \boldsymbol{\nabla}\times\boldsymbol{v}_{\mathcal{T}}) + (\boldsymbol{v}_{\mathcal{T}}, \boldsymbol{\nabla}\phi_{\mathcal{T}}) = (\boldsymbol{u}_{\mathcal{T}}, \boldsymbol{v}_{\mathcal{T}})_{\Omega_c} \quad \forall\, \boldsymbol{v}_{\mathcal{T}} \in \boldsymbol{V}_{\mathcal{T}}, \\ (\boldsymbol{y}_{\mathcal{T}}, \boldsymbol{\nabla} q_{\mathcal{T}}) = 0 \quad \forall\, q_{\mathcal{T}} \in S_{\mathcal{T}}. \end{cases} \tag{2.10}$$

As in the continuous case, there exists a unique minimizer $\boldsymbol{u}_{\mathcal{T}}^* \in \boldsymbol{U}_{\mathcal{T}}$ and a corresponding pair $(\boldsymbol{y}_{\mathcal{T}}^*, \phi_{\mathcal{T}}^*) \in \boldsymbol{V}_{\mathcal{T}} \times S_{\mathcal{T}}$ to the problem (2.9) and (2.10) [28], based on the following discrete Poincaré inequality and the inf-sup condition [6] [16] [28]:

$$\|\boldsymbol{v}\|_0 \le C\|\boldsymbol{\nabla}\times\boldsymbol{v}\|_0 \quad \forall\, \boldsymbol{v} \in \boldsymbol{X}_{\mathcal{T}}, \tag{2.11}$$

$$\sup_{\boldsymbol{0}\neq\boldsymbol{v}\in\boldsymbol{V}_{\mathcal{T}}} \frac{(\boldsymbol{v}, \boldsymbol{\nabla} q)}{\|\boldsymbol{v}\|_{\boldsymbol{H}(\mathbf{curl})}} \ge C\|q\|_1 \quad \forall\, q \in S_{\mathcal{T}} \tag{2.12}$$

where constant C depends only on Ω and the shape-regularity of $\mathcal{T}$, and

$$\boldsymbol{X}_{\mathcal{T}} := \{\boldsymbol{v} \in \boldsymbol{V}_{\mathcal{T}} \mid (\boldsymbol{v}, \boldsymbol{\nabla} q) = 0,\ \forall\, q \in S_{\mathcal{T}}\}.$$

By introducing a costate $\boldsymbol{p}_{\mathcal{T}}^* \in \boldsymbol{V}_{\mathcal{T}}$ and a corresponding multiplier $\psi_{\mathcal{T}}^* \in S_{\mathcal{T}}$, we have the optimality conditions for the solution $(\boldsymbol{u}_{\mathcal{T}}^*, \boldsymbol{y}_{\mathcal{T}}^*, \phi_{\mathcal{T}}^*) \in \boldsymbol{U}_{\mathcal{T}} \times \boldsymbol{V}_{\mathcal{T}} \times S_{\mathcal{T}}$ to the problem (2.9)-(2.10):

$$\begin{cases} (\nu\boldsymbol{\nabla}\times\boldsymbol{y}_{\mathcal{T}}^*, \boldsymbol{\nabla}\times\boldsymbol{v}_{\mathcal{T}}) + (\boldsymbol{v}_{\mathcal{T}}, \boldsymbol{\nabla}\phi_{\mathcal{T}}^*) = (\boldsymbol{u}_{\mathcal{T}}^*, \boldsymbol{v}_{\mathcal{T}})_{\Omega_c} \quad \forall\, \boldsymbol{v}_{\mathcal{T}} \in \boldsymbol{V}_{\mathcal{T}}, \\ (\boldsymbol{y}_{\mathcal{T}}^*, \boldsymbol{\nabla} q_{\mathcal{T}}) = 0 \quad \forall\, q_{\mathcal{T}} \in S_{\mathcal{T}}, \end{cases} \tag{2.13}$$

$$\begin{cases} (\nu\boldsymbol{\nabla}\times\boldsymbol{p}_{\mathcal{T}}^*, \boldsymbol{\nabla}\times\boldsymbol{v}_{\mathcal{T}}) + (\boldsymbol{v}_{\mathcal{T}}, \boldsymbol{\nabla}\psi_{\mathcal{T}}^*) = (\boldsymbol{\nabla}\times\boldsymbol{y}_{\mathcal{T}}^* - \boldsymbol{\nabla}\times\boldsymbol{y}_d, \boldsymbol{\nabla}\times\boldsymbol{v}_{\mathcal{T}}) \quad \forall\, \boldsymbol{v}_{\mathcal{T}} \in \boldsymbol{V}_{\mathcal{T}}, \\ (\boldsymbol{p}_{\mathcal{T}}^*, \boldsymbol{\nabla} q_{\mathcal{T}}) = 0 \quad \forall\, q_{\mathcal{T}} \in S_{\mathcal{T}}, \end{cases} \tag{2.14}$$

$$(\boldsymbol{u}_{\mathcal{T}}^* + \gamma^{-1}\boldsymbol{p}_{\mathcal{T}}^*, \boldsymbol{u}_{\mathcal{T}})_{\Omega_c} = 0 \quad \forall\, \boldsymbol{u}_{\mathcal{T}} \in \boldsymbol{U}_{\mathcal{T}}. \tag{2.15}$$

As for the continuous case, we can easily see $\phi_{\mathcal{T}}^* = 0$ and $\psi_{\mathcal{T}}^* = 0$ by taking $\boldsymbol{v}_{\mathcal{T}} = \boldsymbol{\nabla}\phi_{\mathcal{T}}^*$ and $\boldsymbol{\nabla}\psi_{\mathcal{T}}^*$ in (2.13) and (2.14) respectively.

We shall need the following stability results for finite element solutions to the problem (2.10), (2.13) and (2.14), which are consequences of the Babuska-Brezzi theory:

$$\|\boldsymbol{y}_{\mathcal{T}}\|_{\boldsymbol{H}(\mathbf{curl})} + |\phi_{\mathcal{T}}|_1 \leq C\|\boldsymbol{u}_{\mathcal{T}}\|_{0,\Omega_c}, \tag{2.16}$$

$$\|\boldsymbol{y}^*_{\mathcal{T}}\|_{\boldsymbol{H}(\mathbf{curl})} \leq C\|\boldsymbol{u}^*_{\mathcal{T}}\|_{0,\Omega_c}, \quad \|\boldsymbol{p}^*_{\mathcal{T}}\|_{\boldsymbol{H}(\mathbf{curl})} \leq C(\|\boldsymbol{u}^*_{\mathcal{T}}\|_{0,\Omega_c} + \|\boldsymbol{\nabla}\times \boldsymbol{y}_d\|_0). \tag{2.17}$$

Remark 2.1. The derivation of (2.6) is mainly based on the Helmholtz decomposition [28]:

$$\boldsymbol{H}(\mathbf{curl};\Omega_c) = \boldsymbol{X}^c \oplus \boldsymbol{\nabla} H^1(\Omega_c). \tag{2.18}$$

Using this, the costate $\boldsymbol{p}^*|_{\Omega_c}$ can be expressed as $-\gamma^{-1}\boldsymbol{p}^*|_{\Omega_c} = \boldsymbol{w}+\boldsymbol{\nabla}\xi^*$with $\boldsymbol{w}\in \boldsymbol{X}^c$ and $\xi^*\in H^1(\Omega_c)$. Then $\boldsymbol{w}$ can be shown to be the optimal control $\boldsymbol{u}^*$ by the nonnegative Gâteaux derivative of $\mathcal{J}$ at $\boldsymbol{u}^*$ [28], i.e.,

$$-\gamma^{-1}\boldsymbol{p}^*|_{\Omega_c} = \boldsymbol{u}^* + \boldsymbol{\nabla}\xi^*. \tag{2.19}$$

Similarly, the equation (2.15) is derived on the basis of a discrete Helmholtz decomposition:

$$\boldsymbol{V}^c_{\mathcal{T}} = \boldsymbol{U}_{\mathcal{T}} \oplus \boldsymbol{\nabla} S^c_{\mathcal{T}}. \tag{2.20}$$

As a result, there exists a $\xi^*_{\mathcal{T}^c}\in S^c_{\mathcal{T}}$ such that

$$-\gamma^{-1}\boldsymbol{p}^*_{\mathcal{T}}|_{\Omega_c} = \boldsymbol{u}^*_{\mathcal{T}} + \boldsymbol{\nabla}\xi^*_{\mathcal{T}^c}. \tag{2.21}$$

3 A posteriori error estimate and its reliability and efficiency

In this section, we introduce a residual-type a posteriori error estimator for the discrete problem (2.13)-(2.15) and show its reliability and efficiency with respect to errors of the control, state and costate. For this purpose, some more notation and definitions are needed.

The collection of all faces (resp. all interior faces) in $\mathcal{T}$ is denoted by $\mathcal{F}_{\mathcal{T}}$ (resp. $\mathcal{F}_{\mathcal{T}}(\Omega)$) and its restriction on $\overline{\Omega}_c$ (resp. Ω_c) by $\mathcal{F}_{\mathcal{T}}(\overline{\Omega}_c)$ (resp. $\mathcal{F}_{\mathcal{T}}(\Omega_c)$). The scalar $h_F := |F|^{1/2}$ stands for the diameter of $F\in\mathcal{F}_{\mathcal{T}}$, which is associated with a fixed normal unit vector $\boldsymbol{n}_F$ in $\overline{\Omega}$ with $\boldsymbol{n}_F$ being the unit outward normal on $\partial\Omega_c$ and $\boldsymbol{n}_F = \boldsymbol{n}$ on the boundary $\partial\Omega$. We use D_T (resp. D_F) for the union of all elements in $\mathcal{T}$ with non-empty intersection with element $T\in\mathcal{T}$ (resp. $F\in\mathcal{F}_{\mathcal{T}}$). Furthermore, for any $T\in\mathcal{T}$ we denote by ω_T the union of elements in $\mathcal{T}$ sharing a common face with T, while for any $F\in\mathcal{F}_{\mathcal{T}}(\Omega)$ (resp. $F\subset\partial\Omega$) we denote by ω_F the union of two elements in $\mathcal{T}$ sharing the common face F (resp. the element with F as a face).

For the solution $(\boldsymbol{u}^*_{\mathcal{T}}, \boldsymbol{y}^*_{\mathcal{T}}, \boldsymbol{p}^*_{\mathcal{T}})$ to the problem (2.13)-(2.15), we define two element residuals for each $T\in\mathcal{T}$ by

$$R_{T,1}(\boldsymbol{y}^*_{\mathcal{T}},\boldsymbol{u}^*_{\mathcal{T}}) := \chi_c\boldsymbol{u}^*_{\mathcal{T}} - \boldsymbol{\nabla}\times(\nu\boldsymbol{\nabla}\times\boldsymbol{y}^*_{\mathcal{T}}), \quad R_{T,2}(\boldsymbol{p}^*_{\mathcal{T}}) := -\boldsymbol{\nabla}\times\boldsymbol{\nabla}\times\boldsymbol{y}_d - \boldsymbol{\nabla}\times(\nu\boldsymbol{\nabla}\times\boldsymbol{p}^*_{\mathcal{T}})$$

and some face residuals for each $F\in\mathcal{F}_{\mathcal{T}}(\Omega)$ by

$$J_{F,1}(\boldsymbol{y}^*_{\mathcal{T}}) := [(\nu\boldsymbol{\nabla}\times\boldsymbol{y}^*_{\mathcal{T}})\times\boldsymbol{n}_F], \quad J_{F,2}(\boldsymbol{y}^*_{\mathcal{T}}) := [\boldsymbol{y}^*_{\mathcal{T}}\cdot\boldsymbol{n}_F],$$

$$J_{F,3}(\boldsymbol{u}^*_{\mathcal{T}}) := \begin{cases} [\boldsymbol{u}^*_{\mathcal{T}}\cdot\boldsymbol{n}_F] & \text{for } F\in\mathcal{F}_{\mathcal{T}}(\Omega_c), \\ \boldsymbol{u}^*_{\mathcal{T}}\cdot\boldsymbol{n}_F & \text{for } F\in\mathcal{F}_{\mathcal{T}}(\overline{\Omega}_c)\setminus\mathcal{F}_{\mathcal{T}}(\Omega_c), \\ 0 & \text{for } F\in\mathcal{F}_{\mathcal{T}}(\Omega)\setminus\mathcal{F}_{\mathcal{T}}(\overline{\Omega}_c), \end{cases}$$

$$J_{F,4}(\boldsymbol{p}^*_{\mathcal{T}},\boldsymbol{y}^*_{\mathcal{T}}) := [(\boldsymbol{\nabla}\times\boldsymbol{y}^*_{\mathcal{T}})\times\boldsymbol{n}_F - (\nu\boldsymbol{\nabla}\times\boldsymbol{p}^*_{\mathcal{T}})\times\boldsymbol{n}_F], \quad J_{F,5}(\boldsymbol{p}^*_{\mathcal{T}}) := [\boldsymbol{p}^*_{\mathcal{T}}\cdot\boldsymbol{n}_F],$$

where $[\cdot]$ denotes jumps across interior faces F. For any $\mathcal{M} \subseteq \mathcal{T}$, we introduce our error estimator

$$\begin{aligned}\eta^2_{\mathcal{T}}(\boldsymbol{u}^*_{\mathcal{T}}, \boldsymbol{y}^*_{\mathcal{T}}, \boldsymbol{p}^*_{\mathcal{T}}, \mathcal{M}) &:= \sum_{T\in\mathcal{M}} \eta^2_{\mathcal{T}}(\boldsymbol{u}^*_{\mathcal{T}}, \boldsymbol{y}^*_{\mathcal{T}}, \boldsymbol{p}^*_{\mathcal{T}}, T) \\ &:= \sum_{T\in\mathcal{M}} (\eta^2_{\mathcal{T},1}(\boldsymbol{y}^*_{\mathcal{T}}, \boldsymbol{u}^*_{\mathcal{T}}, T) + \eta^2_{\mathcal{T},2}(\boldsymbol{p}^*_{\mathcal{T}}, \boldsymbol{y}^*_{\mathcal{T}}, T) + \eta^2_{\mathcal{T},3}(\boldsymbol{u}^*_{\mathcal{T}}, T))\end{aligned}$$

where $\eta^2_{\mathcal{T},3}(\boldsymbol{u}^*_{\mathcal{T}}, T) := \sum_{F\subset\partial T\cap\Omega} h_F \|J_{F,3}\|^2_{0,F}$, and

$$\eta^2_{\mathcal{T},1}(\boldsymbol{y}^*_{\mathcal{T}}, \boldsymbol{u}^*_{\mathcal{T}}, T) := h^2_T \|R_{T,1}\|^2_{0,T} + \sum_{F\subset\partial T\cap\Omega} (h_F \|J_{F,1}\|^2_{0,F} + h_F \|J_{F,2}\|^2_{0,F}),$$

$$\eta^2_{\mathcal{T},2}(\boldsymbol{p}^*_{\mathcal{T}}, \boldsymbol{y}^*_{\mathcal{T}}, T) := h^2_T \|R_{T,2}\|^2_{0,T} + \sum_{F\subset\partial T\cap\Omega} (h_F \|J_{F,4}\|^2_{0,F} + h_F \|J_{F,5}\|^2_{0,F})$$

and four oscillation errors that involve the given data and the related elementwise projections:

$$\operatorname{osc}^2_{\mathcal{T}}(\boldsymbol{y}^*_{\mathcal{T}}, \boldsymbol{u}^*_{\mathcal{T}}, \mathcal{M}) := \sum_{T\in\mathcal{M}} h^2_T \|R_{T,1} - \bar{R}_{T,1}\|^2_{0,T},$$

$$\operatorname{osc}^2_{\mathcal{T}}(\boldsymbol{p}^*_{\mathcal{T}}, \mathcal{M}) := \sum_{T\in\mathcal{M}} h^2_T \|R_{T,2} - \bar{R}_{T,2}\|^2_{0,T},$$

$$\operatorname{osc}^2_{\mathcal{T}}(\boldsymbol{y}^*_{\mathcal{T}}, \mathcal{S}) := \sum_{F\in\mathcal{S}} h_F \|J_{F,1} - \bar{J}_{F,1}\|^2_{0,F}, \quad \operatorname{osc}^2_{\mathcal{T}}(\boldsymbol{p}^*_{\mathcal{T}}, \boldsymbol{y}^*_{\mathcal{T}}, \mathcal{S}) := \sum_{F\in\mathcal{S}} h_F \|J_{F,4} - \bar{J}_{F,4}\|^2_{0,F}$$

for some $\mathcal{M} \subseteq \mathcal{T}$ and $\mathcal{S} \subseteq \mathcal{F}_{\mathcal{T}}(\Omega)$, where $\bar{R}_{T,1}$ and $\bar{R}_{T,2}$ (resp. $\bar{J}_{F,1}$ and $\bar{J}_{F,4}$) are integral averages of $R_{T,1}$ and $R_{T,2}$ (resp. $J_{F,1}$ and $J_{F,2}$) over T (resp. F). When $\mathcal{M} = \mathcal{T}$ or $\mathcal{S} = \mathcal{F}_{\mathcal{T}}(\Omega)$, $\mathcal{M}$ or $\mathcal{S}$ will be dropped in the parameter list of the error estimator or the oscillation errors above.

