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Fourier spectral projection method and nonlinear convergence analysis for Navier–Stokes equations

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Abstract

In this paper, we investigate the convergence rate of the Fourier spectral projection methods for the periodic problem of n-dimensional Navier–Stokes equations. Based on some alternative formulations of the Navier–Stokes equations and the related projection methods, the error estimates are carried out by a global nonlinear error analysis. It simplifies the analysis, relaxes the restriction on the time step size, weakens the regularity requirements on the genuine solution, and leads to some improved convergence results. A new correction technique is proposed for improving the accuracy of the numerical pressure.

Keywords: Navier–Stokes equations; Projection method; Equivalent formulation; Convergence rate; Nonlinear error analysis; Numerical pressure correction

1. Introduction

The projection method was introduced by Chorin [4–6] and Téman [20] as an efficient algorithm for the numerical solution of Navier–Stokes equations, based on time splitting

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discretization which decouples the computations of the velocity U and the pressure P. At the first step, an intermediate velocity U_* is calculated using the momentum equation and ignoring the incompressible constraint. At the second step, U_* is projected into the space of divergence-free vector fields to get the next updates of U and P. It saves the computational cost and preserves the incompressibility of U physically. The projection method has been successfully applied to the numerical simulations of incompressible viscous flows, see, e.g., [2,3,13,14,16,21].

The convergence analysis of the projection method was first studied by Chorin [6] and Téman [20]. Some results on the convergence rate was obtained by Shen [18,19]. E and Liu [7] investigated the numerical boundary layers caused by the projection method for two-dimensional semi-periodic Navier–Stokes equations, and provided a technique to improve the accuracy of the numerical pressure. E and Liu [8] also applied the Godunov–Ryabenki analysis to the error estimate. Recently, E and Liu [9] studied the projection method combined with the MAC spatial discretization. For other related work, we refer to [1,17,22,23].

The purpose of this work is to study the convergence rate of the fully discrete projection method and to improve the accuracy of the numerical solution by the pressure correction for the n-dimensional periodic Navier–Stokes equations. The main idea is as follows. We first derive some alternative formulations of the Navier–Stokes equations and the related projection schemes. They enable us to derive the error estimates of numerical velocity and numerical pressure separately. Then, a global nonlinear analysis is utilized to estimate the convergence rates. In comparison with the most existing analysis, the global nonlinear analysis enables us to relax the restriction on τ, the step in time, and the requirement on the regularity of the genuine solution. Furthermore, we show that the numerical pressure of the second-order projection method can be corrected so that it achieves the same convergence rate as the numerical velocity. More precisely, let $\Omega = [-\pi, \pi]^n$ and N be the number of terms in the Fourier expansions of the numerical velocity u_N and the numerical pressure p_N. If $U \in L^\infty(0, T; H^{n/2+\delta}(\Omega) \cap H^r(\Omega)) \cap H^2(0, T; L^2(\Omega))$ for some $r \geqslant n/3$ and $\delta > 0$, and $\tau = O(N^{-n/3})$, then the error of u_N for the first-order projection method is of the order $O(\tau + N^{-r})$. It is interesting to note that p_N and ∇p_N have the errors of the same order as u_N, provided that U and P are a little more regular. Moreover, if $U \in L^\infty(0, T; H^{n/2+\delta}(\Omega) \cap H^r(\Omega)) \cap H^3(0, T; L^2(\Omega))$ for some $r \geqslant n/2$ and $\tau = \mathcal{O}(N^{-n/4})$, then the error of u_N of the second-order projection method by Kim and Moin [14] is of the order $O(\tau^2 + N^{-r})$, while the corrected numerical pressure $p_{N,c}$ has the error of the same order as u_N. Finally, for the second-order projection method based on the pressure increment formulation (see [2,21]), the errors of both u_N and p_N are of the same order $O(\tau^2 + N^{-r})$. In other words, the numerical pressure is corrected automatically. The results of this paper indicate that the projection method has more useful features than those pointed out before. In particular, the numerical pressure of the second-order projection method based on the pressure increment formulation appears more efficient than other projection methods.

The rest of the paper is organized as follows. In Section 2, we present the equivalent formulations for the n-dimensional periodic Navier–Stokes equations, and several Fourier spectral projection schemes. In Section 3, we discuss the second-order projection method

(Kim–Moin method). The final section is for the second-order projection method based on the pressure increment formulation.

2. Alternative formulations of projection methods

Let $x = (x_1, x_2, \dots, x_n)^T \in \mathcal{R}^n$ and $\Omega = [-\pi, \pi]^n$. Denote by $U = (u^{(1)}, u^{(2)}, \dots, u^{(n)})^T$, P, and $\nu > 0$ the velocity, the pressure, and the kinetic viscosity, respectively. U^0 and f are given functions with the period 2π in all spatial directions and $\nabla \cdot U^0 = 0$. The periodic problem of the Navier–Stokes equations is to find U and P with the period 2π such that

$$\begin{cases} \frac{\partial U}{\partial t} + (U \cdot \nabla)U + \nabla P - \nu \Delta U = f, & 0 < t \leqslant T, \\ \nabla \cdot U = 0, & 0 < t \leqslant T, \end{cases} \tag{2.1}$$

with $U = U^0$ at $t = 0$. For the uniqueness of P, we require that

$$\int_\Omega P\,dx = 0, \quad 0 \leqslant t \leqslant T. \tag{2.2}$$

We first derive an alternative formulation of (2.1). Taking the divergence on both sides of the first equation of (2.1), we obtain

$$\frac{\partial}{\partial t}(\nabla \cdot U) + \nabla \cdot \big((U \cdot \nabla)U\big) + \Delta P - \nu \Delta(\nabla \cdot U) = \nabla \cdot f. \tag{2.3}$$

This, with the fact that $\nabla \cdot U = 0$ gives

$$\Delta P + \nabla \cdot \big((U \cdot \nabla)U\big) = \nabla \cdot f, \quad 0 < t \leqslant T. \tag{2.4}$$

Conversely, if U and P satisfy the first equation of (2.1) and (2.4), then by (2.3),

$$\frac{\partial}{\partial t}(\nabla \cdot U) - \nu \Delta(\nabla \cdot U) = 0, \quad 0 < t \leqslant T.$$

Since $\nabla \cdot U^0 = 0$, the above has the unique solution $\nabla \cdot U \equiv 0$. Thus (2.1) is equivalent to

$$\begin{cases} \frac{\partial U}{\partial t} + (U \cdot \nabla)U + \nabla P - \nu \Delta U = f, & 0 < t \leqslant T, \\ \Delta P + \nabla \cdot ((U \cdot \nabla)U) = \nabla \cdot f, & 0 < t \leqslant T, \end{cases} \tag{2.5}$$

coupled with (2.2) and $U = U^0$ at $t = 0$.

Now, let M be any positive integer and $T = M\tau$. Denote by U^k, P^k, and f^k the values of U, P, and f at $t = k\tau$. u^k and p^k are the numerical solutions of U^k and P^k, u_*^k is the predicted value of u^k. The first-order projection method is to find u^k, u_*^k, and p^k such that $u^0 \in U^0$ and

$$\begin{cases} \frac{1}{\tau}(u_*^{k+1} - u^k) + (u^k \cdot \nabla)u^k - \nu \Delta u_*^{k+1} = f^{k+1}, & 0 \leqslant k \leqslant M-1, \\ \frac{1}{\tau}(u^{k+1} - u_*^{k+1}) + \nabla p^{k+1} = 0, & 0 \leqslant k \leqslant M-1, \\ \nabla \cdot u^k = 0, & 0 \leqslant k \leqslant M. \end{cases} \tag{2.6}$$

In addition, we require that

$$\int_\Omega p^k\,dx = 0, \quad 0 \leqslant k \leqslant M. \tag{2.7}$$

Taking the divergence on both sides of the second equation of (2.6) yields

$$\Delta p^{k+1} = \frac{1}{\tau}\nabla\cdot u_*^{k+1}. \tag{2.8}$$

Obviously,

$$u^{k+1} = u_*^{k+1} - \tau\nabla p^{k+1}. \tag{2.9}$$

We now derive an equivalent form of (2.6). Let

$$\delta_t v^k = \frac{1}{\tau}(v^{k+1} - v^k).$$

Using (2.9), we get from (2.6) that

$$\delta_t u^k + (u^k\cdot\nabla)u^k + (1-\nu\tau\Delta)\nabla p^{k+1} - \nu\Delta u^{k+1} = f^{k+1}. \tag{2.10}$$