We shall need the following regular decomposition of vector fields in $\boldsymbol{H}_0(\mathbf{curl};\Omega)$ [7] [16]:

Lemma 3.1. *Let Ω be a bounded Lipschitz domain, then for any $\boldsymbol{v} \in \boldsymbol{H}_0(\mathbf{curl};\Omega)$ there exist some $\boldsymbol{z} \in \boldsymbol{H}^1(\Omega) \cap \boldsymbol{H}_0(\mathbf{curl};\Omega)$ and $\varphi \in H^1_0(\Omega)$ such that*

$$\boldsymbol{v} = \boldsymbol{z} + \boldsymbol{\nabla}\varphi \tag{3.1}$$

with the estimate

$$\|\boldsymbol{z}\|_1 + \|\phi\|_1 \le C \|\boldsymbol{v}\|_{\boldsymbol{H}(\mathbf{curl})}. \tag{3.2}$$

To relate two parts in the splitting (3.1) to discrete spaces, we need two quasi-interpolation operators $\boldsymbol{\Pi}_{\mathcal{T}} : \boldsymbol{H}^1(\Omega) \cap \boldsymbol{H}_0(\mathbf{curl};\Omega) \to \boldsymbol{V}_{\mathcal{T}}$ [2] and $I_{\mathcal{T}} : H^1_0(\Omega) \to S_{\mathcal{T}}$ [20], which have the following estimates for any $T \in \mathcal{T}$ and any $F \in \mathcal{F}_{\mathcal{T}}$:

$$\|\boldsymbol{v} - \boldsymbol{\Pi}_{\mathcal{T}}\boldsymbol{v}\|_{0,T} \le C h_T |\boldsymbol{v}|_{1,D_T}, \quad \|\boldsymbol{v} - \boldsymbol{\Pi}_{\mathcal{T}}\boldsymbol{v}\|_{0,F} \le C h_F^{1/2} |\boldsymbol{v}|_{1,D_F}, \tag{3.3}$$

$$\|q - I_{\mathcal{T}}q\|_{0,T} \le C h_T |q|_{1,D_T}, \quad \|q - I_{\mathcal{T}}q\|_{0,F} \le C h_F^{1/2} |q|_{1,D_F}. \tag{3.4}$$

With the above preparations, we are ready to provide an upper bound of the error between the true solutions to the problem (2.4)-(2.6) and the problem (2.13)-(2.15). As the state $\boldsymbol{y}^*$ and the discrete state $\boldsymbol{y}^*_{\mathcal{T}}$ depend on different controls, the so-called Galerkin orthogonality, essential to the a posteriori error estimates for elliptic equations, does not hold in the current situation. We start our analysis with two auxiliary saddle-point systems: find $(\boldsymbol{y}(\boldsymbol{u}^*_{\mathcal{T}}), \phi(\boldsymbol{u}^*_{\mathcal{T}})) \in \boldsymbol{H}_0(\mathbf{curl};\Omega) \times H^1_0(\Omega)$ such that

$$\begin{cases} (\nu\boldsymbol{\nabla} \times \boldsymbol{y}(\boldsymbol{u}^*_{\mathcal{T}}), \boldsymbol{\nabla} \times \boldsymbol{v}) + (\boldsymbol{v}, \boldsymbol{\nabla}\phi(\boldsymbol{u}^*_{\mathcal{T}})) = (\boldsymbol{u}^*_{\mathcal{T}}, \boldsymbol{v})_{\Omega_c} \quad \forall\ \boldsymbol{v} \in \boldsymbol{H}_0(\mathbf{curl};\Omega), \\ (\boldsymbol{y}(\boldsymbol{u}^*_{\mathcal{T}}), \boldsymbol{\nabla}q) = 0 \quad \forall\ q \in H^1_0(\Omega) \end{cases} \tag{3.5}$$

and find $(\boldsymbol{p}(\boldsymbol{u}_{\mathcal{T}}^*), \psi(\boldsymbol{u}_{\mathcal{T}}^*)) \in \boldsymbol{H}_0(\mathbf{curl};\Omega) \times H_0^1(\Omega)$ such that

$$\begin{cases} (\nu \boldsymbol{\nabla} \times \boldsymbol{p}(\boldsymbol{u}_{\mathcal{T}}^*), \boldsymbol{\nabla} \times \boldsymbol{v}) + (\boldsymbol{v}, \boldsymbol{\nabla}\psi(\boldsymbol{u}_{\mathcal{T}}^*)) = (\boldsymbol{\nabla} \times (\boldsymbol{y}(\boldsymbol{u}_{\mathcal{T}}^*) - \boldsymbol{y}_d), \boldsymbol{\nabla} \times \boldsymbol{v}) \, \forall \, \boldsymbol{v} \in \boldsymbol{H}_0(\mathbf{curl};\Omega), \\ (\boldsymbol{p}(\boldsymbol{u}_{\mathcal{T}}^*), \boldsymbol{\nabla} q) = 0 \quad \forall \, q \in H_0^1(\Omega). \end{cases} \tag{3.6}$$

Unique solvability of the problems (3.5) and (3.6) is guaranteed by the inequality (2.2) and the inf-sup condition (2.3).

Lemma 3.2. *Let* $(\boldsymbol{u}_{\mathcal{T}}^*, \boldsymbol{y}_{\mathcal{T}}^*)$ *be the solution to the problem* (2.9)-(2.10) *and* $\boldsymbol{y}(\boldsymbol{u}_{\mathcal{T}}^*)$ *be defined by* (3.5) *respectively, then*

$$\|\boldsymbol{y}(\boldsymbol{u}_{\mathcal{T}}^*) - \boldsymbol{y}_{\mathcal{T}}^*\|_{\boldsymbol{H}(\mathbf{curl})}^2 \le C(\eta_{\mathcal{T},1}^2(\boldsymbol{y}_{\mathcal{T}}^*, \boldsymbol{u}_{\mathcal{T}}^*) + \eta_{\mathcal{T},3}^2(\boldsymbol{u}_{\mathcal{T}}^*)). \tag{3.7}$$

Proof. By the Helmholtz decomposition $\boldsymbol{H}_0(\mathbf{curl};\Omega) = \boldsymbol{X} \oplus \boldsymbol{\nabla} H_0^1(\Omega)$ (cf. [16]), we can write

$$\boldsymbol{y}(\boldsymbol{u}_{\mathcal{T}}^*) - \boldsymbol{y}_{\mathcal{T}}^* = \boldsymbol{w} + \boldsymbol{\nabla} p \quad \text{for} \quad \boldsymbol{w} \in \boldsymbol{X}, \; p \in H_0^1(\Omega).$$

Noting $(\boldsymbol{w}, \boldsymbol{\nabla} q) = 0$ for any $q \in H_0^1(\Omega)$, we take $\boldsymbol{v} = \boldsymbol{w}$ in the first equation of (3.5) to get

$$\begin{aligned} &\|\nu^{1/2} \boldsymbol{\nabla} \times (\boldsymbol{y}(\boldsymbol{u}_{\mathcal{T}}^*) - \boldsymbol{y}_{\mathcal{T}}^*)\|_0^2 = (\nu \boldsymbol{\nabla} \times (\boldsymbol{y}(\boldsymbol{u}_{\mathcal{T}}^*) - \boldsymbol{y}_{\mathcal{T}}^*), \boldsymbol{\nabla} \times \boldsymbol{w}) \\ &= (\nu \boldsymbol{\nabla} \times (\boldsymbol{y}(\boldsymbol{u}_{\mathcal{T}}^*) - \boldsymbol{y}_{\mathcal{T}}^*), \boldsymbol{\nabla} \times \boldsymbol{w}) + (\boldsymbol{w}, \boldsymbol{\nabla}(\phi(\boldsymbol{u}_{\mathcal{T}}^*) - \phi_{\mathcal{T}}^*)) \\ &= (\boldsymbol{u}_{\mathcal{T}}^*, \boldsymbol{w})_{\Omega_c} - (\nu \boldsymbol{\nabla} \times \boldsymbol{y}_{\mathcal{T}}^*, \boldsymbol{\nabla} \times \boldsymbol{w}) - (\boldsymbol{w}, \boldsymbol{\nabla} \phi_{\mathcal{T}}^*) \\ &= \{(\boldsymbol{u}_{\mathcal{T}}^*, \boldsymbol{z})_{\Omega_c} - (\nu \boldsymbol{\nabla} \times \boldsymbol{y}_{\mathcal{T}}^*, \boldsymbol{\nabla} \times \boldsymbol{z}) - (\boldsymbol{z}, \boldsymbol{\nabla} \phi_{\mathcal{T}}^*)\} \\ &\quad + \{(\boldsymbol{u}_{\mathcal{T}}^*, \boldsymbol{\nabla}\varphi)_{\Omega_c} - (\boldsymbol{\nabla}\varphi, \boldsymbol{\nabla}\phi_{\mathcal{T}}^*)\} := \mathrm{I}_1 + \mathrm{I}_2, \end{aligned} \tag{3.8}$$

where we have used the decomposition $\boldsymbol{w} = \boldsymbol{\nabla}\varphi + \boldsymbol{z}$ with $\boldsymbol{z} \in \boldsymbol{H}^1(\Omega) \cap \boldsymbol{H}_0(\mathbf{curl};\Omega)$ and $\varphi \in H_0^1(\Omega)$ (see Lemma 3.1). Applying operator $\boldsymbol{\Pi}_{\mathcal{T}}$ to $\boldsymbol{z}$, using the first equation of the system (2.13) and noting $\phi_{\mathcal{T}}^* = 0$, we deduce

$$\begin{aligned} \mathrm{I}_1 &= (\boldsymbol{u}_{\mathcal{T}}^*, \boldsymbol{z} - \boldsymbol{\Pi}_{\mathcal{T}}\boldsymbol{z})_{\Omega_c} - (\nu \boldsymbol{\nabla} \times \boldsymbol{y}_{\mathcal{T}}^*, \boldsymbol{\nabla} \times (\boldsymbol{z} - \boldsymbol{\Pi}_{\mathcal{T}}\boldsymbol{z})) - (\boldsymbol{z} - \boldsymbol{\Pi}_{\mathcal{T}}\boldsymbol{z}, \boldsymbol{\nabla}\phi_{\mathcal{T}}^*) \\ &= \sum_{T \in \mathcal{T}} (\chi_c \boldsymbol{u}_{\mathcal{T}}^* - \boldsymbol{\nabla} \times (\nu \boldsymbol{\nabla} \times \boldsymbol{y}_{\mathcal{T}}^*), \boldsymbol{z} - \boldsymbol{\Pi}_{\mathcal{T}}\boldsymbol{z})_T - \sum_{F \in \mathcal{F}_{\mathcal{T}}(\Omega)} ([(\nu \boldsymbol{\nabla} \times \boldsymbol{y}_{\mathcal{T}}^*) \times \boldsymbol{n}_F], \boldsymbol{z} - \boldsymbol{\Pi}_{\mathcal{T}}\boldsymbol{z})_F \\ &\le \sum_{T \in \mathcal{T}} \|R_{T,1}\|_{0,T} \|\boldsymbol{z} - \boldsymbol{\Pi}_{\mathcal{T}}\boldsymbol{z}\|_{0,T} + \sum_{F \in \mathcal{F}_{\mathcal{T}}(\Omega)} \|J_{F,1}\|_{0,F} \|\boldsymbol{z} - \boldsymbol{\Pi}_{\mathcal{T}}\boldsymbol{z}\|_{0,F}\,. \end{aligned}$$

Similarly, by using (2.7), the first equation of (2.13) with $\boldsymbol{v}_{\mathcal{T}} = \boldsymbol{\nabla} I_{\mathcal{T}}\varphi$ and the fact that $\boldsymbol{\nabla} \cdot \boldsymbol{u}_{\mathcal{T}}^*$ vanishes on each element $T \in \mathcal{T}^c$ and $\phi_{\mathcal{T}}^* = 0$ we derive

$$\begin{aligned} \mathrm{I}_2 &= (\boldsymbol{u}_{\mathcal{T}}^*, \boldsymbol{\nabla}(\varphi - I_{\mathcal{T}}\varphi))_{\Omega_c} - (\boldsymbol{\nabla}(\varphi - I_{\mathcal{T}}\varphi), \boldsymbol{\nabla}\phi_{\mathcal{T}}^*) \\ &= \sum_{F \in \mathcal{F}_{\mathcal{T}}(\overline{\Omega}_c)} (J_{F,3}, \varphi - I_{\mathcal{T}}\varphi)_F \le \sum_{F \in \mathcal{F}_{\mathcal{T}}(\overline{\Omega}_c)} \|J_{F,3}\|_{0,F} \|\varphi - I_{\mathcal{T}}\varphi\|_{0,F}. \end{aligned}$$

It follows further from (3.3), (3.4) and the fact that $J_{F,3} = 0$ on $F \in \mathcal{F}_{\mathcal{T}}(\Omega) \setminus \mathcal{F}_{\mathcal{T}}(\overline{\Omega}_c)$ that

$$|\mathrm{I}_1| \le C\Big(\sum_{T \in \mathcal{T}} (h_T^2 \|R_{T,1}\|_{0,T}^2 + \sum_{F \subset \partial T} h_F \|J_{F,1}\|_{0,F}^2)\Big)^{1/2} |\boldsymbol{z}|_1 \tag{3.9}$$

$$|\mathrm{I}_2| \le C \eta_{\mathcal{T},3}(\boldsymbol{u}_{\mathcal{T}}^*) |\varphi|_1. \tag{3.10}$$

Now with the help of (3.2), the norm equivalence between $\|\boldsymbol{w}\|_{\boldsymbol{H}(\mathbf{curl})}$ and $\|\boldsymbol{\nabla} \times \boldsymbol{w}\|_0$ and the fact that $\boldsymbol{\nabla} \times \boldsymbol{w} = \boldsymbol{\nabla} \times (\boldsymbol{y}(\boldsymbol{u}_{\mathcal{T}}^*) - \boldsymbol{y}_{\mathcal{T}}^*)$, we obtain from (3.8)-(3.10) that

$$\|\boldsymbol{\nabla} \times (\boldsymbol{y}(\boldsymbol{u}_{\mathcal{T}}^*) - \boldsymbol{y}_{\mathcal{T}}^*)\|_0 \le C\Big(\sum_{T \in \mathcal{T}} (h_T^2 \|R_{T,1}\|_{0,T}^2 + \sum_{F \subset \partial T \cap \Omega} h_F \|J_{F,1}\|_{0,F}^2) + \eta_{\mathcal{T},3}^2(\boldsymbol{u}_{\mathcal{T}}^*)\Big)^{1/2}. \tag{3.11}$$

On the other hand, we deduce from the second equation of the problems (3.5), (2.13) with $q = p$ and $q_{\mathcal{T}} = I_{\mathcal{T}} p$ respectively and the fact that $\boldsymbol{\nabla} \cdot \boldsymbol{y}_{\mathcal{T}}^* = 0$ on each $T \in \mathcal{T}$ that

$$
\begin{aligned}
(\boldsymbol{y}(\boldsymbol{u}_{\mathcal{T}}^*) - \boldsymbol{y}_{\mathcal{T}}^*, \boldsymbol{\nabla} p) &= -(\boldsymbol{y}_{\mathcal{T}}^*, \boldsymbol{\nabla} p) = -(\boldsymbol{y}_{\mathcal{T}}^*, \boldsymbol{\nabla}(p - I_{\mathcal{T}} p)) \\
&= -\sum_{F \in \mathcal{F}_{\mathcal{T}}(\Omega)} (J_{F,2}, p - I_{\mathcal{T}} p)_F \leq \sum_{F \in \mathcal{F}_{\mathcal{T}}(\Omega)} \|J_{F,2}\|_{0,F} \|p - I_{\mathcal{T}} p\|_{0,F}.
\end{aligned}
$$

But for the projection of $\boldsymbol{y}(\boldsymbol{u}_{\mathcal{T}}^*) - \boldsymbol{y}_{\mathcal{T}}^*$ in $\boldsymbol{\nabla} H_0^1(\Omega)$, we note that $(\boldsymbol{y}(\boldsymbol{u}_{\mathcal{T}}^*) - \boldsymbol{y}_{\mathcal{T}}^*)^0 = \boldsymbol{\nabla} p$, then along with the second estimate in (3.4) we get

$$
\|(\boldsymbol{y}(\boldsymbol{u}_{\mathcal{T}}^*) - \boldsymbol{y}_{\mathcal{T}}^*)^0\|_0 \leq C\Big(\sum_{T \in \mathcal{T}} \sum_{F \subset \partial T \cap \Omega} h_F \|J_{F,2}\|_{0,F}^2\Big)^{1/2}. \tag{3.12}
$$

So we conclude from (3.11)-(3.12) that

$$
\|\boldsymbol{\nabla} \times (\boldsymbol{y}(\boldsymbol{u}_{\mathcal{T}}^*) - \boldsymbol{y}_{\mathcal{T}}^*)\|_2^2 + \|(\boldsymbol{y}(\boldsymbol{u}_{\mathcal{T}}^*) - \boldsymbol{y}_{\mathcal{T}}^*)^0\|_0^2 \leq C(\eta_{\mathcal{T},1}^2(\boldsymbol{y}_{\mathcal{T}}^*, \boldsymbol{u}_{\mathcal{T}}^*) + \eta_{\mathcal{T},3}^2(\boldsymbol{u}_{\mathcal{T}}^*)).
$$

The desired estimate follows now from the norm equivalence between $(\|\boldsymbol{\nabla} \times \cdot\|_0^2 + \| \cdot^0 \|_0^2)^{1/2}$ and $\| \cdot \|_{\boldsymbol{H}(\mathbf{curl})}$. □

Likewise, we may obtain an estimate for the discrete costate $\boldsymbol{p}_{\mathcal{T}}^*$.

Lemma 3.3. *Let $\boldsymbol{p}_{\mathcal{T}}^*$ be the solution to the problem* (2.14) *and $\boldsymbol{p}(\boldsymbol{u}_{\mathcal{T}}^*)$ be defined by* (3.6)*, then*

$$
\|\boldsymbol{p}(\boldsymbol{u}_{\mathcal{T}}^*) - \boldsymbol{p}_{\mathcal{T}}^*\|_{\boldsymbol{H}(\mathbf{curl})}^2 \leq C\eta_{\mathcal{T}}^2(\boldsymbol{u}_{\mathcal{T}}^*, \boldsymbol{y}_{\mathcal{T}}^*, \boldsymbol{p}_{\mathcal{T}}^*). \tag{3.13}
$$

Proof. The proof is almost the same as that of Lemma 3.2. The Helmholtz decomposition and the regular decomposition give

$$
\boldsymbol{p}(\boldsymbol{u}_{\mathcal{T}}^*) - \boldsymbol{p}_{\mathcal{T}}^* = \boldsymbol{w} + \boldsymbol{\nabla} p = \boldsymbol{z} + \boldsymbol{\nabla}\varphi + \boldsymbol{\nabla} p
$$

for some $\boldsymbol{w} \in \boldsymbol{X}$, $\boldsymbol{z} \in \boldsymbol{H}^1(\Omega) \cap \boldsymbol{H}_0(\mathbf{curl}; \Omega)$ and $\varphi, p \in H_0^1(\Omega)$. Then using the first equation of (3.6) we derive as in (3.8) that

$$
\begin{aligned}
&\|\nu^{1/2} \boldsymbol{\nabla} \times (\boldsymbol{p}(\boldsymbol{u}_{\mathcal{T}}^*) - \boldsymbol{p}_{\mathcal{T}}^*)\|_0^2 = (\nu \boldsymbol{\nabla} \times (\boldsymbol{p}(\boldsymbol{u}_{\mathcal{T}}^*) - \boldsymbol{p}_{\mathcal{T}}^*), \boldsymbol{\nabla} \times \boldsymbol{w}) \\
&= (\nu \boldsymbol{\nabla} \times (\boldsymbol{p}(\boldsymbol{u}_{\mathcal{T}}^*) - \boldsymbol{p}_{\mathcal{T}}^*), \boldsymbol{\nabla} \times \boldsymbol{w}) + (\boldsymbol{w}, \boldsymbol{\nabla}(\psi(\boldsymbol{u}_{\mathcal{T}}^*) - \psi_{\mathcal{T}}^*)) \\
&= (\boldsymbol{\nabla} \times \boldsymbol{y}(\boldsymbol{u}_{\mathcal{T}}^*) - \boldsymbol{\nabla} \times \boldsymbol{y}_d, \boldsymbol{\nabla} \times \boldsymbol{w}) - (\nu \boldsymbol{\nabla} \times \boldsymbol{p}_{\mathcal{T}}^*, \boldsymbol{\nabla} \times \boldsymbol{w}) - (\boldsymbol{w}, \boldsymbol{\nabla}\psi_{\mathcal{T}}^*) \\
&= (\boldsymbol{\nabla} \times (\boldsymbol{y}(\boldsymbol{u}_{\mathcal{T}}^*) - \boldsymbol{y}_{\mathcal{T}}^*), \boldsymbol{\nabla} \times \boldsymbol{w}) + (\boldsymbol{\nabla} \times \boldsymbol{y}_{\mathcal{T}}^* - \boldsymbol{\nabla} \times \boldsymbol{y}_d, \boldsymbol{\nabla} \times \boldsymbol{w}) \\
&\quad - (\nu \boldsymbol{\nabla} \times \boldsymbol{p}_{\mathcal{T}}^*, \boldsymbol{\nabla} \times \boldsymbol{w}) - (\boldsymbol{w}, \boldsymbol{\nabla}\psi_{\mathcal{T}}^*) \\
&= (\boldsymbol{\nabla} \times (\boldsymbol{y}(\boldsymbol{u}_{\mathcal{T}}^*) - \boldsymbol{y}_{\mathcal{T}}^*), \boldsymbol{\nabla} \times \boldsymbol{w}) + (-(\boldsymbol{\nabla}\varphi, \boldsymbol{\nabla}\psi_{\mathcal{T}}^*)) \\
&\quad + (\boldsymbol{\nabla} \times \boldsymbol{y}_{\mathcal{T}}^* - \boldsymbol{\nabla} \times \boldsymbol{y}_d, \boldsymbol{\nabla} \times \boldsymbol{z}) - (\nu \boldsymbol{\nabla} \times \boldsymbol{p}_{\mathcal{T}}^*, \boldsymbol{\nabla} \times \boldsymbol{z}) - (\boldsymbol{z}, \boldsymbol{\nabla}\psi_{\mathcal{T}}^*) \\
&:= \mathrm{II}_1 + \mathrm{II}_2 + \mathrm{II}_3.
\end{aligned}
$$

Then by the Cauchy-Schwarz inequality,

$$
|\mathrm{II}_1| \leq \|\boldsymbol{\nabla} \times (\boldsymbol{y}(\boldsymbol{u}_{\mathcal{T}}^*) - \boldsymbol{y}_{\mathcal{T}}^*)\|_0 \|\boldsymbol{\nabla} \times \boldsymbol{w}\|_0 \leq \|\boldsymbol{\nabla} \times (\boldsymbol{y}(\boldsymbol{u}_{\mathcal{T}}^*) - \boldsymbol{y}_{\mathcal{T}}^*)\|_0 \|\boldsymbol{\nabla} \times (\boldsymbol{p}(\boldsymbol{u}_{\mathcal{T}}^*) - \boldsymbol{p}_{\mathcal{T}}^*)\|_0.
$$

Using the fact that $\psi_{\mathcal{T}}^* = 0$ we know $\mathrm{II}_2 = 0$. But for II_3, taking $\boldsymbol{v} = \boldsymbol{\Pi}_{\mathcal{T}} \boldsymbol{z}$ in the first equation of (2.14), applying the estimate (3.3) and noting $\psi_{\mathcal{T}}^* = 0$ we derive

$$
\begin{aligned}
|\mathrm{II}_3| &= |(\boldsymbol{\nabla} \times \boldsymbol{y}_{\mathcal{T}}^* - \boldsymbol{\nabla} \times \boldsymbol{y}_d, \boldsymbol{\nabla} \times (\boldsymbol{z} - \boldsymbol{\Pi}_{\mathcal{T}} \boldsymbol{z})) \\
&\quad - (\nu \boldsymbol{\nabla} \times \boldsymbol{p}_{\mathcal{T}}^*, \boldsymbol{\nabla} \times (\boldsymbol{z} - \boldsymbol{\Pi}_{\mathcal{T}} \boldsymbol{z})) - (\boldsymbol{z} - \boldsymbol{\Pi}_{\mathcal{T}} \boldsymbol{z}, \boldsymbol{\nabla}\psi_{\mathcal{T}}^*)| \\
&\leq C\Big(\sum_{T \in \mathcal{T}} (h_T^2 \|R_{T,2}\|_{0,T}^2 + \sum_{F \subset \partial T \cap \Omega} h_F \|J_{F,4}\|_{0,F}^2)\Big)^{1/2} |\boldsymbol{z}|_1.
\end{aligned}
$$

Now collecting the above two inequalities and using the estimates (2.2) and (3.2), we obtain

$$\begin{aligned}\|\boldsymbol{\nabla}\times(\boldsymbol{p}(\boldsymbol{u}_{\mathcal{T}}^*)-\boldsymbol{p}_{\mathcal{T}}^*)\|_0^2 \le C(\|\boldsymbol{\nabla}\times(\boldsymbol{y}(\boldsymbol{u}_{\mathcal{T}}^*)-\boldsymbol{y}_{\mathcal{T}}^*)\|_0^2 \\ +\sum_{T\in\mathcal{T}}(h_T^2\|R_{T,2}\|_{0,T}^2+\sum_{F\subset\partial T\cap\Omega}h_F\|J_{F,4}\|_{0,F}^2).\end{aligned} \tag{3.14}$$

Furthermore, we can deduce from the second equations of the problems (3.6) and (2.14) with $q=p$ and $q_{\mathcal{T}}=I_{\mathcal{T}}p$ respectively as well as the estimate (3.4) that

$$\begin{aligned}|p|_1^2 &= (\boldsymbol{p}(\boldsymbol{u}_{\mathcal{T}}^*)-\boldsymbol{p}_{\mathcal{T}}^*,\boldsymbol{\nabla}p)=-(\boldsymbol{p}_{\mathcal{T}}^*,\boldsymbol{\nabla}p)=-(\boldsymbol{p}_{\mathcal{T}}^*,\boldsymbol{\nabla}(p-I_{\mathcal{T}}p)) \\ &\le \|J_{F,5}\|_{0,F}\|p-I_{\mathcal{T}}p\|_{0,F}\le C(\sum_{T\in\mathcal{T}}\sum_{F\subset\partial T\cap\Omega}h_F\|J_{F,5}\|_{0,F}^2)^{1/2}|p|_1.\end{aligned} \tag{3.15}$$

The proof is now completed by a combination of (3.14)-(3.15) and the estimate (3.7). □

Now we are in a position to establish the reliability of the error estimator $\eta_{\mathcal{T}}$.

Theorem 3.1. *Let $(\boldsymbol{u}^*,\boldsymbol{y}^*,\boldsymbol{p}^*)$ and $(\boldsymbol{u}_{\mathcal{T}}^*,\boldsymbol{y}_{\mathcal{T}}^*,\boldsymbol{p}_{\mathcal{T}}^*)$ be the solutions to the problems* (2.4)-(2.6) *and* (2.13)-(2.15) *respectively, then there holds that*

$$\|\boldsymbol{y}^*-\boldsymbol{y}_{\mathcal{T}}^*\|_{\boldsymbol{H}(\mathbf{curl})}^2+\|\boldsymbol{p}^*-\boldsymbol{p}_{\mathcal{T}}^*\|_{\boldsymbol{H}(\mathbf{curl})}^2+\|\boldsymbol{u}^*-\boldsymbol{u}_{\mathcal{T}}^*\|_{0,\Omega_c}^2\le C\eta_{\mathcal{T}}^2(\boldsymbol{u}_{\mathcal{T}}^*,\boldsymbol{y}_{\mathcal{T}}^*,\boldsymbol{p}_{\mathcal{T}}^*), \tag{3.16}$$

where the constant C depends only on ν, γ and the shape-regularity of $\mathcal{T}$.

Proof. We start with the estimate of $\boldsymbol{u}^*-\boldsymbol{u}_{\mathcal{T}}^*$. As we know from Remark 2.1, there exist $\xi^*\in H_1(\Omega_c)$ and $\xi_{\mathcal{T}^c}^*\in S_{\mathcal{T}}^c$ such that

$$\boldsymbol{u}^*=-\gamma^{-1}\boldsymbol{p}^*|_{\Omega_c}-\boldsymbol{\nabla}\xi^*,\quad \boldsymbol{u}_{\mathcal{T}}^*=-\gamma^{-1}\boldsymbol{p}_{\mathcal{T}}^*|_{\Omega_c}-\boldsymbol{\nabla}\xi_{\mathcal{T}^c}^*. \tag{3.17}$$

Subtracting the first equations of (3.5) and (3.6) from (2.4) and (2.5) respectively, we get for any $\boldsymbol{v}\in\boldsymbol{X}$ that

$$(\nu\boldsymbol{\nabla}\times(\boldsymbol{y}^*-\boldsymbol{y}(\boldsymbol{u}_{\mathcal{T}}^*)),\boldsymbol{\nabla}\times\boldsymbol{v})=(\boldsymbol{u}^*-\boldsymbol{u}_{\mathcal{T}}^*,\boldsymbol{v})_{\Omega_c}, \tag{3.18}$$

$$(\nu\boldsymbol{\nabla}\times(\boldsymbol{p}^*-\boldsymbol{p}(\boldsymbol{u}_{\mathcal{T}}^*)),\boldsymbol{\nabla}\times\boldsymbol{v})=(\boldsymbol{\nabla}\times(\boldsymbol{y}^*-\boldsymbol{y}(\boldsymbol{u}_{\mathcal{T}}^*)),\boldsymbol{\nabla}\times\boldsymbol{v}). \tag{3.19}$$

Taking $\boldsymbol{v}=\boldsymbol{p}^*-\boldsymbol{p}(\boldsymbol{u}_{\mathcal{T}}^*)$, $\boldsymbol{v}=\boldsymbol{y}^*-\boldsymbol{y}(\boldsymbol{u}_{\mathcal{T}}^*)$ in (3.18)-(3.19) respectively and noting (3.17), we find

$$\begin{aligned}\|\boldsymbol{\nabla}\times(\boldsymbol{y}^*-\boldsymbol{y}(\boldsymbol{u}_{\mathcal{T}}^*))\|_0^2 &= (\boldsymbol{u}^*-\boldsymbol{u}_{\mathcal{T}}^*,\boldsymbol{p}^*-\boldsymbol{p}(\boldsymbol{u}_{\mathcal{T}}^*))_{\Omega_c} \\ &= (\boldsymbol{u}^*-\boldsymbol{u}_{\mathcal{T}}^*,\boldsymbol{p}^*-\boldsymbol{p}_{\mathcal{T}}^*)_{\Omega_c}+(\boldsymbol{u}^*-\boldsymbol{u}_{\mathcal{T}}^*,\boldsymbol{p}_{\mathcal{T}}^*-\boldsymbol{p}(\boldsymbol{u}_{\mathcal{T}}^*))_{\Omega_c} \\ &= \gamma(\boldsymbol{u}^*-\boldsymbol{u}_{\mathcal{T}}^*,\boldsymbol{u}_{\mathcal{T}}^*-\boldsymbol{u}^*)_{\Omega_c}+\gamma(\boldsymbol{u}^*-\boldsymbol{u}_{\mathcal{T}}^*,\boldsymbol{\nabla}(\xi_{\mathcal{T}^c}^*-\xi^*))_{\Omega_c} \\ &\quad +(\boldsymbol{u}^*-\boldsymbol{u}_{\mathcal{T}}^*,\boldsymbol{p}_{\mathcal{T}}^*-\boldsymbol{p}(\boldsymbol{u}_{\mathcal{T}}^*))_{\Omega_c}\end{aligned}$$

Thus by the definitions of $\boldsymbol{U}$ and $\boldsymbol{U}_{\mathcal{T}}$ (cf. (1.3) and (2.8)) and the Scott-Zhang interpolation $I_{\mathcal{T}}^c: H^1(\Omega_c)\to S_{\mathcal{T}}^c$ [20],

$$\begin{aligned}&\gamma\|\boldsymbol{u}^*-\boldsymbol{u}_{\mathcal{T}}^*\|_{0,\Omega_c}^2+\|\boldsymbol{\nabla}\times(\boldsymbol{y}^*-\boldsymbol{y}(\boldsymbol{u}_{\mathcal{T}}^*))\|_0^2 \\ =\;&\gamma(\boldsymbol{u}_{\mathcal{T}}^*,\boldsymbol{\nabla}(I_{\mathcal{T}}^c(\xi_{\mathcal{T}^c}^*-\xi^*)-(\xi_{\mathcal{T}^c}^*-\xi^*)))_{\Omega_c}+(\boldsymbol{u}^*-\boldsymbol{u}_{\mathcal{T}}^*,\boldsymbol{p}_{\mathcal{T}}^*-\boldsymbol{p}(\boldsymbol{u}_{\mathcal{T}}^*))_{\Omega_c} \\ \le\;&C\gamma\eta_{\mathcal{T},3}(\boldsymbol{u}_{\mathcal{T}}^*)|\xi^*-\xi_{\mathcal{T}^c}^*|_{1,\Omega_c}+\|\boldsymbol{u}^*-\boldsymbol{u}_{\mathcal{T}}^*\|_{0,\Omega_c}\|\boldsymbol{p}_{\mathcal{T}}^*-\boldsymbol{p}(\boldsymbol{u}_{\mathcal{T}}^*)\|_0,\end{aligned}$$

which, together with (3.17) and (3.13), yields

$$\begin{aligned}\gamma\|\boldsymbol{u}^*-\boldsymbol{u}_{\mathcal{T}}^*\|_{0,\Omega_c}^2 &\le C(\gamma\eta_{\mathcal{T},3}(\boldsymbol{u}_{\mathcal{T}}^*)\|\boldsymbol{u}^*-\boldsymbol{u}_{\mathcal{T}}^*\|_{0,\Omega_c}+\eta_{\mathcal{T},3}(\boldsymbol{u}_{\mathcal{T}}^*)\|\boldsymbol{p}^*-\boldsymbol{p}_{\mathcal{T}}^*\|_0) \\ &\quad +\|\boldsymbol{u}^*-\boldsymbol{u}_{\mathcal{T}}^*\|_{0,\Omega_c}\|\boldsymbol{p}_{\mathcal{T}}^*-\boldsymbol{p}(\boldsymbol{u}_{\mathcal{T}}^*)\|_0 \\ &\le C(\eta_{\mathcal{T}}(\boldsymbol{u}_{\mathcal{T}}^*,\boldsymbol{y}_{\mathcal{T}}^*,\boldsymbol{p}_{\mathcal{T}}^*)\|\boldsymbol{u}^*-\boldsymbol{u}_{\mathcal{T}}^*\|_{0,\Omega_c}+\eta_{\mathcal{T},3}(\boldsymbol{u}_{\mathcal{T}}^*)\|\boldsymbol{p}^*-\boldsymbol{p}_{\mathcal{T}}^*\|_0)\,.\end{aligned}$$

Then by Young's inequality, we have

$$\|\boldsymbol{u}^* - \boldsymbol{u}_{\mathcal{T}}^*\|_{0,\Omega_c}^2 \le C(\eta_{\mathcal{T}}^2(\boldsymbol{u}_{\mathcal{T}}^*, \boldsymbol{y}_{\mathcal{T}}^*, \boldsymbol{p}_{\mathcal{T}}^*) + \eta_{\mathcal{T},3}(\boldsymbol{u}_{\mathcal{T}}^*)\|\boldsymbol{p}^* - \boldsymbol{p}_{\mathcal{T}}^*\|_0). \tag{3.20}$$

Noting $\boldsymbol{y}^* - \boldsymbol{y}(\boldsymbol{u}_{\mathcal{T}}^*) \in \boldsymbol{X}$, taking $\boldsymbol{v} = \boldsymbol{y}^* - \boldsymbol{y}(\boldsymbol{u}_{\mathcal{T}}^*)$ in (3.18) and using (2.2) we deduce

$$\|\boldsymbol{y}^* - \boldsymbol{y}(\boldsymbol{u}_{\mathcal{T}}^*)\|_{\boldsymbol{H}(\mathbf{curl})} \le C\|\boldsymbol{\nabla} \times (\boldsymbol{y}^* - \boldsymbol{y}(\boldsymbol{u}_{\mathcal{T}}^*))\|_0 \le C\|\boldsymbol{u}^* - \boldsymbol{u}_{\mathcal{T}}^*\|_{0,\Omega_c}. \tag{3.21}$$

With $\boldsymbol{v} = \boldsymbol{p}^* - \boldsymbol{p}(\boldsymbol{u}_{\mathcal{T}}^*) \in \boldsymbol{X}$ in (3.19) and by (2.2) as well as (3.21) there holds

$$\|\boldsymbol{p}^* - \boldsymbol{p}(\boldsymbol{u}_{\mathcal{T}}^*)\|_{\boldsymbol{H}(\mathbf{curl})}^2 \le C\|\boldsymbol{\nabla} \times (\boldsymbol{p}^* - \boldsymbol{p}(\boldsymbol{u}_{\mathcal{T}}^*))\|_0^2 \le C\|\boldsymbol{\nabla} \times (\boldsymbol{y}^* - \boldsymbol{y}(\boldsymbol{u}_{\mathcal{T}}^*))\|_0^2 \le C\|\boldsymbol{u}^* - \boldsymbol{u}_{\mathcal{T}}^*\|_{0,\Omega_c}^2,$$

which, along with (3.13) in Lemma 3.3, (3.20) and Young's inequality, yields

$$\|\boldsymbol{p}^* - \boldsymbol{p}_{\mathcal{T}}^*\|_{\boldsymbol{H}(\mathbf{curl})}^2 \le C\eta_{\mathcal{T}}^2(\boldsymbol{u}_{\mathcal{T}}^*, \boldsymbol{y}_{\mathcal{T}}^*, \boldsymbol{p}_{\mathcal{T}}^*). \tag{3.22}$$

Then it follows readily from (3.20) and (3.22) that

$$\|\boldsymbol{u}^* - \boldsymbol{u}_{\mathcal{T}}^*\|_{0,\Omega_c}^2 \le C\eta_{\mathcal{T}}^2(\boldsymbol{u}_{\mathcal{T}}^*, \boldsymbol{y}_{\mathcal{T}}^*, \boldsymbol{p}_{\mathcal{T}}^*). \tag{3.23}$$

Using (3.21), (3.23) and (3.7) in Lemma 3.2, we are led to

$$\|\boldsymbol{y}^* - \boldsymbol{y}_{\mathcal{T}}^*\|_{\boldsymbol{H}(\mathbf{curl})}^2 \le C\eta_{\mathcal{T}}^2(\boldsymbol{u}_{\mathcal{T}}^*, \boldsymbol{y}_{\mathcal{T}}^*, \boldsymbol{p}_{\mathcal{T}}^*). \tag{3.24}$$

Clearly the desired estimate follows now from (3.22)-(3.24). □

By the bubble function techniques [24], we may bound the estimator locally from above by the errors up to the oscillation terms, i.e., the following efficiency theorem.

Theorem 3.2. *Let $(\boldsymbol{u}^*, \boldsymbol{y}^*, \boldsymbol{p}^*)$ and $(\boldsymbol{u}_{\mathcal{T}}^*, \boldsymbol{y}_{\mathcal{T}}^*, \boldsymbol{p}_{\mathcal{T}}^*)$ be the solutions to the problems* (2.4)-(2.6) *and* (2.13)-(2.15) *respectively. Then there holds for any $T \in \mathcal{T}$ that*

$$\begin{aligned}
&\eta_{\mathcal{T}}^2(\boldsymbol{u}_{\mathcal{T}}^*, \boldsymbol{y}_{\mathcal{T}}^*, \boldsymbol{p}_{\mathcal{T}}^*, T)\\
\le & C(\|\boldsymbol{y}^* - \boldsymbol{y}_{\mathcal{T}}^*\|_{\boldsymbol{H}(\mathbf{curl}),\omega_T}^2 + \|\boldsymbol{p}^* - \boldsymbol{p}_{\mathcal{T}}^*\|_{\boldsymbol{H}(\mathbf{curl}),\omega_T}^2 + \|\chi_c\boldsymbol{u}^* - \chi_c\boldsymbol{u}_{\mathcal{T}}^*\|_{0,\omega_T}^2\\
&\quad + \mathrm{osc}_{\mathcal{T}}^2(\boldsymbol{y}_{\mathcal{T}}^*, \boldsymbol{u}_{\mathcal{T}}^*, \omega_T) + \mathrm{osc}_{\mathcal{T}}^2(\boldsymbol{p}_{\mathcal{T}}^*, \omega_T) + \mathrm{osc}_{\mathcal{T}}^2(\boldsymbol{y}_{\mathcal{T}}^*, \partial T) + \mathrm{osc}_{\mathcal{T}}^2(\boldsymbol{p}_{\mathcal{T}}^*, \boldsymbol{y}_{\mathcal{T}}^*, \partial T)),
\end{aligned} \tag{3.25}$$

where the constant C depends only on ν and the shape-regularity of $\mathcal{T}$.