Taking the divergence on both sides of (2.10), we obtain

$$(1-\nu\tau\Delta)\Delta p^{k+1} + \nabla\cdot\big((u^k\cdot\nabla)u^k\big) = \nabla\cdot f^{k+1}. \tag{2.11}$$

Conversely, for any u^k and p^k satisfying (2.10), we have

$$\delta_t(\nabla\cdot u^k) + \nabla\cdot\big((u^k\cdot\nabla)u^k\big) + (1-\nu\tau\Delta)\Delta p^{k+1} - \nu\Delta(\nabla\cdot u^{k+1}) = \nabla\cdot f^{k+1}.$$

If u^k and p^k also fulfill (2.11), then

$$\delta_t(\nabla\cdot u^k) - \nu\Delta(\nabla\cdot u^{k+1}) = 0.$$

By induction with $\nabla\cdot u^0 = 0$, we assert that $\nabla\cdot u^k \equiv 0$ for $0\leqslant k\leqslant M$. The previous statements imply that (2.6) is equivalent to the system (2.10), (2.11) with (2.7), and $u^0 = U^0$.

We now turn to the Kim–Moin method [14]. It is to find u^k, u_*^k, and $p^{k+1/2}$ such that $u^0 = U^0$, $\nabla\cdot u^M = 0$, and for $1\leqslant k\leqslant M-1$,

$$\begin{cases} \frac{1}{\tau}(u_*^{k+1} - u^k) + \frac{3}{2}(u^k\cdot\nabla)u^k - \frac{1}{2}(u^{k-1}\cdot\nabla)u^{k-1} - \frac{1}{2}\nu\Delta(u^k + u_*^{k+1}) \\ \quad = \frac{3}{2}f^k - \frac{1}{2}f^{k-1}, \\ \frac{1}{\tau}(u^{k+1} - u_*^{k+1}) + \nabla p^{k+1/2} = 0, \\ \nabla\cdot u^k = 0, \end{cases} \tag{2.12}$$

coupled with (2.7). By the first two equations of (2.12), we deduce that

$$\begin{aligned} &\delta_t u^k + \frac{3}{2}(u^k\cdot\nabla)u^k - \frac{1}{2}(u^{k-1}\cdot\nabla)u^{k-1} + \left(1-\frac{1}{2}\nu\tau\Delta\right)\nabla p^{k+1/2} \\ &\quad - \frac{1}{2}\nu\Delta(u^k + u^{k+1}) = \frac{3}{2}f^k - \frac{1}{2}f^{k-1}. \end{aligned} \tag{2.13}$$

Taking the divergence on both sides of (2.13) gives

$$\begin{aligned} &\left(1-\frac{1}{2}\nu\tau\Delta\right)\Delta p^{k+1/2} + \nabla\cdot\left(\frac{3}{2}(u^k\cdot\nabla)u^k - \frac{1}{2}(u^{k-1}\cdot\nabla)u^{k-1}\right) \\ &\quad = \nabla\cdot\left(\frac{3}{2}f^k - \frac{1}{2}f^{k-1}\right). \end{aligned} \tag{2.14}$$

Conversely, if u^k and p^k satisfy (2.13) and (2.14), then

$$\delta_t(\nabla\cdot u^k)-\frac{1}{2}\nu\Delta(\nabla\cdot u^k+\nabla\cdot u^{k+1})=0.$$

By induction with $\nabla\cdot u^0=0$, we know that $\nabla\cdot u^k\equiv 0$ for $0\leqslant k\leqslant M$. Therefore (2.12) is equivalent to system (2.13), (2.14) with (2.7), and $u^0=U^0$.

Finally we discuss the second-order projection method based on the pressure increment formulation. It is to find u^k, u_*^k, and $p^{k+1/2}$ such that $u^0=U^0$, $\nabla\cdot u^M=0$, and for $1\leqslant k\leqslant M-1$,

$$\begin{cases}\frac{1}{\tau}(u_*^{k+1}-u^k)+\frac{3}{2}(u^k\cdot\nabla)u^k-\frac{1}{2}(u^{k-1}\cdot\nabla)u^{k-1}+\nabla p^{k-1/2}-\frac{1}{2}\nu\Delta(u^k+u_*^{k+1})\\ \quad=\frac{3}{2}f^k-\frac{1}{2}f^{k-1},\\ \frac{1}{\tau}(u^{k+1}-u_*^{k+1})+\nabla(p^{k+1/2}-p^{k-1/2})=0,\\ \nabla\cdot u^k=0,\end{cases}\tag{2.15}$$

coupled with (2.7). We have from (2.15) that

$$\begin{aligned}&\delta_t u^k+\frac{3}{2}(u^k\cdot\nabla)u^k-\frac{1}{2}(u^{k-1}\cdot\nabla)u^{k-1}+\nabla p^{k+1/2}-\frac{1}{2}\nu\tau^2\delta_t\Delta(\nabla p^{k-1/2})\\ &\quad-\frac{1}{2}\nu\Delta(u^k+u^{k+1})=\frac{3}{2}f^k-\frac{1}{2}f^{k-1}.\end{aligned}\tag{2.16}$$