Proof. For $T \in \mathcal{T}$, let b_T be the usual tetrahedral bubble function on T [24]. With $\boldsymbol{w}_T = \bar{R}_{T,1}b_T$, the standard scaling argument and the definition of $\bar{R}_{T,1}$ imply

$$\begin{aligned}
C\|\bar{R}_{T,1}\|_{0,T}^2 \le (\bar{R}_{T,1}, \boldsymbol{w}_T)_T &= (\bar{R}_{T,1} - R_{T,1}, \boldsymbol{w}_T)_T + (R_{T,1}, \boldsymbol{w}_T)_T\\
&= (\chi_c\boldsymbol{u}_{\mathcal{T}}^* - \boldsymbol{\nabla} \times (\nu\boldsymbol{\nabla} \times \boldsymbol{y}_{\mathcal{T}}^*), \boldsymbol{w}_T)_T + (\bar{R}_{T,1} - R_{T,1}, \boldsymbol{w}_T)_T.
\end{aligned} \tag{3.26}$$

Integrating by parts and invoking the first equation of (2.4) with $\boldsymbol{v} = \boldsymbol{w}_T \in \boldsymbol{H}_0^1(T)$ and $\phi^* = 0$ admit

$$\begin{aligned}
&(\chi_c\boldsymbol{u}_{\mathcal{T}}^* - \boldsymbol{\nabla} \times (\nu\boldsymbol{\nabla} \times \boldsymbol{y}_{\mathcal{T}}^*), \boldsymbol{w}_T)_T = (\chi_c\boldsymbol{u}_{\mathcal{T}}^*, \boldsymbol{w}_T)_T - (\nu\boldsymbol{\nabla} \times \boldsymbol{y}_{\mathcal{T}}^*, \boldsymbol{\nabla} \times \boldsymbol{w}_T)_T\\
=& (\nu\boldsymbol{\nabla} \times (\boldsymbol{y}^* - \boldsymbol{y}_{\mathcal{T}}^*), \boldsymbol{\nabla} \times \boldsymbol{w}_T)_T - (\chi_c\boldsymbol{u}^* - \chi_c\boldsymbol{u}_{\mathcal{T}}^*, \boldsymbol{w}_T)_T.
\end{aligned} \tag{3.27}$$

By the Cauchy-Schwarz inequality, the inverse estimate, the scaling argument and the triangle inequality, we see from (3.26) and (3.27) that

$$h_T^2\|R_{T,1}\|_{0,T}^2 \le C(\|\boldsymbol{\nabla} \times (\boldsymbol{y}^* - \boldsymbol{y}_{\mathcal{T}}^*)\|_{0,T}^2 + h_T^2\|\chi_c\boldsymbol{u}^* - \chi_c\boldsymbol{u}_{\mathcal{T}}^*\|_{0,T}^2 + h_T^2\|R_{T,1} - \bar{R}_{T,1}\|_{0,T}^2). \tag{3.28}$$

For $F \in \mathcal{F}_\mathcal{T}(\Omega)$, we make use of the face bubble function b_F [24], which vanishes on $\partial\omega_F$, to construct $\boldsymbol{w}_F := \bar{J}_{F,1} b_F \in \boldsymbol{H}_0^1(\omega_F)$. By arguments similar to those for (3.26)-(3.27) we find

$$C\|\bar{J}_{F,1}\|_{0,F}^2 \leq (\bar{J}_{F,1}, \boldsymbol{w}_F)_F = (J_{F,1}, \boldsymbol{w}_F)_F + (\bar{J}_{F,1} - J_{F,1}, \boldsymbol{w}_F)_F,$$
$$(J_{F,1}, \boldsymbol{w}_F)_F = (\chi_c \boldsymbol{u}^* - \chi_c \boldsymbol{u}_\mathcal{T}^*, \boldsymbol{w}_F)_{\omega_F} - (\nu \boldsymbol{\nabla} \times (\boldsymbol{y}^* - \boldsymbol{y}_\mathcal{T}^*), \boldsymbol{\nabla} \times \boldsymbol{w}_F)_{\omega_F} + (R_{T,1}, \boldsymbol{w}_F)_{\omega_F},$$

which, together with (3.28), the inverse estimate, the estimate $\|\boldsymbol{w}_F\|_{0,\omega_F} \leq C h_F^{1/2} \|\bar{J}_{F,1}\|_{0,F}$ and the triangle inequality, yields

$$\begin{aligned} h_F \|J_{F,1}\|_F^2 \leq C\Big(\sum_{T \in \omega_F} (\|\boldsymbol{\nabla} \times (\boldsymbol{y}^* - \boldsymbol{y}_\mathcal{T}^*)\|_{0,T}^2 + h_T^2 \|\chi_c \boldsymbol{u}^* - \chi_c \boldsymbol{u}_\mathcal{T}^*\|_{0,T}^2 \\ + h_T^2 \|R_{T,1} - \bar{R}_{T,1}\|_{0,T}^2) + h_F \|J_{F,1} - \bar{J}_{F,1}\|_{0,F}^2 \Big). \end{aligned} \tag{3.29}$$

For the jump $J_{F,2}$, we extend it constantly along the normal to F to get an extension $E_F(J_{F,1})$ over ω_F. Then taking $\boldsymbol{v} = E_F(J_{F,2}) b_F \in H_0^1(\omega_F)$ in the second equation of (2.4) and applying the same arguments as above, we obtain

$$h_F \|J_{F,2}\|_{0,F}^2 \leq C \sum_{T \in \omega_F} \|\boldsymbol{y}^* - \boldsymbol{y}_\mathcal{T}^*\|_{0,T}^2. \tag{3.30}$$

For $R_{T,2}$, $J_{F,4}$ and $J_{F,5}$, it is not difficult to deduce their bounds in a similar manner:

$$h_T^2 \|R_{T,2}\|_{0,T}^2 \leq C(\|\boldsymbol{\nabla} \times (\boldsymbol{p}^* - \boldsymbol{p}_\mathcal{T}^*)\|_{0,T}^2 + \|\boldsymbol{\nabla} \times (\boldsymbol{y}^* - \boldsymbol{y}_\mathcal{T}^*)\|_{0,T}^2 + h_T^2 \|R_{T,2} - \bar{R}_{T,2}\|_{0,T}^2), \tag{3.31}$$

$$\begin{aligned} h_F \|J_{F,4}\|_F^2 \leq C\Big(\sum_{T \in \omega_F} (\|\boldsymbol{\nabla} \times (\boldsymbol{p}^* - \boldsymbol{p}_\mathcal{T}^*)\|_{0,T}^2 + \|\boldsymbol{\nabla} \times (\boldsymbol{y}^* - \boldsymbol{y}_\mathcal{T}^*)\|_{0,T}^2 \\ + h_T^2 \|R_{T,2} - \bar{R}_{T,2}\|_{0,T}^2) + h_F \|J_{F,4} - \bar{J}_{F,4}\|_{0,F}^2 \Big), \end{aligned} \tag{3.32}$$

$$h_F \|J_{F,5}\|_{0,F}^2 \leq C \sum_{T \in \omega_F} \|\boldsymbol{p}^* - \boldsymbol{p}_\mathcal{T}^*\|_{0,T}^2. \tag{3.33}$$

Finally, by virtue of the constraint $(\boldsymbol{u}^*, \boldsymbol{\nabla} q)_{\Omega_c} = 0$ for any $q \in H^1(\Omega_c)$ (cf. (1.3)), we choose $q = q_F := \chi_c E_F(J_{F,3}) b_F$ and proceed for some given $F \in \mathcal{F}_\mathcal{T}(\bar{\Omega}_c)$,

$$C\|J_{F,3}\|_{0,F}^2 \leq (J_{F,3}, q_F)_F = (\chi_c \boldsymbol{u}_\mathcal{T}^* - \chi_c \boldsymbol{u}^*, \boldsymbol{\nabla} q_F)_{\omega_F}.$$

Then the inverse estimate and $\|q_F\|_{0,\omega_F} \leq C h_F^{1/2} \|J_{F,3}\|_{0,F}$ give

$$h_F \|J_{F,3}\|_{0,F}^2 \leq C \sum_{T \in \omega_F} \|\chi_c \boldsymbol{u}^* - \chi_c \boldsymbol{u}_\mathcal{T}^*\|_{0,T}^2. \tag{3.34}$$

Now the desired lower bound follows from (3.28)-(3.34). □

We end this section with the following useful stability results for the error estimator $\eta_\mathcal{T}$, which are the direct consequences of the inverse estimate, the local quasi-uniformity of $\mathcal{T}$ and the assumption on the coefficient ν.

Lemma 3.4. *Let $(\boldsymbol{u}_\mathcal{T}^*, \boldsymbol{y}_\mathcal{T}^*, \boldsymbol{p}_\mathcal{T}^*)$ be the solutions to the problem* (2.13)-(2.15)*. Then for the error indicators $\eta_{\mathcal{T},1}$, $\eta_{\mathcal{T},2}$ and $\eta_{\mathcal{T},3}$, there hold that for any $T \in \mathcal{T}$,*

$$\eta_{\mathcal{T},1}^2(\boldsymbol{y}_\mathcal{T}^*, \boldsymbol{u}_\mathcal{T}^*, T) \leq C(\|\boldsymbol{\nabla} \times \boldsymbol{y}_\mathcal{T}^*\|_{0,\omega_T}^2 + \|\boldsymbol{y}_\mathcal{T}^*\|_{0,\omega_T}^2 + \|\chi_c \boldsymbol{u}_\mathcal{T}^*\|_{0,\omega_T}^2), \tag{3.35}$$

$$\eta_{\mathcal{T},2}^2(\boldsymbol{p}_\mathcal{T}^*, \boldsymbol{y}_\mathcal{T}^*, T) \leq C(\|\boldsymbol{\nabla} \times \boldsymbol{y}_\mathcal{T}^*\|_{0,\omega_T}^2 + \|\boldsymbol{\nabla} \times \boldsymbol{p}_\mathcal{T}^*\|_{0,\omega_T}^2 + \|\boldsymbol{p}_\mathcal{T}^*\|_{0,\omega_T}^2 + h_T^2 \|\boldsymbol{\nabla} \times \boldsymbol{\nabla} \times \boldsymbol{y}_d\|_{0,T}^2), \tag{3.36}$$

$$\eta_{\mathcal{T},3}^2(\boldsymbol{u}_\mathcal{T}^*, T) \leq C\|\chi_c \boldsymbol{u}_\mathcal{T}^*\|_{0,\omega_T}^2. \tag{3.37}$$

4 Adaptive algorithm

In this section we present an adaptive algorithm for the problem (1.2) and (2.1), based on the a posteriori error estimator defined in the first part of Section 3 for the discrete problem (2.13)-(2.15). Some more definitions and notation are needed. Let $\mathbb{T}$ be the set of all possible conforming triangulations of $\overline{\Omega}$ obtained from some shape-regular initial mesh by the bisection successively [14] [18]. This refinement process ensures that the set $\mathbb{T}$ is uniformly shape regular [18] [22], thus all the constants depend only on the initial mesh and the given data, not on any particular mesh from the refinement. For any $\mathcal{T}$, $\mathcal{T}' \in \mathbb{T}$, we call $\mathcal{T}'$ a refinement of $\mathcal{T}$ if $\mathcal{T}'$ is produced from $\mathcal{T}$ by a finite number of bisections.

For any triangulation sequence $\{\mathcal{T}_k\} \subset \mathbb{T}$ with $\mathcal{T}_{k+1}$ being a refinement of $\mathcal{T}_k$, we define

$$\mathcal{T}_k^+ := \bigcap_{l \geq k} \mathcal{T}_l, \quad \mathcal{T}_k^0 := \mathcal{T}_k \setminus \mathcal{T}_k^+, \quad \Omega_k^+ := \bigcup_{T \in \mathcal{T}_k^+} D_T, \quad \Omega_k^0 := \bigcup_{T \in \mathcal{T}_k^0} D_T.$$

That is, $\mathcal{T}_k^+$ consists of all elements not refined after the k-th iteration while all elements in $\mathcal{T}_k^0$ are refined at least once after the k-th iteration. It is easy to see $\mathcal{T}_l^+ \subset \mathcal{T}_k^+$ for $l < k$. We also define a mesh-size function $h_k : \overline{\Omega} \to \mathbb{R}^+$ almost everywhere by $h_k(x) = h_T$ for x in the interior of an element $T \in \mathcal{T}_k$ and $h_k(x) = h_F$ for x in the relative interior of a face $F \in \mathcal{F}_k$. Letting χ_k^0 be the characteristic function of Ω_k^0, then the mesh-size function $h_k(x)$ has the property [17] [21]:

$$\lim_{k \to \infty} \|h_k \chi_k^0\|_\infty = 0\,. \tag{4.1}$$

Now we are ready to propose an adaptive algorithm featured by an iteration of (1.4). In what follows, all dependences on triangulations are indicated by the number of refinements, e.g., the error estimator $\eta_{\mathcal{T}}(\boldsymbol{u}_{\mathcal{T}}^*, \boldsymbol{y}_{\mathcal{T}}^*, \boldsymbol{p}_{\mathcal{T}}^*)$ is rewritten as $\eta_k(\boldsymbol{u}_k^*, \boldsymbol{y}_k^*, \boldsymbol{p}_k^*)$ when $\mathcal{T}$ is replaced by $\mathcal{T}_k$.

Algorithm 4.1. *Given a conforming initial mesh $\mathcal{T}_0$. Set $k := 0$.*

1. (SOLVE) *Solve the discrete problem* (2.9)-(2.10) *on $\mathcal{T}_k$ for the minimizer $(\boldsymbol{u}_k^*, \boldsymbol{y}_k^*) \in \boldsymbol{U}_k \times \boldsymbol{V}_k$ and the discrete adjoint problem* (2.14) *for $\boldsymbol{p}_k^* \in \boldsymbol{V}_k$.*

2. (ESTIMATE) *Compute the error estimator $\eta_k(\boldsymbol{u}_k^*, \boldsymbol{y}_k^*, \boldsymbol{p}_k^*)$.*

3. (MARK) *Mark a subset $\mathcal{M}_k \subset \mathcal{T}_k$ containing at least one element $\widetilde{T} \in \mathcal{T}_k$ such that*

$$\eta_k(\boldsymbol{u}_k^*, \boldsymbol{y}_k^*, \boldsymbol{p}_k^*, \widetilde{T}) = \max_{T \in \mathcal{T}_k} \eta_k(\boldsymbol{u}_k^*, \boldsymbol{y}_k^*, \boldsymbol{p}_k^*, T). \tag{4.2}$$

4. (REFINE) *Refine each triangle $T \in \mathcal{M}_k$ by bisection to get $\mathcal{T}_{k+1}$.*

5. *Set $k := k + 1$ and go to Step 1.*

We note that Algorithm 4.1 can also produce a sequence of linear element spaces S_k, and the Lagrange multipliers ϕ_k^* and ψ_k^* from the second equations in (2.13) and (2.14). But as ϕ_k^* and ψ_k^* are both equal to zero, they are not included in the module of SOLVE. The requirement in the module MARK is clearly met by several practical marking strategies in computations, such as the maximum strategy, the equi-distribution strategy and the modified equi-distribution strategy. In practice, it is often required in Dörfler's strategy that the element with the maximal error indicator be included in $\mathcal{M}_k$ for computing efficiency, that is, it holds that

$$\min_{T \in \mathcal{M}_k} \eta_k(\boldsymbol{u}_k^*, \boldsymbol{y}_k^*, \boldsymbol{p}_k^*, T) \geq \max_{T \in \mathcal{T}_k \setminus \mathcal{M}_k} \eta_k(\boldsymbol{u}_k^*, \boldsymbol{y}_k^*, \boldsymbol{p}_k^*, T)\,.$$

5 Limiting problems

This section explores certain limits of sequences of approximate solutions and discrete spaces given by the adaptive Algorithm 4.1, which are the basis of convergence analysis in the next section. To this end, we define the following limiting spaces

$$\boldsymbol{V}_\infty := \overline{\bigcup_{k\geq 0} \boldsymbol{V}_k} \text{ (in } \boldsymbol{H}(\mathbf{curl})\text{-norm)}, \quad S_\infty := \overline{\bigcup_{k\geq 0} S_k} \text{ (in } H^1\text{-norm)},$$

$$\boldsymbol{V}^c_\infty := \overline{\bigcup_{k\geq 0} \boldsymbol{V}^c_k} \text{ (in } \boldsymbol{H}(\mathbf{curl})\text{-norm)}, \quad S^c_\infty := \overline{\bigcup_{k\geq 0} S^c_k} \text{ (in } H^1\text{-norm)},$$

$$\begin{aligned} \boldsymbol{X}_\infty &:= \{\boldsymbol{v} \in \boldsymbol{V}_\infty \mid (\boldsymbol{v}, \boldsymbol{\nabla} q) = 0 \ \forall\, q \in S_\infty\}, \\ \boldsymbol{U}_\infty &:= \{\boldsymbol{v} \in \boldsymbol{V}^c_\infty \mid (\boldsymbol{v}, \boldsymbol{\nabla} q)_{\Omega_c} = 0 \ \forall\, q \in S^c_\infty\}, \end{aligned}$$

where $\{\boldsymbol{V}_k\}$ and $\{S_k\}$ are both generated by Algorithm 4.1. These spaces have similar properties to those of discrete ones. In fact, we easily observe from definitions of $\boldsymbol{V}_\infty$ and S_∞ and (2.7) that

$$\boldsymbol{\nabla} S_\infty \subset \boldsymbol{V}_\infty, \tag{5.1}$$

$$\sup_{\mathbf{0}\neq \boldsymbol{v}\in \boldsymbol{V}_\infty} \frac{(\boldsymbol{v}, \boldsymbol{\nabla} q)}{\|\boldsymbol{v}\|_{\boldsymbol{H}(\mathbf{curl})}} \geq C\|q\|_1 \quad \forall\, q \in S_\infty \tag{5.2}$$

with the constant C depending only on Ω. In addition, since $\boldsymbol{\nabla} S^c_\infty$ is a closed subspace of $\boldsymbol{V}^c_\infty$ we have a decomposition similar to (2.20):

$$\boldsymbol{V}^c_\infty = \boldsymbol{U}_\infty \oplus \boldsymbol{\nabla} S^c_\infty. \tag{5.3}$$

Finally we note that $\boldsymbol{X}_0 \not\subset \boldsymbol{X}_1 \not\subset \cdots \not\subset \boldsymbol{X}_\infty \not\subset \boldsymbol{X}$, but over $\boldsymbol{X}_\infty$ there hold the following important density result and a counterpart of (2.2).