Taking the divergence on both sides of (2.16) leads to

$$\begin{aligned}&\Delta p^{k+1/2}+\nabla\cdot\left(\frac{3}{2}(u^k\cdot\nabla)u^k-\frac{1}{2}(u^{k-1}\cdot\nabla)u^{k-1}\right)-\frac{1}{2}\nu\tau^2\delta_t(\Delta^2p^{k-1/2})\\ &\quad=\nabla\cdot\left(\frac{3}{2}f^k-\frac{1}{2}f^{k-1}\right).\end{aligned}\tag{2.17}$$

We can verify that (2.15) is equivalent to system (2.16), (2.17) with (2.7), and $u^0=U^0$.

3. First-order Fourier spectral projection method

The Fourier spectral method has been successfully used for the numerical solution of the periodic problem of Navier–Stokes equations, see, e.g., [11,15]. However, there seems no work for the Fourier spectral projection method. We now develop the first-order Fourier spectral projection method.

3.1. Some notations and auxiliary results

For any $r\geqslant 0$, we use $H_p^r(\Omega)$ to denote the subspace of $H^r(\Omega)$ consisting of all functions with the period 2π in all spatial directions, $|\cdot|_r$ and $\|\cdot\|_r$ to denote the semi-norm and norm of $H_p^r(\Omega)$, respectively. Furthermore, $(\cdot,\cdot)$ and $\|\cdot\|$ represent the scalar product and norm of the space $L_p^2(\Omega)$, respectively. In addition,

$$L_{p,0}^2(\Omega)=\left\{v\mid v\in L_p^2(\Omega)\text{ and }\int_\Omega v\,dx=0\right\}.$$

Let $l=(l_1,l_2,\dots,l_n)^T$ and N be any positive integer. Then we define

$$\tilde V_N=\mathrm{span}\{e^{il\cdot x},\ 0\leqslant |l|\leqslant N\},$$

and V_N is the subspace of $\tilde V_N$, consisting of all real-valued functions. Moreover $V_{N,0}=V_N\cap L^2_{p,0}(\Omega)$. Throughout the paper, c or C, with or without subscripts, will always denote a generic positive constant independent of τ, N, and any function. We have that for any $\phi\in V_N$, $1\leqslant p\leqslant q\leqslant\infty$, and $r\geqslant 0$,

$$\|\phi\|_{L^q}\leqslant cN^{n/p-n/q}\|\phi\|_{L^p},\qquad |\phi|_r\leqslant\sqrt{n}N^r\|\phi\|. \tag{3.1}$$

Next, let Π_N be the $L^2_p(\Omega)$-orthogonal projection such that for any $v\in L^2_p(\Omega)$,

$$(v-\Pi_N v,\phi)=0,\quad \forall\phi\in V_N.$$

As is well known (see [12]) that for any $v\in H^r_p(\Omega)$, $r\geqslant 0$, and $\mu\leqslant r$,

$$\|v-\Pi_N v\|_\mu\leqslant cN^{\mu-r}\|v\|_r. \tag{3.2}$$

It is easy to see that for any $v\in L^2_{p,0}(\Omega)$, $\Pi_N v\in V_{N,0}$.

The following lemma will play an important role in the subsequent analysis; see [12].

Lemma 3.1. *Let E^k be a nonnegative function of k, G^k and F^k be two functions of E^j $(0\leqslant j\leqslant k)$, and $d>0$, $\lambda>0$, and $b\geqslant 0$ are three constants. Furthermore, we assume that*

(i) *If $E^j\leqslant d$ for all $j\leqslant k-1$, then $G^k\geqslant\lambda E^k$ and $F^{k-1}\leqslant bE^{k-1}$;*
(ii) *For a nondecreasing function ρ^k and all $1\leqslant k\leqslant M$, $G^k\leqslant\tau\sum_{j=0}^{k-1}F^j+\rho^k$;*
(iii) *$\lambda E^0\leqslant\rho^M\leqslant\lambda d e^{-bM\tau/\lambda}$.*

Then for all $k\leqslant M$, $E^k\leqslant(1/\lambda)\rho^M e^{bM\tau/\lambda}$.

3.2. First-order spectral projection method

Let u_N and p_N be the approximations to U and P, respectively, and $u_{N,*}$ the predicted value of u_N. Their values at $t=k\tau$ are denoted by u^k_N, $u^k_{N,*}$, and p^k_N. Then the first-order Fourier spectral projection scheme is to find $u^k_N, u^k_{N,*}\in V_N$, and $p^*_N\in V_{N,0}$ such that $u^0_N=\Pi_N U^0$, and

$$\begin{cases}\frac{1}{\tau}(u^{k+1}_{N,*}-u^k_N)+\Pi_N(u^k_N\cdot\nabla)u^k_N-\nu\Delta u^{k+1}_{N,*}=\Pi_N f^{k+1}, & 0\leqslant k\leqslant M-1,\\ \frac{1}{\tau}(u^{k+1}_N-u^{k+1}_{N,*})+\nabla p^{k+1}_N=0, & 0\leqslant k\leqslant M-1,\\ \nabla\cdot u^k_N=0, & 0\leqslant k\leqslant M.\end{cases} \tag{3.3}$$

We have from the second equation of (3.3) that

$$u^{k+1}_N=u^{k+1}_{N,*}-\tau\nabla p^{k+1}_N, \tag{3.4}$$

$$\Delta p^{k+1}_N=\frac{1}{\tau}\nabla\cdot u^{k+1}_{N,*}. \tag{3.5}$$

Inserting (3.4) into the first equation of (3.3) yields

$$\delta_t u_N^k + \Pi_N\big(u_N^k \cdot \nabla\big)u_N^k + (1 - \nu\tau\Delta)\nabla p_N^{k+1} - \nu\Delta u_N^{k+1} = \Pi_N f^{k+1}. \tag{3.6}$$

Then taking the divergence on both sides, we obtain

$$(1 - \nu\tau\Delta)\Delta p_N^{k+1} + \Pi_N \nabla \cdot \big(\big(u_N^k \cdot \nabla\big)u_N^k\big) = \Pi_N(\nabla \cdot f^{k+1}). \tag{3.7}$$

It is easy to verify that (3.3) is equivalent to system (3.6)–(3.7) with $u_N^0 = \Pi_N U^0$.

We are now in a position for the error analysis on scheme (3.4), see Theorems 3.1–3.3.

Theorem 3.1. *Assume that for some $r \geqslant n/3$ and an arbitrary small constant $\delta > 0$,*

$$U \in L^\infty\big(0, T; H_p^{n/2+\delta}(\Omega) \cap H_p^r(\Omega)\big) \cap H^2\big(0, T; L_p^2(\Omega)\big).$$

Then there exist positive constants c_1 and c_2, depending only on ν, T, and the norms of U in the mentioned spaces, such that for all $\tau \leqslant c_1 N^{-n/3}$ and $k \leqslant M$,

$$\big\|U^k - u_N^k\big\|^2 \leqslant c_2(\tau^2 + N^{-2r}).$$

If, in addition, $U \in L^2(0, T; H_p^{r+1}(\Omega))$, then

$$\tau \sum_{j=1}^{k} \big|U^j - u_N^j\big|_1^2 \leqslant c_3(\tau^2 + N^{-2r}),$$

where c_3 depends on c_2 and $\|U\|_{L^2(0,T;H_p^{r+1}(\Omega))}$.

Proof. We divide the proof into two parts. In Part 1, we compare the numerical solution with the $L_p^2(\Omega)$-orthogonal projection of the exact solution, and build up a basic energy inequality. In Part 2, we deal with the nonlinear error terms. Then we complete the proof by means of the global nonlinear error analysis initiated by Guo [10].