Lemma 5.1. *For any $\boldsymbol{v} \in \boldsymbol{X}_\infty$ there exists a sequence $\{\boldsymbol{w}_k\}$ with $\boldsymbol{w}_k \in \boldsymbol{X}_k$ such that*

$$\boldsymbol{w}_k \to \boldsymbol{v} \quad \textit{in } \boldsymbol{H}_0(\mathbf{curl}; \Omega) \tag{5.4}$$

and there is a constant C depending only on the shape-regularity of $\mathcal{T}_0$ such that

$$\|\boldsymbol{v}\|_0 \leq C\|\boldsymbol{\nabla} \times \boldsymbol{v}\|_0. \tag{5.5}$$

Proof. Since $\boldsymbol{X}_\infty \subset \boldsymbol{V}_\infty$, for any $\boldsymbol{v} \in \boldsymbol{X}_\infty$ there exists a sequence $\{\boldsymbol{v}_k\} \subset \bigcup_{k\geq 0} \boldsymbol{V}_k$ such that

$$\boldsymbol{v}_k \to \boldsymbol{v} \quad \text{in } \boldsymbol{H}_0(\mathbf{curl}; \Omega). \tag{5.6}$$

By the discrete Helmholtz decomposition [16], $\boldsymbol{V}_k = \boldsymbol{X}_k \oplus \boldsymbol{\nabla} S_k$, we may split $\boldsymbol{v}_k$ as $\boldsymbol{v}_k = \boldsymbol{w}_k + \boldsymbol{\nabla} q_k$ with $\boldsymbol{w}_k \in \boldsymbol{X}_k$ and $q_k \in S_k$. Using the orthogonality of $\boldsymbol{X}_\infty$ to $\boldsymbol{\nabla} S_k$, we have

$$(\boldsymbol{\nabla} q_k, \boldsymbol{\nabla} q_k) = (\boldsymbol{v}_k, \boldsymbol{\nabla} q_k) = (\boldsymbol{v}_k - \boldsymbol{v}, \boldsymbol{\nabla} q_k) \leq \|\boldsymbol{v} - \boldsymbol{v}_k\|_0 \|\boldsymbol{\nabla} q_k\|_0.$$

In light of (5.6), $\|\boldsymbol{\nabla} q_k\|_0 \leq \|\boldsymbol{v} - \boldsymbol{v}_k\|_0 \to 0$ as $k \to \infty$. Therefore,

$$\|\boldsymbol{v} - \boldsymbol{w}_k\|_0 \leq \|\boldsymbol{v} - \boldsymbol{v}_k\|_0 + \|\boldsymbol{\nabla} q_k\|_0 \to 0 \quad \text{as } k \to \infty. \tag{5.7}$$

Noting $\boldsymbol{\nabla} \times \boldsymbol{v}_k = \boldsymbol{\nabla} \times \boldsymbol{w}_k$ and using (5.6) again, we get

$$\|\boldsymbol{\nabla} \times (\boldsymbol{v} - \boldsymbol{w}_k)\|_0 = \|\boldsymbol{\nabla} \times (\boldsymbol{v} - \boldsymbol{v}_k)\|_0 \to 0 \quad \text{as } k \to \infty. \tag{5.8}$$

Then it follows from the discrete Poincaré inequality (2.11) that

$$\|\boldsymbol{w}_k\|_0 \leq C\|\boldsymbol{\nabla} \times \boldsymbol{w}_k\|_0. \tag{5.9}$$

The proof is completed by collecting (5.7)-(5.9). □

Now we introduce a limiting minimization problem over $\boldsymbol{U}_\infty$:

$$\min_{\boldsymbol{u}_\infty \in \boldsymbol{U}_\infty} \mathcal{J}_\infty(\boldsymbol{u}_\infty) := \frac{1}{2}\|\boldsymbol{\nabla}\times \boldsymbol{y}_\infty(\boldsymbol{u}_\infty) - \boldsymbol{\nabla}\times \boldsymbol{y}_d\|_0^2 + \frac{\gamma}{2}\|\boldsymbol{u}_\infty\|_{0,\Omega_c}^2, \tag{5.10}$$

where $\boldsymbol{y}_\infty := \boldsymbol{y}_\infty(\boldsymbol{u}_\infty) \in \boldsymbol{V}_\infty$ and $\phi_\infty := \phi_\infty(\boldsymbol{u}_\infty) \in S_\infty$ satisfy a limiting variational problem:

$$\begin{cases} (\nu\boldsymbol{\nabla}\times \boldsymbol{y}_\infty, \boldsymbol{\nabla}\times \boldsymbol{v}) + (\boldsymbol{v}, \boldsymbol{\nabla}\phi_\infty) = (\boldsymbol{u}_\infty, \boldsymbol{v})_{\Omega_c} \quad \forall\, \boldsymbol{v}\in \boldsymbol{V}_\infty, \\ (\boldsymbol{y}_\infty, \boldsymbol{\nabla} q) = 0 \quad \forall\, q \in S_\infty. \end{cases} \tag{5.11}$$

The unique solvability of the saddle-point system (5.11) is guaranteed by (5.2) and (5.5).

To formulate the optimality conditions of the problem (5.10)-(5.11), we invoke its adjoint problem: find $(\boldsymbol{p}_\infty, \psi_\infty) \in \boldsymbol{V}_\infty \times S_\infty$ such that

$$\begin{cases} (\nu\boldsymbol{\nabla}\times \boldsymbol{p}_\infty, \boldsymbol{\nabla}\times \boldsymbol{v}) + (\boldsymbol{v}, \boldsymbol{\nabla}\psi_\infty) = (\boldsymbol{\nabla}\times \boldsymbol{y}_\infty - \boldsymbol{\nabla}\times \boldsymbol{y}_d, \boldsymbol{\nabla}\times \boldsymbol{v}) \quad \forall\, \boldsymbol{v}\in \boldsymbol{V}_\infty, \\ (\boldsymbol{p}_\infty, \boldsymbol{\nabla} q) = 0 \quad \forall\, q \in S_\infty. \end{cases} \tag{5.12}$$

This problem has a unique solution due to (5.2) and (5.5). Moreover, we know from (5.11) the Gâteaux derivative of $\boldsymbol{y}_\infty(\boldsymbol{u}_\infty)$ at $\boldsymbol{u}_\infty \in \boldsymbol{U}_\infty$ in a direction $\boldsymbol{w}\in \boldsymbol{U}_\infty$, i.e., $\boldsymbol{y}'_\infty(\boldsymbol{u}_\infty)\boldsymbol{w} \in \boldsymbol{X}_\infty$, solves

$$(\nu\boldsymbol{\nabla}\times (\boldsymbol{y}'_\infty(\boldsymbol{u}_\infty)\boldsymbol{w}), \boldsymbol{\nabla}\times \boldsymbol{v}) = (\boldsymbol{w}, \boldsymbol{v})_{\Omega_c} \quad \forall\, \boldsymbol{v}\in \boldsymbol{X}_\infty. \tag{5.13}$$

By virtue of (5.12)-(5.13) and $(\boldsymbol{y}'_\infty(\boldsymbol{u}_\infty)\boldsymbol{w}, \boldsymbol{\nabla} q) = 0$ for any $q \in \boldsymbol{S}_\infty$, we can compute the Gâteaux derivative of $\mathcal{J}_\infty(\boldsymbol{u}_\infty)$ with respect to any $\boldsymbol{w}\in \boldsymbol{U}_\infty$:

$$\begin{aligned} \mathcal{J}'_\infty(\boldsymbol{u}_\infty)\boldsymbol{w} &= (\boldsymbol{\nabla}\times \boldsymbol{y}_\infty - \boldsymbol{\nabla}\times \boldsymbol{y}_d, \boldsymbol{\nabla}\times (\boldsymbol{y}'_\infty(\boldsymbol{u}_\infty)\boldsymbol{w})) + \gamma(\boldsymbol{u}_\infty, \boldsymbol{w})_{\Omega_c} \\ &= (\nu\boldsymbol{\nabla}\times \boldsymbol{p}_\infty, \boldsymbol{\nabla}\times (\boldsymbol{y}'_\infty(\boldsymbol{u}_\infty)\boldsymbol{w})) + (\boldsymbol{y}'_\infty(\boldsymbol{u}_\infty)\boldsymbol{w}, \boldsymbol{\nabla}\psi_\infty) + \gamma(\boldsymbol{u}_\infty, \boldsymbol{w})_{\Omega_c} \\ &= (\nu\boldsymbol{\nabla}\times (\boldsymbol{y}'_\infty(\boldsymbol{u}_\infty)\boldsymbol{w}), \boldsymbol{\nabla}\times \boldsymbol{p}_\infty) + \gamma(\boldsymbol{u}_\infty, \boldsymbol{w})_{\Omega_c} \\ &= (\boldsymbol{p}_\infty + \gamma \boldsymbol{u}_\infty, \boldsymbol{w})_{\Omega_c}. \end{aligned}$$

Noting $\boldsymbol{p}_\infty|_{\Omega_c} \in \boldsymbol{V}^c_\infty$ and the decomposition (5.3), we further see $-\boldsymbol{p}_\infty|_{\Omega_c} = \gamma \boldsymbol{z}_\infty + \gamma\boldsymbol{\nabla}\xi_\infty$ with $\boldsymbol{z}_\infty \in \boldsymbol{U}_\infty$ and $\xi_\infty \in S^c_\infty$, which, along with $(\boldsymbol{w}, \boldsymbol{\nabla}\xi_\infty)_{\Omega_c} = 0$, implies

$$\mathcal{J}'_\infty(\boldsymbol{u}_\infty)\boldsymbol{w} = \gamma(\boldsymbol{u}_\infty - \boldsymbol{z}_\infty, \boldsymbol{w})_{\Omega_c} \quad \forall\, \boldsymbol{w}\in \boldsymbol{U}_\infty. \tag{5.14}$$

Now with a costate $\boldsymbol{p}^*_\infty \in \boldsymbol{V}_\infty$ given by (5.12) with $\boldsymbol{y}_\infty = \boldsymbol{y}^*_\infty(\boldsymbol{u}^*_\infty)$ in the right-hand side, we know from (5.14) and the standard convex analysis that the equivalent condition for the minimizer $\boldsymbol{u}^*_\infty \in \boldsymbol{U}_\infty$ to the problem (5.10)-(5.11) is given by

$$\mathcal{J}'_\infty(\boldsymbol{u}^*_\infty)(\boldsymbol{w} - \boldsymbol{u}^*_\infty) = \gamma(\boldsymbol{u}^*_\infty - \boldsymbol{z}^*_\infty, \boldsymbol{w} - \boldsymbol{u}^*_\infty)_{\Omega_c} \geq 0 \quad \forall\, \boldsymbol{w}\in \boldsymbol{U}_\infty,$$

where $-\boldsymbol{p}^*_\infty|_{\Omega_c} = \gamma\boldsymbol{z}^*_\infty + \gamma\boldsymbol{\nabla}\xi^*_\infty$ with $\boldsymbol{z}^*_\infty \in \boldsymbol{U}_\infty$ and $\xi^*_\infty \in S^c_\infty$. Hence $\boldsymbol{u}^*_\infty = \boldsymbol{z}^*_\infty$, i.e.,

$$-\gamma^{-1}\boldsymbol{p}^*_\infty|_{\Omega_c} = \boldsymbol{u}^*_\infty + \boldsymbol{\nabla}\xi^*_\infty. \tag{5.15}$$

Summarizing the above analysis, we conclude the equivalent optimality conditions for the constrained minimization problem (5.10)-(5.11): the triplet $(\boldsymbol{u}^*_\infty, \boldsymbol{y}^*_\infty, \boldsymbol{p}^*_\infty) \in \boldsymbol{U}_\infty \times \boldsymbol{V}_\infty \times \boldsymbol{V}_\infty$ and corresponding multipliers $(\phi^*_\infty, \psi^*_\infty) \in S_\infty \times S_\infty$ satisfy

$$\begin{cases} (\nu\boldsymbol{\nabla}\times \boldsymbol{y}^*_\infty, \boldsymbol{\nabla}\times \boldsymbol{v}) + (\boldsymbol{v}, \boldsymbol{\nabla}\phi^*_\infty) = (\boldsymbol{u}^*_\infty, \boldsymbol{v})_{\Omega_c} \quad \forall\, \boldsymbol{v}\in \boldsymbol{V}_\infty, \\ (\boldsymbol{y}^*_\infty, \boldsymbol{\nabla} q) = 0 \quad \forall\, q \in S_\infty. \end{cases} \tag{5.16}$$

$$\begin{cases} (\nu\boldsymbol{\nabla}\times \boldsymbol{p}^*_\infty, \boldsymbol{\nabla}\times \boldsymbol{v}) + (\boldsymbol{v}, \boldsymbol{\nabla}\psi^*_\infty) = (\boldsymbol{\nabla}\times \boldsymbol{y}^*_\infty - \boldsymbol{\nabla}\times \boldsymbol{y}_d, \boldsymbol{\nabla}\times \boldsymbol{v}) \quad \forall\, \boldsymbol{v}\in \boldsymbol{V}_\infty, \\ (\boldsymbol{p}^*_\infty, \boldsymbol{\nabla} q) = 0 \quad \forall\, q \in S_\infty. \end{cases} \tag{5.17}$$

$$(\boldsymbol{u}^*_\infty + \gamma^{-1}\boldsymbol{p}^*_\infty, \boldsymbol{w})_{\Omega_c} = 0 \quad \forall\, \boldsymbol{w}\in \boldsymbol{U}_\infty. \tag{5.18}$$

Noting that $\boldsymbol{\nabla} S_\infty \subset \boldsymbol{V}_\infty$ and $\boldsymbol{u}_\infty^* \in \boldsymbol{U}_\infty$, we can easily get $\phi_\infty^* = 0$ and $\psi_\infty^* = 0$ by taking $\boldsymbol{v} = \boldsymbol{\nabla}\phi_\infty^*$ and $\boldsymbol{\nabla}\psi_\infty^*$ respectively in (5.16) and (5.17).

Next we investigate the unique solvability of the limiting optimal control problem (5.10)-(5.11) and present an auxiliary result: the sequence $\{\boldsymbol{u}_k^*\}$ generated by Algorithm 4.1 converges strongly in $\boldsymbol{L}^2(\Omega_c)$ to a limiting minimizer. Unlike the traditional approach, for proving existence we shall resort to the adaptive solution sequence and extract some weakly convergent subsequence, the limit of which will be shown to be the minimizer. For this purpose, some auxiliary results are needed. We define a projection operator $\boldsymbol{P}_k : \boldsymbol{V}_\infty^c \to \boldsymbol{V}_k^c$ by

$$(\boldsymbol{P}_k\boldsymbol{u}, \boldsymbol{v})_{\Omega_c} = (\boldsymbol{u}, \boldsymbol{v})_{\Omega_c} \quad \forall\, \boldsymbol{v} \in \boldsymbol{V}_k^c. \tag{5.19}$$

Lemma 5.2. *If $\boldsymbol{u} \in \boldsymbol{U}_\infty$, then $\boldsymbol{P}_k\boldsymbol{u} \in \boldsymbol{U}_k$. Moreover,*

$$\|\boldsymbol{u} - \boldsymbol{P}_k\boldsymbol{u}\|_{0,\Omega_c} \to 0 \quad \text{as } k \to \infty. \tag{5.20}$$

Proof. As $\boldsymbol{\nabla} S_k^c \subset \boldsymbol{V}_k^c$, it is easy to check that for $\boldsymbol{u} \in \boldsymbol{U}_\infty$,

$$(\boldsymbol{P}_k\boldsymbol{u}, \boldsymbol{\nabla} q)_{\Omega_c} = (\boldsymbol{u}, \boldsymbol{\nabla} q)_{\Omega_c} = 0 \quad \forall\, q \in S_k^c.$$

From the optimal error estimate of the operator $\boldsymbol{P}_k$, we know

$$\|\boldsymbol{u} - \boldsymbol{P}_k\boldsymbol{u}\|_{0,\Omega_c} \le \inf_{\boldsymbol{v}\in\boldsymbol{V}_k^c} \|\boldsymbol{u} - \boldsymbol{v}\|_{0,\Omega_c},$$

which converges to zero due to the density of $\bigcup_{k\ge 0} \boldsymbol{V}_k^c$ in $\boldsymbol{V}_\infty^c$. ☐

Lemma 5.3. *Let $\{\boldsymbol{V}_k, S_k\}$ be a sequence of discrete spaces given by Algorithm 4.1. If the sequence $\{\boldsymbol{u}_k\} \subset \bigcup_{k\ge 0} \boldsymbol{V}_k^c$ converges strongly to some $\boldsymbol{u} \in \boldsymbol{V}_\infty^c$ in $\boldsymbol{L}^2(\Omega_c)$, then for the sequence $\{(\boldsymbol{y}_k(\boldsymbol{u}_k), \phi_k(\boldsymbol{u}_k)\} \subset \bigcup_{k\ge 0} \boldsymbol{V}_k \times S_k$ given by* (2.10) *and for $(\boldsymbol{y}_\infty(\boldsymbol{u}), \phi_\infty(\boldsymbol{u})) \in \boldsymbol{V}_\infty \times S_\infty$ given by* (5.11)*, there holds*

$$\|\boldsymbol{y}_k(\boldsymbol{u}_k) - \boldsymbol{y}_\infty(\boldsymbol{u})\|_{\boldsymbol{H}(\mathbf{curl})} \to 0 \quad \text{and} \quad |\phi_k(\boldsymbol{u}_k) - \phi_\infty(\boldsymbol{u})|_1 \to 0 \quad \text{as} \quad k \to \infty\,. \tag{5.21}$$

Proof. We begin with an auxiliary discrete problem: find $(\boldsymbol{y}_k(\boldsymbol{u}), \phi_k(\boldsymbol{u})) \in \boldsymbol{V}_k \times S_k$ such that

$$\begin{cases} (\nu\boldsymbol{\nabla}\times\boldsymbol{y}_k(\boldsymbol{u}), \boldsymbol{\nabla}\times\boldsymbol{v}) + (\boldsymbol{v}, \boldsymbol{\nabla}\phi_k(\boldsymbol{u})) = (\boldsymbol{u}, \boldsymbol{v})_{\Omega_c} \quad \forall\, \boldsymbol{v} \in \boldsymbol{V}_k, \\ (\boldsymbol{y}_k(\boldsymbol{u}), \boldsymbol{\nabla} q) = 0 \quad \forall\, q \in S_k\,. \end{cases} \tag{5.22}$$

In view of (5.2) and (5.5), this problem is well-posed. The Babuska-Brezzi theory admits a quasi-optimal approximation property

$$\begin{aligned} &\|\boldsymbol{y}_\infty(\boldsymbol{u}) - \boldsymbol{y}_k(\boldsymbol{u})\|_{\boldsymbol{H}(\mathbf{curl})} + |\phi_\infty(\boldsymbol{u}) - \phi_k(\boldsymbol{u})|_1 \\ \le& C(\inf_{\boldsymbol{v}\in\boldsymbol{V}_k} \|\boldsymbol{y}_\infty(\boldsymbol{u}) - \boldsymbol{v}\|_{\boldsymbol{H}(\mathbf{curl})} + \inf_{q\in S_k} |\phi_\infty(\boldsymbol{u}) - q|_1). \end{aligned}$$