Part 1. Let $U_N^k = \Pi_N U^k$ and $P_N^k = \Pi_N P^k$. Taking the $L_p^2(\Omega)$-orthogonal projection on both sides of (2.5) at the time $t = k\tau + \tau$, we obtain

$$\begin{cases} \delta_t U_N^k + \Pi_N((U_N^k \cdot \nabla)U_N^k) + \nabla P_N^{k+1} - \nu\Delta U_N^{k+1} \\ \quad = \Pi_N(R_{N,0}^k + R_{N,1}^k + R_{N,2}^k + f^{k+1}), \\ \Delta P_N^{k+1} + \Pi_N \nabla \cdot ((U_N^k \cdot \nabla)U_N^k) = \Pi_N \nabla \cdot (R_{N,1}^k + R_{N,2}^k + f^{k+1}), \end{cases} \tag{3.8}$$

where $R_{N,0}^k$, $R_{N,1}^k$, and $R_{N,2}^k$ are given by

$$R_{N,0}^k = \delta_t U_N^k - \frac{\partial}{\partial t} U_N^{k+1},$$
$$R_{N,1}^k = \big((U^k - U^{k+1}) \cdot \nabla\big)U^{k+1} + (U^k \cdot \nabla)(U^k - U^{k+1}),$$
$$R_{N,2}^k = \big(\big(U_N^k - U^k\big) \cdot \nabla\big)U_N^k + (U^k \cdot \nabla)\big(U_N^k - U^k\big).$$

Now, let $\tilde{u}_N^k = u_N^k - U_N^k$ and $\tilde{p}_N^k = p_N^k - P_N^k$. Subtracting the first equation of (3.8) from (3.6), and the second one from (3.7), respectively, we arrive at

$$\begin{cases} \delta_t \tilde{u}_N^k + \Pi_N A_N^k + \nabla \tilde{p}_N^{k+1} - \nu\tau\Delta(\nabla p_N^{k+1}) - \nu\Delta(\tilde{u}_N^k + \tau\delta_t\tilde{u}_N^k) \\ \quad = -\Pi_N(R_{N,0}^k + R_{N,1}^k + R_{N,2}^k), \\ (1 - \nu\tau\Delta)\Delta\tilde{p}_N^{k+1} + \Pi_N \nabla \cdot A_N^k - \nu\tau\Delta^2 P_N^{k+1} = -\Pi_N \nabla \cdot (R_{N,1}^k + R_{N,2}^k), \end{cases} \tag{3.9}$$

where $A_N^k = A_{N,1}^k + A_{N,2}^k + A_{N,3}^k$, and

$$A_{N,1}^k = (\tilde{u}_N^k \cdot \nabla) U_N^k, \qquad A_{N,2}^k = (U_N^k \cdot \nabla)\tilde{u}_N^k, \qquad A_{N,3}^k = (\tilde{u}_N^k \cdot \nabla)\tilde{u}_N^k.$$

Clearly, $\nabla \cdot \tilde{u}_N^k = 0$, $\tilde{p}_N^k \in L_{p,0}^2(\Omega)$, and for any $v \in L_p^2(\Omega)$,

$$\begin{aligned} &2(\delta_t v^k, v^k) = \delta_t \|v^k\|^2 - \tau \|\delta_t v^k\|^2, \\ &2(\delta_t v^k, v^{k+1}) = \delta_t \|v^k\|^2 + \tau \|\delta_t v^k\|^2. \end{aligned} \tag{3.10}$$

By taking the scalar product on the first equation of (3.9) with $2\tilde{u}_N^k$, we get from (3.10) that

$$\begin{aligned} &\delta_t \|\tilde{u}_N^k\|^2 - \tau \|\delta_t \tilde{u}_N^k\|^2 + 2\nu |\tilde{u}_N^k|_1^2 + \nu\tau\delta_t |\tilde{u}_N^k|_1^2 - \nu\tau^2 |\delta_t \tilde{u}_N^k|_1^2 \\ &\quad = -2\big(A_N^k + R_{N,0}^k + R_{N,1}^k + R_{N,2}^k, \tilde{u}_N^k\big). \end{aligned} \tag{3.11}$$

Next, let ξ be a certain undetermined positive constant. Taking the scalar product on both sides of the first equation of (3.9) with $\xi\tau\delta_t \tilde{U}_N^k$ leads to

$$\begin{aligned} &\xi\tau \|\delta_t \tilde{u}_N^k\|^2 + \frac{1}{2}\xi\nu\tau\delta_t |\tilde{u}_N^k|_1^2 + \frac{1}{2}\xi\nu\tau^2 |\delta_t \tilde{u}_N^k|_1^2 \\ &\quad = -\xi\tau\big(A_N^k + R_{N,0}^k + R_{N,1}^k + R_{N,2}^k, \delta_t \tilde{u}_N^k\big). \end{aligned} \tag{3.12}$$