On the other hand, we substract (2.10) from (5.22) to get a stability result

$$\|\boldsymbol{y}_k(\boldsymbol{u}) - \boldsymbol{y}_k(\boldsymbol{u}_k)\|_{\boldsymbol{H}(\mathbf{curl})} + |\phi_k(\boldsymbol{u}) - \phi_k(\boldsymbol{u}_k)|_1 \le C\|\boldsymbol{u}_k - \boldsymbol{u}\|_{0,\Omega_c}.$$

Then (5.21) comes from the above estimates and the density of $\bigcup_k \boldsymbol{V}_k$ and $\bigcup_k S_k$ in $\boldsymbol{V}_\infty$ and S_∞. ☐

Now we are able to show

Theorem 5.1. *There exists a unique minimizer to the optimization problem* (5.10)-(5.11)*.*

Proof. Let $\{\boldsymbol{u}_k^*\}$ be the sequence of minimizers to the discrete problem (2.9)-(2.10) given by Algorithm 4.1. Noting that $\mathcal{J}_k(\boldsymbol{u}_k^*)$ is the minimum over $\boldsymbol{U}_k$ and $\boldsymbol{0} \in \boldsymbol{U}_k$, we obtain

$$\frac{\gamma}{2}\|\boldsymbol{u}_k^*\|_{0,\Omega_c}^2 \leq \mathcal{J}_k(\boldsymbol{u}_k^*) \leq \mathcal{J}_k(\boldsymbol{0}) = \frac{1}{2}\|\boldsymbol{\nabla} \times \boldsymbol{y}_d\|_0^2 \quad \forall\ k, \tag{5.23}$$

which, together with the stability (2.16), implies

$$\|\boldsymbol{y}_k^*\|_{\boldsymbol{H}(\mathbf{curl})} + |\phi_k^*|_1 \leq C \quad \forall\ k. \tag{5.24}$$

Furthermore, by (2.17) the sequence of discrete costates is also bounded, i.e.,

$$\|\boldsymbol{p}_k^*\|_{\boldsymbol{H}(\mathbf{curl})} \leq C \quad \forall\ k.$$

Then it follows from (2.21) that

$$\|\boldsymbol{\nabla} \times \boldsymbol{u}_k^*\|_{0,\Omega_c} \leq C \quad \forall\ k. \tag{5.25}$$

In view of (5.23)-(5.25), there exist three subsequences $\{\boldsymbol{u}_{k_n}^*\}$, $\{\boldsymbol{y}_{k_n}^*\}$, $\{\phi_{k_n}^*\}$ and three functions $\boldsymbol{w} \in \boldsymbol{V}_\infty^c$, $\boldsymbol{y} \in \boldsymbol{V}_\infty$, $\phi \in S_\infty$ such that the following weak convergences hold

$$\boldsymbol{u}_{k_n}^* \rightharpoonup \boldsymbol{w} \quad \text{ in } \boldsymbol{L}^2(\Omega_c), \tag{5.26}$$

$$\boldsymbol{y}_{k_n}^* \rightharpoonup \boldsymbol{y} \quad \text{ in } \boldsymbol{H}_0(\mathbf{curl};\Omega) \ \text{ and } \ \phi_{k_n}^* \rightharpoonup \phi \quad \text{ in } H_0^1(\Omega). \tag{5.27}$$

To obtain the existence of a minimizer to the problem (5.10)-(5.11), we only need to prove the following two claims: $(\boldsymbol{y}, \phi)$ satisfies the problem (5.11) with $\boldsymbol{u}_\infty = \boldsymbol{w}$; and $\mathcal{J}_\infty(\boldsymbol{w})$ attains the minimum over $\boldsymbol{U}_\infty$. For any integer $l \geq 0$, (2.10) implies for $k_n \geq l$ that

$$\begin{cases} (\nu\boldsymbol{\nabla} \times \boldsymbol{y}_{k_n}^*, \boldsymbol{\nabla} \times \boldsymbol{v}_l) + (\boldsymbol{v}_l, \boldsymbol{\nabla}\phi_{k_n}^*) = (\boldsymbol{u}_{k_n}^*, \boldsymbol{v}_l)_{\Omega_c} \quad \forall\ \boldsymbol{v}_l \in \boldsymbol{V}_l, \\ (\boldsymbol{y}_{k_n}^*, \boldsymbol{\nabla} q_l) = 0 \quad \forall\ q_l \in S_l, \end{cases}$$

which, along with the weak convergence (5.26) and (5.27), yields

$$\begin{cases} (\nu\boldsymbol{\nabla} \times \boldsymbol{y}, \boldsymbol{\nabla} \times \boldsymbol{v}_l) + (\boldsymbol{v}_l, \boldsymbol{\nabla}\phi) = (\boldsymbol{w}, \boldsymbol{v}_l)_{\Omega_c} \quad \forall\ \boldsymbol{v}_l \in \boldsymbol{V}_l, \\ (\boldsymbol{y}, \boldsymbol{\nabla} q_l) = 0 \quad \forall\ q_l \in S_l. \end{cases}$$

Since $(\boldsymbol{v}_l, q_l)$ is arbitrary, from the density definitions of $\boldsymbol{V}_\infty$ and S_∞ we find

$$\begin{cases} (\nu\boldsymbol{\nabla} \times \boldsymbol{y}, \boldsymbol{\nabla} \times \boldsymbol{v}) + (\boldsymbol{v}, \boldsymbol{\nabla}\phi) = (\boldsymbol{w}, \boldsymbol{v})_{\Omega_c} \quad \forall\ \boldsymbol{v} \in \boldsymbol{V}_\infty, \\ (\boldsymbol{y}, \boldsymbol{\nabla} q) = 0 \quad \forall\ q \in S_\infty. \end{cases}$$

Then the first claim holds with $(\boldsymbol{y}_\infty(\boldsymbol{w}), \phi_\infty(\boldsymbol{w})) = (\boldsymbol{y}, \phi)$. Noting $\boldsymbol{u}_{k_n}^* \in \boldsymbol{U}_{k_n}$ as well as the density of $\bigcup_{l\geq 0} S_l^c$ in S_∞^c and using the weak convergence (5.26), we have $(\boldsymbol{w}, \boldsymbol{\nabla} q) = 0$ for any $q \in S_\infty^c$, i.e., $\boldsymbol{w} \in \boldsymbol{U}_\infty$. With the projection operator $\boldsymbol{P}_k$ applied to any $\boldsymbol{u} \in \boldsymbol{U}_\infty$, we know $\boldsymbol{u}_k := \boldsymbol{P}_k\boldsymbol{u} \in \boldsymbol{U}_k$ by Lemma 5.2 and

$$\|\boldsymbol{u} - \boldsymbol{u}_k\|_{0,\Omega_c} \to 0, \tag{5.28}$$

from which and Lemma 5.3, we further get

$$\|\boldsymbol{\nabla} \times (\boldsymbol{y}_k(\boldsymbol{u}_k) - \boldsymbol{y}_\infty(\boldsymbol{u}))\|_0 \to 0. \tag{5.29}$$

Now it follows from (5.26)-(5.29) and the fact $\{\boldsymbol{u}_k^*\}$ is a sequence of discrete minimizers to the cost functional sequence $\{\mathcal{J}_k(\cdot)\}$ over $\{\boldsymbol{U}_k\}$ that

$$\begin{aligned}\mathcal{J}_\infty(\boldsymbol{w}) &= \frac{1}{2}\|\boldsymbol{\nabla}\times\boldsymbol{y}_\infty(\boldsymbol{w})-\boldsymbol{\nabla}\times\boldsymbol{y}_d\|_0^2+\frac{\gamma}{2}\|\boldsymbol{w}\|_{0,\Omega_c}^2\\ &\le \frac{1}{2}\liminf_{n\to\infty}\|\boldsymbol{\nabla}\times\boldsymbol{y}_{k_n}^*-\boldsymbol{\nabla}\times\boldsymbol{y}_d\|_0^2+\frac{\gamma}{2}\liminf_{n\to\infty}\|\boldsymbol{u}_{k_n}^*\|_{0,\Omega_c}^2\le\liminf_{n\to\infty}\mathcal{J}_{k_n}(\boldsymbol{u}_{k_n}^*)\\ &\le\limsup_{n\to\infty}\mathcal{J}_{k_n}(\boldsymbol{u}_{k_n}^*)\le\limsup_{k\to\infty}\mathcal{J}_k(\boldsymbol{u}_k^*)\le\limsup_{k\to\infty}\mathcal{J}_k(\boldsymbol{u}_k)=\mathcal{J}_\infty(\boldsymbol{u})\quad\forall\,\boldsymbol{u}\in\boldsymbol{U}_\infty.\end{aligned}\tag{5.30}$$

Hence $\boldsymbol{u}_\infty^* := \boldsymbol{w}\in\boldsymbol{U}_\infty$ and $\boldsymbol{y}_\infty^* := \boldsymbol{y}\in\boldsymbol{V}_\infty$ are a solution of the problem (5.10)-(5.11). The uniqueness is guaranteed by the strict convexity of $\mathcal{J}_\infty(\cdot)$. □

In the proof of Theorem 5.1, the choice $\boldsymbol{u}=\boldsymbol{u}_\infty^*$ in (5.30) implies that

$$\lim_{n\to\infty}\mathcal{J}_{k_n}(\boldsymbol{u}_{k_n}^*)=\mathcal{J}_\infty(\boldsymbol{u}_\infty^*)=\min_{\boldsymbol{u}_\infty\in\boldsymbol{U}_\infty}\mathcal{J}_\infty(\boldsymbol{u}_\infty)$$

and there further holds for the whole sequence $\{\boldsymbol{u}_k^*\}$ that

$$\lim_{k\to\infty}\mathcal{J}_k(\boldsymbol{u}_k^*)=\mathcal{J}_\infty(\boldsymbol{u}_\infty^*).\tag{5.31}$$

On the other hand, the uniqueness of minimizer to the limiting problem (5.10)-(5.11) asserts that the weak convergence in (5.26) and (5.27) are true for the whole sequence, i.e.,

$$\boldsymbol{u}_k^*\rightharpoonup\boldsymbol{u}_\infty^*\quad\text{in }\boldsymbol{L}^2(\Omega_c),\qquad\boldsymbol{y}_k^*\rightharpoonup\boldsymbol{y}_\infty^*\quad\text{in }\boldsymbol{H}_0(\mathbf{curl};\Omega).\tag{5.32}$$

As a consequence, the elementary equality

$$\begin{aligned}&\|\boldsymbol{\nabla}\times(\boldsymbol{y}_k^*-\boldsymbol{y}_\infty^*)\|_0^2+\gamma\|\boldsymbol{u}_k^*-\boldsymbol{u}_\infty^*\|_{0,\Omega_c}^2\\ =&2\mathcal{J}_k(\boldsymbol{u}_k^*)+2\mathcal{J}_\infty(\boldsymbol{u}_\infty^*)-2(\boldsymbol{\nabla}\times\boldsymbol{y}_k^*-\boldsymbol{\nabla}\times\boldsymbol{y}_d,\boldsymbol{\nabla}\times\boldsymbol{y}_\infty^*-\boldsymbol{\nabla}\times\boldsymbol{y}_d)-2\gamma(\boldsymbol{u}_k^*,\boldsymbol{u}_\infty^*)_{\Omega_c}\end{aligned}$$

and the three limits in (5.31) and (5.32) lead to the following first main result of this section.

Theorem 5.2. *Let* $\{\boldsymbol{u}_k^*\}$ *be a sequence of minimizers to the discrete problem* (2.9)-(2.10) *given by Algorithm 4.1 and* $\boldsymbol{u}_\infty^*$ *be the minimizer to the problem* (5.10)-(5.11)*. Then*

$$\|\boldsymbol{u}_k^*-\boldsymbol{u}_\infty^*\|_{0,\Omega_c}\to 0\quad\text{as } k\to\infty.\tag{5.33}$$

Remark 5.1. In the case of uniform refinements , i.e., $\|h_k\|_\infty\to 0$, Theorem 5.2 was established in [28], where the main tool is the discrete compactness of the Nédélec edge elements [16]. To our best knowledge, it remains still open whether this discrete compactness property is true for adaptively generated meshes since it may not hold in this case that $\|h_k\|_\infty\to 0$. So we resort here to the weak convergence of the sequences of the discrete states and controls in combination with the convergence of the cost functionals.

The following second main result of this section holds for the optimality conditions (5.16)-(5.18).

Theorem 5.3. *Let* $\{\boldsymbol{U}_k,\boldsymbol{V}_k,S_k\}$ *be the sequence of discrete spaces generated by Algorithm 4.1, then their corresponding discrete solutions* $\{(\boldsymbol{u}_k^*,\boldsymbol{y}_k^*,\boldsymbol{p}_k^*)\}$ *converges to the solution to the problem* (5.16)-(5.18) *in the following sense:*

$$\|\boldsymbol{u}_k^*-\boldsymbol{u}_\infty^*\|_{0,\Omega_c}\to 0,\quad\|\boldsymbol{y}_k^*-\boldsymbol{y}_\infty^*\|_{\boldsymbol{H}(\mathbf{curl})}\to 0,\quad\|\boldsymbol{p}_k^*-\boldsymbol{p}_\infty^*\|_{\boldsymbol{H}(\mathbf{curl})}\to 0\quad\text{as } k\to\infty.\tag{5.34}$$

Proof. The first convergence is just the conclusion of Theorem 5.2, while the second one follows from Lemma 5.3. It remains to prove the third result. We argue in a manner similar to that in the proof of Lemma 5.3 by introducing a discrete auxiliary problem: find $(\widetilde{\boldsymbol{p}}_k, \widetilde{\psi}_k) \in \boldsymbol{V}_k \times S_k$ such that

$$\begin{cases} (\nu \boldsymbol{\nabla} \times \widetilde{\boldsymbol{p}}_k, \boldsymbol{\nabla} \times \boldsymbol{v}) + (\boldsymbol{v}, \boldsymbol{\nabla}\widetilde{\psi}_k) = (\boldsymbol{\nabla} \times \boldsymbol{y}^*_\infty - \boldsymbol{\nabla} \times \boldsymbol{y}_d, \boldsymbol{\nabla} \times \boldsymbol{v})_{\Omega_c} \quad \forall\, \boldsymbol{v} \in \boldsymbol{V}_k, \\ (\widetilde{\boldsymbol{p}}_k, \boldsymbol{\nabla} q) = 0 \quad \forall\, q \in S_k\,. \end{cases} \tag{5.35}$$

It is not difficult to observe that this problem is a finite element approximation of the mixed formulation (5.17), so by the Babuska-Brezzi theory and the fact that $\psi^*_\infty = 0$ we have the error estimate

$$\|\boldsymbol{p}^*_\infty - \widetilde{\boldsymbol{p}}_k\|_{\boldsymbol{H}(\mathbf{curl})} \le C\big(\inf_{\boldsymbol{v}\in\boldsymbol{V}_k} \|\boldsymbol{p}^*_\infty - \boldsymbol{v}\|_{\boldsymbol{H}(\mathbf{curl})} + \inf_{q\in S_k} |\psi^*_\infty - q|_1\big) \le C \inf_{\boldsymbol{v}\in\boldsymbol{V}_k} \|\boldsymbol{p}^*_\infty - \boldsymbol{v}\|_{\boldsymbol{H}(\mathbf{curl})}\,, \tag{5.36}$$

and the stability estimate by a substraction of (2.14) from (5.35)

$$\|\widetilde{\boldsymbol{p}}_k - \boldsymbol{p}^*_k\|_{\boldsymbol{H}(\mathbf{curl})} \le C\|\boldsymbol{\nabla} \times (\boldsymbol{y}^*_\infty - \boldsymbol{y}^*_k)\|_0\,. \tag{5.37}$$

Now the third convergence in (5.34) follows from (5.36)-(5.37), the second convergence in (5.34) and the density of $\boldsymbol{V}_\infty$. □

6 Convergence of adaptively generated solutions

This section aims at the ultimate goal of this paper: the sequence $\{\boldsymbol{u}^*_k\}$ generated by the adaptive Algorithm 4.1 converges strongly in $\boldsymbol{L}^2(\Omega_c)$ to the minimizer $\boldsymbol{u}^*$ of problem (1.2) and (2.1). In view of Theorems 5.2 and 5.3, this is realised if $(\boldsymbol{u}^*_\infty, \boldsymbol{y}^*_\infty, \boldsymbol{p}^*_\infty)$ is shown to be the same as $(\boldsymbol{u}^*, \boldsymbol{y}^*, \boldsymbol{p}^*)$, the solution to the optimality conditions (2.4)-(2.6). For this purpose, the error estimator introduced in section 3 and the marking requirement (4.2) in Algorithm 4.1 are involved in the relevant arguments, whereas the analysis in the previous section does not depend on these two points. We first show the following fact on the maximal error indicator in the set of marked elements.

Lemma 6.1. *Let $\{\mathcal{T}_k, \boldsymbol{U}_k \times \boldsymbol{V}_k, (\boldsymbol{u}^*_k, \boldsymbol{y}^*_k, \boldsymbol{p}^*_k)\}$ be the sequence of meshes, finite element spaces and discrete solutions generated by Algorithm 4.1 and $\mathcal{M}_k$ be the set of marked elements given by (4.2). Then there holds*

$$\lim_{k\to\infty} \max_{T\in\mathcal{M}_k} \eta_k(\boldsymbol{u}^*_k, \boldsymbol{y}^*_k, \boldsymbol{p}^*_k, T) = 0. \tag{6.1}$$

Proof. Let $\widetilde{T}_k$ be the element with the largest error indicator among $\mathcal{M}_k$. Using $\widetilde{T}_k \in \mathcal{T}^0_k$, the local quasi-uniformity and (4.1), we derive

$$|\omega_{\widetilde{T}_k}| \le C|\widetilde{T}_k| \le C\|h_k\|^3_{\infty,\Omega^0_k} \to 0 \quad \text{as } k \to \infty. \tag{6.2}$$

But by Lemma 3.4 and the triangle inequality, we have

$$\eta_{k,1}(\boldsymbol{y}^*_k, \boldsymbol{u}^*_k, \widetilde{T}_k) \le C(\|\boldsymbol{y}^*_k - \boldsymbol{y}^*_\infty\|_{\boldsymbol{H}(\mathbf{curl})} + \|\boldsymbol{y}^*_\infty\|_{\boldsymbol{H}(\mathbf{curl}),\omega_{\widetilde{T}_k}} + \|\boldsymbol{u}^*_k - \boldsymbol{u}^*_\infty\|_{0,\Omega_c} + \|\boldsymbol{u}^*_\infty\|_{0,\omega_{\widetilde{T}_k}})\,.$$

Now the right-hand side goes to zero by means of the results from Theorems 5.2 and 5.3, (6.2) and the absolute continuity of $\|\cdot\|_{\boldsymbol{H}(\mathbf{curl})}$ and $\|\cdot\|_0$ with respect to the Lebesgue measure. Similar arguments apply to the other two terms $\eta_{k,2}$ and $\eta_{k,3}$. This completes the proof of the desired vanishing limit. □

Our main convergence will be conducted in a series of lemmas. First we show that the limiting minimizer $\boldsymbol{u}^*_\infty$ is in $\boldsymbol{U}$ (Lemma 6.2), then the residuals with respect to adaptive solutions $\{(\boldsymbol{u}^*_k, \boldsymbol{y}^*_k, \boldsymbol{p}^*_k)\}$ are shown to vanish in the limit (Lemma 6.4), by which and Theorem 5.3 we will find that $(\boldsymbol{u}^*_\infty, \boldsymbol{y}^*_\infty, \boldsymbol{p}^*_\infty)$ satisfy problems (2.4)-(2.5) with $\boldsymbol{u}^*_\infty$ and $\boldsymbol{y}^*_\infty$ in the right-hand side respectively (Lemma 6.5). The desired result is then obtained by proving $\boldsymbol{u}^*_\infty = \boldsymbol{u}^*$ (Lemma 6.6).

Lemma 6.2. *The minimizer* $\boldsymbol{u}_\infty^*$ *to the problem* (5.10)-(5.11) *also belongs to* $\boldsymbol{U}$*.*

Proof. By the definition of $\boldsymbol{U}$, it only needs to show $(\boldsymbol{u}_\infty^*, \boldsymbol{\nabla} q)_{\Omega_c} = 0$ for any $q \in H^1(\Omega_c)$. Since $\boldsymbol{u}_k^* \in \boldsymbol{U}_k$ we utilize the usual nodal interpolation operator $\widetilde{I}_k^c$ [9] and the Scott-Zhang operator I_k^c associated with S_k^c to deduce for any $q \in C^\infty(\overline{\Omega}_c)$ and $k > l$ that

$$\begin{aligned}
(\boldsymbol{u}_k^*, \boldsymbol{\nabla} q)_{\Omega_c} &= (\boldsymbol{u}_k^*, \boldsymbol{\nabla}(q - \widetilde{I}_k^c q))_{\Omega_c} = (\boldsymbol{u}_k^*, \boldsymbol{\nabla}((q - \widetilde{I}_k^c q) - I_k^c(q - \widetilde{I}_k^c q)))_{\Omega_c} \\
&\leq C \sum_{T \in \mathcal{T}_k^c} \eta_{k,3}(\boldsymbol{u}_k^*, T) \|q - \widetilde{I}_k^c q\|_{1,D_T} \\
&= C\Big(\sum_{T \in \mathcal{T}_k^c \setminus \mathcal{T}_l^+} \eta_{k,3}(\boldsymbol{u}_k^*, T) \|q - \widetilde{I}_k^c q\|_{1,D_T} + \sum_{T \in \mathcal{T}_k^c \cap \mathcal{T}_l^+} \eta_{k,3}(\boldsymbol{u}_k^*, T) \|q - \widetilde{I}_k^c q\|_{1,D_T} \Big) \\
&\leq C\Big(\big(\sum_{T \in \mathcal{T}_k^c \setminus \mathcal{T}_l^+} \eta_{k,3}^2(\boldsymbol{u}_k^*, T) \big)^{1/2} \|q - \widetilde{I}_k^c q\|_{1,\Omega_l^0} + \big(\sum_{T \in \mathcal{T}_k^c \cap \mathcal{T}_l^+} \eta_{k,3}^2(\boldsymbol{u}_k^*, T) \big)^{1/2} \|q - \widetilde{I}_k^c q\|_{1,\Omega_l^+} \Big),
\end{aligned}$$

where in the third inequality we have used the elementwise integration by parts and error estimates for I_k^c [20]. Noting (3.37) in Lemma 3.4, the uniform boundedness of $\{\boldsymbol{u}_k^*\}$ (cf. (5.23)), the error estimate for $\widetilde{I}_k^c$ [9] and the fact that $J_{F,3} = 0$ on $F \in \mathcal{F}_k(\Omega) \setminus \mathcal{F}_k(\overline{\Omega}_c)$, we proceed to see

$$(\boldsymbol{u}_k^*, \boldsymbol{\nabla} q)_{\Omega_c} \leq C_1 \|h_l\|_{\infty,\Omega_l^0} \|q\|_2 + C_2 \big(\sum_{T \in \mathcal{T}_l^+} \eta_{k,3}^2(\boldsymbol{u}_k^*, T) \big)^{1/2} \|q\|_2.$$

Thanks to (4.1), we know for sufficiently large l that $C_1 \|h_l\|_{\infty,\Omega_l^0} \|q\|_2 < \varepsilon/2$ for any given $\varepsilon > 0$. On the other hand, since $\mathcal{T}_l^+ \subset \mathcal{T}_k^+ \subset \mathcal{T}_k$ for $k > l$ the marking property (4.2) implies that

$$\big(\sum_{T \in \mathcal{T}_l^+} \eta_{k,3}^2(\boldsymbol{u}_k^*, T) \big)^{1/2} \leq \sqrt{|\mathcal{T}_l^+|} \max_{T \in \mathcal{T}_l^+} \eta_{k,3}(\boldsymbol{u}_k^*, T) \leq \sqrt{|\mathcal{T}_l^+|} \max_{T \in \mathcal{M}_k} \eta_k(\boldsymbol{y}_k^*, \boldsymbol{u}_k^*, \boldsymbol{p}_k^*, T).$$

Using Lemma 6.1 we may choose some $K > l$ for a fixed l such that there holds when $k \geq K$,

$$C_2 \big(\sum_{T \in \mathcal{T}_l^+} \eta_{k,3}^2(\boldsymbol{u}_k^*, T) \big)^{1/2} \|q\|_2 < \varepsilon/2.$$

Thus the density of $C^\infty(\overline{\Omega}_c)$ in $H^1(\Omega_c)$ gives

$$\lim_{k \to \infty} (\boldsymbol{u}_k^*, \boldsymbol{\nabla} q) = 0 \quad \forall\, q \in H^1(\Omega_c),$$

which, together with the convergence (5.33) in Theorem 5.2 and the elementary equality

$$(\boldsymbol{u}_\infty^*, \boldsymbol{\nabla} q)_{\Omega_c} = (\boldsymbol{u}_\infty^* - \boldsymbol{u}_k^*, \boldsymbol{\nabla} q)_{\Omega_c} + (\boldsymbol{u}_k^*, \boldsymbol{\nabla} q)_{\Omega_c},$$

leads to the desired claim. □

One may see from the proof of Lemma 6.2 that the key point lies in a density argument for the limiting behaviour of $\{\boldsymbol{u}_k^*\}$ projected on $\boldsymbol{\nabla} H^1(\Omega_c)$. And $\eta_{k,3}$ was split based on two parts Ω_l^0 and Ω_l^+ of $\mathcal{T}_k$, then local approximation properties and uniform convergence (4.1) were applied to the former while the marking property (4.2) was used for the latter. This idea will be also employed to verify the limiting triplet $(\boldsymbol{u}_\infty^*, \boldsymbol{y}_\infty^*, \boldsymbol{p}_\infty^*)$ with respect to the continuous system (2.4)-(2.5). We now define two residuals:

$$\langle \mathcal{R}_1(\boldsymbol{y}_k^*, \boldsymbol{u}_k^*), \boldsymbol{v} \rangle := (\boldsymbol{u}_k^*, \boldsymbol{v})_{\Omega_c} - (\nu \boldsymbol{\nabla} \times \boldsymbol{y}_k^*, \boldsymbol{\nabla} \times \boldsymbol{v}) \quad \forall\, \boldsymbol{v} \in \boldsymbol{H}_0(\mathbf{curl}; \Omega),$$

$$\langle \mathcal{R}_2(\boldsymbol{p}_k^*, \boldsymbol{y}_k^*), \boldsymbol{v}\rangle := (\boldsymbol{\nabla}\times \boldsymbol{y}_k^* - \boldsymbol{\nabla}\times \boldsymbol{y}_d, \boldsymbol{v}) - (\nu \boldsymbol{\nabla}\times \boldsymbol{p}_k^*, \boldsymbol{\nabla}\times \boldsymbol{v}) \quad \forall\, \boldsymbol{v}\in \boldsymbol{H}_0(\mathbf{curl};\Omega).$$

As ϕ_k^* and ψ_k^* are both zero, it is clear from the first equations of (2.13) and (2.14) respectively that

$$\langle \mathcal{R}_1(\boldsymbol{y}_k^*, \boldsymbol{u}_k^*), \boldsymbol{v}\rangle = \langle \mathcal{R}_2(\boldsymbol{p}_k^*, \boldsymbol{y}_k^*), \boldsymbol{v}\rangle = 0 \quad \forall\, \boldsymbol{v}\in \boldsymbol{V}_k. \tag{6.3}$$

To control the above two residuals, we shall use the following local regular decomposition [19].

Lemma 6.3. *There exists a quasi-interpolation operator $\boldsymbol{\Pi}_k^s : \boldsymbol{H}_0(\mathbf{curl};\Omega)\to \boldsymbol{V}_k$ such that for every $\boldsymbol{v}\in \boldsymbol{H}_0(\mathbf{curl};\Omega)$ there exist $\boldsymbol{z}\in \boldsymbol{H}_0^1(\Omega)$ and $\varphi\in H_0^1(\Omega)$ satisfying*

$$\boldsymbol{v} - \boldsymbol{\Pi}_k^s \boldsymbol{v} = \boldsymbol{z} + \boldsymbol{\nabla}\varphi\,, \tag{6.4}$$

with the stability estimates

$$h_T^{-1}\|\boldsymbol{z}\|_{0,T} + |\boldsymbol{z}|_{1,T} \le C\|\boldsymbol{\nabla}\times \boldsymbol{v}\|_{0,\widetilde{D}_T}, \tag{6.5}$$

$$h_T^{-1}\|\varphi\|_{0,T} + |\varphi|_{1,T} \le C\|\boldsymbol{v}\|_{0,\widetilde{D}_T} \tag{6.6}$$

where constant C depends only on the shape of the elements in the enlarged element patch $\widetilde{D}_T := \cup\{T'\in\mathcal{T}_k \mid T'\cap D_T \neq \emptyset\}$, not on the global shape of domain Ω or the size of $\widetilde{D}_T$.

As some elements in $\widetilde{D}_T$ for $T\in\mathcal{T}_k\setminus\mathcal{T}_l^+$ may not be in Ω_l^0 for $l<k$, we can not directly use (4.1) as in the proof of Lemma 6.2 when estimating $\mathcal{R}_1$ and $\mathcal{R}_2$. This difficulty motivates us to define a buffer layer of elements between $\mathcal{T}_l$ and $\mathcal{T}_k$ for $k>l$,

$$\mathcal{T}_{k,l}^b := \{T\in\mathcal{T}_k\setminus\mathcal{T}_l^+ \mid T\cap T'\neq\emptyset,\ \forall\, T'\in\mathcal{T}_l^+\}.$$

We know from $\mathcal{T}_l^+\subset\mathcal{T}_k^+\subset\mathcal{T}_k$ and the uniform shape-regularity of $\{\mathcal{T}_k\}$ that

$$|\mathcal{T}_{k,l}^b| \le C|\mathcal{T}_l^+| \tag{6.7}$$

with constant C depending only on the initial mesh $\mathcal{T}_0$, and $\widetilde{D}_T\subset\Omega_l^0$ for any $T\in\mathcal{T}_k\setminus(\mathcal{T}_l^+\cup\mathcal{T}_{k,l}^b)$.

Lemma 6.4. *The sequence $\{(\boldsymbol{u}_k^*, \boldsymbol{y}_k^*, \boldsymbol{p}_k^*)\}$ produced by Algorithm 4.1 satisfies for any $\boldsymbol{v}\in \boldsymbol{H}_0(\mathbf{curl};\Omega)$ and $q\in H_0^1(\Omega)$ that*

$$\lim_{k\to\infty}\langle \mathcal{R}_1(\boldsymbol{y}_k^*, \boldsymbol{u}_k^*), \boldsymbol{v}\rangle = 0, \quad \lim_{k\to\infty}(\boldsymbol{y}_k^*, \boldsymbol{\nabla} q) = 0, \tag{6.8}$$

$$\lim_{k\to\infty}\langle \mathcal{R}_2(\boldsymbol{p}_k^*, \boldsymbol{y}_k^*), \boldsymbol{v}\rangle = 0, \quad \lim_{k\to\infty}(\boldsymbol{p}_k^*, \boldsymbol{\nabla} q) = 0. \tag{6.9}$$

Proof. We only focus on (6.8) as the other two can be derived in a similar manner. Invoking the canonical edge interpolation operator $\widetilde{\boldsymbol{\Pi}}_k$ [16] associated with $\boldsymbol{V}_k$ and using (6.3), we get

$$\langle \mathcal{R}_1(\boldsymbol{y}_k^*, \boldsymbol{u}_k^*), \boldsymbol{v}\rangle = \langle \mathcal{R}_1(\boldsymbol{y}_k^*, \boldsymbol{u}_k^*), \boldsymbol{v} - \widetilde{\boldsymbol{\Pi}}_k \boldsymbol{v}\rangle \quad \forall\, \boldsymbol{v}\in \boldsymbol{C}_0^\infty(\Omega).$$

Since $\boldsymbol{w} = \boldsymbol{v} - \widetilde{\boldsymbol{\Pi}}_k\boldsymbol{v}\in \boldsymbol{H}_0(\mathbf{curl};\Omega)$, it can be further split by Lemma 6.3 as $\boldsymbol{w} - \boldsymbol{\Pi}_k^s\boldsymbol{w} = \boldsymbol{z} + \boldsymbol{\nabla}\varphi$ with $\boldsymbol{z}\in \boldsymbol{H}_0^1(\Omega)$ and $\varphi\in H_0^1(\Omega)$. With the help of (6.3) again, $\langle \mathcal{R}_1(\boldsymbol{y}_k^*, \boldsymbol{u}_k^*), \boldsymbol{w}\rangle = \langle \mathcal{R}_1(\boldsymbol{y}_k^*, \boldsymbol{u}_k^*), \boldsymbol{w} - \boldsymbol{\Pi}_k^s\boldsymbol{w}\rangle = \langle \mathcal{R}_1(\boldsymbol{y}_k^*, \boldsymbol{u}_k^*), \boldsymbol{z} + \boldsymbol{\nabla}\varphi\rangle$. Then an elementwise integration by parts, the trace theorem and the estimates (6.5)-(6.6) imply that

$$\begin{aligned}
\langle \mathcal{R}_1(\boldsymbol{y}_k^*, \boldsymbol{u}_k^*), \boldsymbol{z}\rangle &= \sum_{T\in\mathcal{T}_k}(R_{T,1}, \boldsymbol{z})_T - \sum_{F\in\mathcal{F}_k(\Omega)}(J_{F,1}, \boldsymbol{z})_F \\
&\le \sum_{T\in\mathcal{T}_k} h_T\|R_{T,1}\|_{0,T} h_T^{-1}\|\boldsymbol{z}\|_{0,T} + \sum_{F\in\mathcal{F}_k(\Omega)} h_F^{1/2}\|J_{F,1}\|_{0,F} h_F^{-1/2}\|\boldsymbol{z}\|_{0,F} \\
&\le C\sum_{T\in\mathcal{T}_k}\big(h_T^2\|R_{T,1}\|_{0,T}^2 + \sum_{F\subset\partial T\cap\Omega} h_F\|J_{F,1}\|_{0,F}^2\big)^{1/2}\big(h_T^{-1}\|\boldsymbol{z}\|_{0,T} + |\boldsymbol{z}|_{1,T}\big) \\
&\le C\sum_{T\in\mathcal{T}_k}\big(h_T^2\|R_{T,1}\|_{0,T}^2 + \sum_{F\subset\partial T\cap\Omega} h_F\|J_{F,1}\|_{0,F}^2\big)^{1/2}\|\boldsymbol{\nabla}\times(\boldsymbol{v} - \widetilde{\boldsymbol{\Pi}}_k\boldsymbol{v})\|_{0,\widetilde{D}_T},
\end{aligned}$$

$$\begin{aligned}\langle \mathcal{R}_1(\boldsymbol{y}_k^*, \boldsymbol{u}_k^*), \boldsymbol{\nabla}\varphi\rangle &= \sum_{F\in\mathcal{F}_k(\Omega)} (J_{F,3}, \varphi)_F \le \sum_{F\in\mathcal{F}_k(\Omega)} h_F^{1/2}\|J_{F,3}\|_{0,F} h_F^{-1/2}\|\varphi\|_{0,F}\\ &\le C\sum_{T\in\mathcal{T}_k}\Big(\sum_{F\subset\partial T\cap\Omega} h_F\|J_{F,3}\|_{0,F}^2\Big)^{1/2}\big(h_T^{-1}\|\varphi\|_{0,T} + |\varphi|_{1,T}\big)\\ &\le C\sum_{T\in\mathcal{T}_k}\Big(\sum_{F\subset\partial T\cap\Omega} h_F\|J_{F,3}\|_{0,F}^2\Big)^{1/2}\|\boldsymbol{v} - \widetilde{\boldsymbol{\Pi}}_k\boldsymbol{v}\|_{0,\widetilde{D}_T}.\end{aligned}$$

Splitting $\mathcal{T}_k$ into $\mathcal{T}_l^+ \cup \mathcal{T}_{k,l}^b$ and $\mathcal{T}_k \setminus (\mathcal{T}_l^+ \cup \mathcal{T}_{k,l}^b)$ for $k > l$, and noting that $\bigcup_{T\in\mathcal{T}_k\setminus(\mathcal{T}_l^+\cup\mathcal{T}_{k,l}^b)} \widetilde{D}_T \subseteq \Omega_l^0$, we can further proceed to derive

$$\begin{aligned}\langle \mathcal{R}_1(\boldsymbol{y}_k^*, \boldsymbol{u}_k^*), \boldsymbol{v}\rangle &\le C\sum_{T\in\mathcal{T}_k} \widetilde{\eta}_{k,1}(\boldsymbol{y}_k^*, \boldsymbol{u}_k^*, T)\|\boldsymbol{v} - \widetilde{\boldsymbol{\Pi}}_k\boldsymbol{v}\|_{\boldsymbol{H}(\mathbf{curl}),\widetilde{D}_T}\\ &\le C\big(\widetilde{\eta}_{k,1}(\boldsymbol{y}_k^*, \boldsymbol{u}_k^*, \mathcal{T}_k\setminus(\mathcal{T}_l^+\cup\mathcal{T}_{k,l}^b))\|\boldsymbol{v} - \widetilde{\boldsymbol{\Pi}}_k\boldsymbol{v}\|_{\boldsymbol{H}(\mathbf{curl}),\Omega_l^0}\\ &\qquad + \widetilde{\eta}_{k,1}(\boldsymbol{y}_k^*, \boldsymbol{u}_k^*, \mathcal{T}_l^+\cup\mathcal{T}_{k,l}^b)\|\boldsymbol{v} - \widetilde{\boldsymbol{\Pi}}_k\boldsymbol{v}\|_{\boldsymbol{H}(\mathbf{curl})}\big),\end{aligned}$$

where we have written $\widetilde{\eta}_{k,1}^2(\boldsymbol{y}_k^*, \boldsymbol{u}_k^*, \mathcal{M}) := \sum_{T\in\mathcal{M}} \widetilde{\eta}_{k,1}^2(\boldsymbol{y}_k^*, \boldsymbol{u}_k^*, T)$ for $\mathcal{M} \subset \mathcal{T}$ with

$$\widetilde{\eta}_{k,1}^2(\boldsymbol{y}_k^*, \boldsymbol{u}_k^*, T) := h_T^2\|R_{T,1}\|_{0,T}^2 + \sum_{F\subset\partial T\cap\Omega}(h_F\|J_{F,1}\|_{0,F}^2 + h_F\|J_{F,3}\|_{0,F}^2)\,.$$

In view of (2.16) and (5.23), the sequences of discrete minimizers $\{\boldsymbol{u}_k^*\}$ and related states $\{\boldsymbol{y}_k^*\}$ are uniformly bounded in $\boldsymbol{L}^2(\Omega_c)$ and $\boldsymbol{H}_0(\mathbf{curl};\Omega)$. Then by virtue of the stability (3.35) in Lemma 3.4 and the error estimates for $\widetilde{\boldsymbol{\Pi}}_k$ [16], we can deduce

$$\langle \mathcal{R}_1(\boldsymbol{y}_k^*, \boldsymbol{u}_k^*), \boldsymbol{v}\rangle \le C_3\|h_l\|_{\Omega_l^0}\|\boldsymbol{v}\|_2 + C_4\widetilde{\eta}_{k,1}(\boldsymbol{y}_k^*, \boldsymbol{u}_k^*, \mathcal{T}_l^+\cup\mathcal{T}_{k,l}^b)\|\boldsymbol{v}\|_2.$$

As it was done earlier, property (4.1) allows the first term to be small enough for sufficiently large l. Noting $\widetilde{\eta}_{k,1}(\boldsymbol{y}_k^*, \boldsymbol{u}_k^*, T) \le \eta_k(\boldsymbol{u}_k^*, \boldsymbol{y}_k^*, \boldsymbol{p}_k^*, T)$ for any $T \in \mathcal{T}_k$ and using (4.2) and (6.7), we can obtain

$$\Big(\sum_{T\in\mathcal{T}_l^+} \widetilde{\eta}_{k,1}^2(\boldsymbol{y}_k^*, \boldsymbol{u}_k^*, T) + \sum_{T\in\mathcal{T}_{k,l}^b} \widetilde{\eta}_{k,1}^2(\boldsymbol{y}_k^*, \boldsymbol{u}_k^*, T)\Big)^{1/2} \le C\sqrt{|\mathcal{T}_l^+|}\max_{T\in\mathcal{M}_k}\eta_k(\boldsymbol{u}_k^*, \boldsymbol{y}_k^*, \boldsymbol{p}_k^*, T),$$

which, aided by (6.1) in Lemma 6.1, indicates that the second term is also small for large k after fixing l. A combination of these two facts yields

$$\lim_{k\to\infty}\langle \mathcal{R}_1(\boldsymbol{y}_k^*, \boldsymbol{u}_k^*), \boldsymbol{v}\rangle = 0 \quad \boldsymbol{v} \in \boldsymbol{C}_0^\infty(\Omega)\,.$$

This, with the density of $\boldsymbol{C}_0^\infty(\Omega)$ in $\boldsymbol{H}_0(\mathbf{curl};\Omega)$, implies the first convergence in (6.8). For the second convergence, we follow similar arguments to those for Lemma 6.2 with $\widetilde{I}_k$, the nodal Lagrange interpolation [9], and the Scott-Zhang operator I_k over S_k replacing $\widetilde{I}_k^c$ and I_k^c respectively. □

Remark 6.1. As can be seen from proofs of Lemmas 6.2 and 6.4, the density argument split all elements over $\mathcal{T}_k$ into two parts by the marking. For the error estimator over unrefined elements after a fixed iteration $l < k$, the marking property (4.2) and Lemma 6.1 indeed guarantee

$$\lim_{k\to\infty}\eta_k(\boldsymbol{u}_k^*, \boldsymbol{y}_k^*, \boldsymbol{p}_k^*, \mathcal{T}_l^+) = 0. \tag{6.10}$$

Lemma 6.5. *The solutions* $(\boldsymbol{u}_\infty^*, \boldsymbol{y}_\infty^*, \boldsymbol{p}_\infty^*)$ *to problems* (5.16)-(5.17) *solve the system*

$$\begin{cases} (\nu \nabla \times \boldsymbol{y}_\infty^*, \nabla \times \boldsymbol{v}) + (\boldsymbol{v}, \nabla \phi_\infty^*) = (\boldsymbol{u}_\infty^*, \boldsymbol{v})_{\Omega_c} \quad \forall\, \boldsymbol{v} \in \boldsymbol{H}_0(\mathbf{curl}; \Omega), \\ (\boldsymbol{y}_\infty^*, \nabla q) = 0 \quad \forall\, q \in H_0^1(\Omega). \end{cases} \tag{6.11}$$

$$\begin{cases} (\nu \nabla \times \boldsymbol{p}_\infty^*, \nabla \times \boldsymbol{v}) + (\boldsymbol{v}, \nabla \psi_\infty^*) = (\nabla \times \boldsymbol{y}_\infty^* - \nabla \times \boldsymbol{y}_d, \nabla \times \boldsymbol{v}) \quad \forall\, \boldsymbol{v} \in \boldsymbol{H}_0(\mathbf{curl}; \Omega), \\ (\boldsymbol{p}_\infty^*, \nabla q) = 0 \quad \forall\, q \in H_0^1(\Omega). \end{cases} \tag{6.12}$$

Proof. Let $\{(\boldsymbol{u}_k^*, \boldsymbol{y}_k^*, \boldsymbol{p}_k^*)\}$ be the convergent sequence generated by Algorithm 4.1. Noting the fact that $\phi_\infty^* = 0$, we then have for any $\boldsymbol{v} \in \boldsymbol{H}_0(\mathbf{curl}; \Omega)$ and $q \in H_0^1(\Omega)$,

$$\begin{aligned} &|(\nu \nabla \times \boldsymbol{y}_\infty^*, \nabla \times \boldsymbol{v}) + (\boldsymbol{v}, \nabla \phi_\infty^*) - (\boldsymbol{u}_\infty^*, \boldsymbol{v})_{\Omega_c}| \\ &= |(\nu \nabla \times (\boldsymbol{y}_\infty^* - \boldsymbol{y}_k^*), \nabla \times \boldsymbol{v}) - (\boldsymbol{u}_\infty^* - \boldsymbol{u}_k^*, \boldsymbol{v})_{\Omega_c} - \langle \mathcal{R}_1(\boldsymbol{y}_k^*, \boldsymbol{u}_k^*), \boldsymbol{v} \rangle| \\ &\le \nu_2 \|\nabla \times (\boldsymbol{y}_\infty^* - \boldsymbol{y}_k^*)\|_0 \|\nabla \times \boldsymbol{v}\|_0 + \|\boldsymbol{u}_\infty^* - \boldsymbol{u}_k^*\|_{0,\Omega_c} \|\boldsymbol{v}\|_0 + |\langle \mathcal{R}_1(\boldsymbol{y}_k^*, \boldsymbol{u}_k^*), \boldsymbol{v} \rangle|, \end{aligned} \tag{6.13}$$

$$|(\boldsymbol{y}_\infty^*, \nabla q)| \le \|\boldsymbol{y}_\infty^* - \boldsymbol{y}_k^*\|_0 |q|_1 + |(\boldsymbol{y}_k^*, \nabla q)|. \tag{6.14}$$

So the problem (6.11) holds true upon observing that every term in the right-hand sides of (6.13) and (6.14) tends to zero due to (5.34) and (6.8). To see (6.12), we may argue similarly that

$$\begin{aligned} &|(\nu \nabla \times \boldsymbol{p}_\infty^*, \nabla \times \boldsymbol{v}) + (\boldsymbol{v}, \nabla \psi_\infty^*) - (\nabla \times \boldsymbol{y}_\infty^* - \nabla \times \boldsymbol{y}_d, \nabla \times \boldsymbol{v})| \\ &= |(\nu \nabla \times (\boldsymbol{p}_\infty^* - \boldsymbol{p}_k^*), \nabla \times \boldsymbol{v}) - (\nabla \times (\boldsymbol{y}_\infty^* - \boldsymbol{y}_k^*), \nabla \times \boldsymbol{v}) - \langle \mathcal{R}_2(\boldsymbol{p}_k^*, \boldsymbol{y}_k^*), \boldsymbol{v} \rangle \\ &\le \nu_2 \|\nabla \times (\boldsymbol{p}_\infty^* - \boldsymbol{p}_k^*)\|_0 \|\nabla \times \boldsymbol{v}\|_0 + \|\nabla \times (\boldsymbol{y}_\infty^* - \boldsymbol{y}_k^*)\|_0 \|\nabla \times \boldsymbol{v}\|_0 + |\langle \mathcal{R}_2(\boldsymbol{p}_k^*, \boldsymbol{y}_k^*), \boldsymbol{v} \rangle|, \\ &|(\boldsymbol{p}_\infty^*, \nabla q)| \le \|\boldsymbol{p}_\infty^* - \boldsymbol{p}_k^*\|_0 |q|_1 + |(\boldsymbol{p}_k^*, \nabla q)|. \end{aligned}$$

Now the desired conclusion comes from Theorem 5.3 and Lemma 6.4. □

Lemma 6.6. *For the solution* $(\boldsymbol{u}^*, \boldsymbol{y}^*, \boldsymbol{p}^*)$ *to the problem* (2.4)-(2.6) *and the solutions* $\boldsymbol{y}_\infty^*$ *and* $\boldsymbol{p}_\infty^*$ *to the problems* (6.11) *and* (6.12), *there hold that*

$$\boldsymbol{u}^* = \boldsymbol{u}_\infty^* \quad \textit{in } \boldsymbol{L}^2(\Omega_c); \quad \boldsymbol{y}^* = \boldsymbol{y}_\infty^* \quad \textit{and} \quad \boldsymbol{p}^* = \boldsymbol{p}_\infty^* \quad \textit{in } \boldsymbol{H}_0(\mathbf{curl}; \Omega). \tag{6.15}$$

Proof. Owing to Lemma 6.5 and the problems (2.4)-(2.5), we only need to prove the first equality. Subtracting (2.4) and (2.5) from (6.11) and (6.12) respectively and noting the facts that $\phi^* = \psi^* = \phi_\infty^* = \psi_\infty^* = 0$, we obtain for any $\boldsymbol{v} \in \boldsymbol{H}_0(\mathbf{curl}; \Omega)$ that

$$(\nu \nabla \times (\boldsymbol{y}^* - \boldsymbol{y}_\infty^*), \nabla \times \boldsymbol{v}) = (\boldsymbol{u}^* - \boldsymbol{u}_\infty^*, \boldsymbol{v})_{\Omega_c}, \tag{6.16}$$

$$(\nu \nabla \times (\boldsymbol{p}^* - \boldsymbol{p}_\infty^*), \nabla \times \boldsymbol{v}) = (\nabla \times (\boldsymbol{y}^* - \boldsymbol{y}_\infty^*), \nabla \times \boldsymbol{v}). \tag{6.17}$$

We know from Lemma 6.5 that both $\boldsymbol{y}_\infty^*$ and $\boldsymbol{p}_\infty^*$ belong to $\boldsymbol{X}$. Taking $\boldsymbol{v}$ to be $\boldsymbol{p}^* - \boldsymbol{p}_\infty^*$ and $\boldsymbol{y}^* - \boldsymbol{y}_\infty^*$ in (6.16) and (6.17) respectively, it further holds

$$\|\nabla \times (\boldsymbol{y}^* - \boldsymbol{y}_\infty^*)\|_0^2 = (\boldsymbol{u}^* - \boldsymbol{u}_\infty^*, \boldsymbol{p}^* - \boldsymbol{p}_\infty^*)_{\Omega_c},$$

which, along with $-\gamma^{-1} \boldsymbol{p}^*|_{\Omega_c} = \boldsymbol{u}^* + \nabla \xi^*$ and $-\gamma^{-1} \boldsymbol{p}_\infty^*|_{\Omega_c} = \boldsymbol{u}_\infty^* + \nabla \xi_\infty^*$ (cf. (2.19), (5.15)), implies

$$\|\nabla \times (\boldsymbol{y}^* - \boldsymbol{y}_\infty^*)\|_0^2 + \gamma \|\boldsymbol{u}^* - \boldsymbol{u}_\infty^*\|_{0,\Omega_c}^2 = \gamma (\boldsymbol{u}^* - \boldsymbol{u}_\infty^*, \nabla \xi_\infty^* - \nabla \xi^*)_{\Omega_c},$$

with $\xi^* \in H^1(\Omega_c)$ and $\xi_\infty^* \in S_\infty^c$. Since $\boldsymbol{u}^* \in \boldsymbol{U}$ and $\boldsymbol{u}_\infty^* \in \boldsymbol{U}$ (cf. Lemma 6.2), we get $\boldsymbol{u}^* = \boldsymbol{u}_\infty^*$ in $\boldsymbol{L}^2(\Omega_c)$. □

Now with the help of Theorem 5.3 and Lemma 6.6, we can conclude our major convergence results for Algorithm 4.1.

Theorem 6.1. *Let $(\boldsymbol{u}^*, \boldsymbol{y}^*, \boldsymbol{p}^*)$ be the solution to the problem (2.4)-(2.6). Then Algorithm 4.1 produces a sequence of discrete solutions $(\boldsymbol{u}_k^*, \boldsymbol{y}_k^*, \boldsymbol{p}_k^*)$ converging to $(\boldsymbol{u}^*, \boldsymbol{y}^*, \boldsymbol{p}^*)$ in the following sense:*

$$\lim_{k\to\infty} \|\boldsymbol{y}^* - \boldsymbol{y}_k^*\|_{\boldsymbol{H}(\mathbf{curl})} = 0, \quad \lim_{k\to\infty} \|\boldsymbol{p}^* - \boldsymbol{p}_k^*\|_{\boldsymbol{H}(\mathbf{curl})} = 0, \quad \lim_{k\to\infty} \|\boldsymbol{u}^* - \boldsymbol{u}_k^*\|_{0,\Omega_c} = 0. \tag{6.18}$$

With Theorems 3.2 and 6.1, we end this work with the convergence of our error estimator.

Theorem 6.2. *The sequence $\{\eta_k(\boldsymbol{u}_k^*, \boldsymbol{y}_k^*, \boldsymbol{p}_k^*)\}$ of the estimators generated by Algorithm 4.1 converges to zero.*

Proof. As in the proof of Lemma 6.4, we rewrite the estimator as two parts

$$\eta_k^2(\boldsymbol{u}_k^*, \boldsymbol{y}_k^*, \boldsymbol{p}_k^*) = \eta_k^2(\boldsymbol{u}_k^*, \boldsymbol{y}_k^*, \boldsymbol{p}_k^*, \mathcal{T}_k \setminus \mathcal{T}_l^+) + \eta_k^2(\boldsymbol{u}_k^*, \boldsymbol{y}_k^*, \boldsymbol{p}_k^*, \mathcal{T}_l^+) \tag{6.19}$$

for $k > l$. Summing up the local lower bound (3.25) over all elements in $\mathcal{T}_k \setminus \mathcal{T}_l^+$, we obtain

$$\eta_k^2(\boldsymbol{u}_k^*, \boldsymbol{y}_k^*, \boldsymbol{p}_k^*, \mathcal{T}_k \setminus \mathcal{T}_l^+) \le C(\|\boldsymbol{y}^* - \boldsymbol{y}_k^*\|^2_{\boldsymbol{H}(\mathbf{curl})} + \|\boldsymbol{p}^* - \boldsymbol{p}_k^*\|^2_{\boldsymbol{H}(\mathbf{curl})} + \|\boldsymbol{u}^* - \boldsymbol{u}_k^*\|^2_{0,\Omega_c} + \mathrm{III}),$$

where the term III is given by

$$\mathrm{III} := \sum_{\mathcal{T}_k \setminus \mathcal{T}_l^+} (\mathrm{osc}^2_{\mathcal{T}}(\boldsymbol{y}^*_{\mathcal{T}}, \boldsymbol{u}^*_{\mathcal{T}}, \omega_T) + \mathrm{osc}^2_{\mathcal{T}}(\boldsymbol{p}^*_{\mathcal{T}}, \omega_T) + \mathrm{osc}^2_{\mathcal{T}}(\boldsymbol{y}^*_{\mathcal{T}}, \partial T) + \mathrm{osc}^2_{\mathcal{T}}(\boldsymbol{p}^*_{\mathcal{T}}, \boldsymbol{y}^*_{\mathcal{T}}, \partial T)).$$

Noting $\bar{R}_{T,1}$ and $\bar{R}_{T,2}$ are the best L^2-projections onto a constant space and the fact $\boldsymbol{\nabla} \times \boldsymbol{\nabla} \times \boldsymbol{y}_k^* = \boldsymbol{\nabla} \times \boldsymbol{\nabla} \times \boldsymbol{p}_k^* = \mathbf{0}$, we have

$$h_T \|R_{T,1} - \bar{R}_{T,1}\|_{0,T} \le h_T \|\chi_c \boldsymbol{u}_k^*\|_{0,T} + \|\boldsymbol{\nabla}\nu\|_{\infty,T} h_T \|\boldsymbol{\nabla} \times \boldsymbol{y}_k^*\|_{0,T},$$

$$h_T \|R_{T,2} - \bar{R}_{T,2}\|_{0,T} \le h_T \|\boldsymbol{\nabla} \times \boldsymbol{y}_d\|_{0,T} + \|\boldsymbol{\nabla}\nu\|_{\infty,T} h_T \|\boldsymbol{\nabla} \times \boldsymbol{p}_k^*\|_{0,T}.$$

Letting $\overline{[\nu]}$ be the average of $[\nu]$ on F, the inverse estimate and the Poincaré inequality imply

$$h_F^{1/2} \|J_{F,1} - \bar{J}_{F,1}\|_{0,F} \le C \|[\nu] - \overline{[\nu]}\|_{\infty,F} \|\boldsymbol{\nabla} \times \boldsymbol{y}_k^*\|_{0,\omega_F} \le C h_F \|\boldsymbol{\nabla} \times \boldsymbol{y}_k^*\|_{0,\omega_F},$$

$$h_F^{1/2} \|J_{F,4} - \bar{J}_{F,4}\|_{0,F} \le C h_F \|\boldsymbol{\nabla} \times \boldsymbol{p}_k^*\|_{0,\omega_F}.$$

Taking (2.17) into account and noting that $\|\boldsymbol{u}_k^*\|_{0,\Omega_c}$ is uniformly bounded (cf. (5.23)) we may infer from the monotonicity of h_k and (4.1) that

$$\mathrm{III} \le C \max_{T \in \mathcal{T}_k \setminus \mathcal{T}_l^+} h_T^2 \le C \|h_l\|^2_{\infty,\Omega_l^0} \to 0 \quad \text{as } l \to \infty,$$

which, together with norm convergences in Theorem 6.1, makes the first term in the right-hand side of (6.19) smaller than any given positive number after a sufficiently large l is chosen. On the other hand, the convergence (6.10) in Remark 6.1 implies that the second term is also smaller than any given positive number for fixed l and sufficiently large k. □

7 Concluding remarks

We have established a residual-type reliable and efficient error estimator for edge element approximations of an optimal control problem governed by an $\boldsymbol{H}(\mathbf{curl})$ saddle-point system. Based on the estimator and a general but practical assumption on the marking, an adaptive algorithm is designed, which is proved to generate a null sequence of estimators and a sequence of discrete solutions strongly converging to the exact minimizer, the corresponding state and costate variables. In the analysis, we have specifically utilized convergence of discrete objective functionals to lift the weak convergence of discrete optimal controls to a strong one so that the discrete compactness property of edge elements is circumvented.

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