For convenience, we let $J_N^k = -(A_N^k + R_{N,0}^k + R_{N,1}^k + R_{N,2}^k, 2\tilde{u}_N^k + \xi\tau\delta_t \tilde{u}_N^k)$. Then adding up (3.11) and (3.12), we deduce that

$$\begin{aligned} &\delta_t \|\tilde{u}_N^k\|^2 + \tau(\xi - 1)\|\delta_t \tilde{u}_N^k\|^2 + 2\nu |\tilde{u}_N^k|_1^2 \\ &\quad + \nu\tau\left(\frac{\xi}{2} + 1\right)\delta_t |\tilde{u}_N^k|_1^2 + \nu\tau^2\left(\frac{\xi}{2} - 1\right)|\delta_t \tilde{u}_N^k|_1^2 = J_N^k. \end{aligned} \tag{3.13}$$

Take $\xi = 2(q+1)$, $q > 0$. Then (3.13) reads

$$\begin{aligned} &\delta_t \|\tilde{u}_N^k\|^2 + \tau(2q+1)\|\delta_t \tilde{u}_N^k\|^2 + 2\nu |\tilde{u}_N^k|_1^2 \\ &\quad + \nu\tau(q+2)\delta_t |\tilde{u}_N^k|_1^2 + q\nu\tau^2 |\delta_t \tilde{u}_N^k|_1^2 = J_N^k. \end{aligned} \tag{3.14}$$

Part 2. We now estimate the terms involved in J_N^k. First, for any $v, w, z \in H_p^1(\Omega)$ with $\nabla \cdot v = 0$,

$$\big((v \cdot \nabla)w, z\big) + \big((v \cdot \nabla)z, w\big) = 0, \qquad \big((v \cdot \nabla)w, w\big) = 0. \tag{3.15}$$

This implies $(A_{N,2}^k, \tilde{u}_N^k) = (A_{N,3}^k, \tilde{u}_N^k) = 0$. Thus, using (3.15), the imbedding theory, and (3.2), we obtain

$$\begin{aligned} 2\big|(A_N^k, \tilde{u}_N^k)\big| &= 2\big|(A_{N,1}^k, \tilde{u}_N^k)\big| = 2\big|\big((\tilde{u}_N^k \cdot \nabla)\tilde{u}_N^k, U_N^k\big)\big| \\ &\leqslant \frac{1}{4}\nu |\tilde{u}_N^k|_1^2 + \frac{c}{\nu}\|U_N^k\|_{n/2+\delta}^2 \|\tilde{u}_N^k\|^2 \leqslant \frac{1}{4}\nu |\tilde{u}_N^k|_1^2 + \frac{c}{\nu}\|U^k\|_{n/2+\delta}^2 \|\tilde{u}_N^k\|^2. \end{aligned} \tag{3.16}$$

By the same reason, we have

$$\xi\tau\big|\big(A_{N,1}^k + A_{N,2}^k, \delta_t \tilde{u}_N^k\big)\big| \leqslant \frac{1}{6}q\nu\tau^2 |\delta_t \tilde{u}_N^k|_1^2 + \frac{c}{q\nu}\|U^k\|_{n/2+\delta}^2 \|\tilde{u}_N^k\|^2. \tag{3.17}$$

Moreover, by (3.1),

$$\xi\tau\left|\left(A_{N,3}^k,\delta_t\tilde{u}_N^k\right)\right| \leqslant q\tau\left\|\delta_t\tilde{u}_N^k\right\|^2+\frac{c\tau}{q}N^n\left\|\tilde{u}_N^k\right\|^2\left|\tilde{u}_N^k\right|_1^2. \tag{3.18}$$

Next, let $\Lambda_k=(k\tau,k\tau+\tau)$. According to the property of the Bochner integral,

$$2\left|\left(R_{N,0}^k,\tilde{u}_N^k\right)\right| \leqslant \left\|\tilde{u}_N^k\right\|^2+\left\|R_{N,0}^k\right\|^2 \leqslant \left\|\tilde{u}_N^k\right\|^2+c\tau\|U\|^2_{H^2(\Lambda_k;L_p^2(\Omega))}. \tag{3.19}$$

By virtue of (3.15), the imbedding theory and (3.2),

$$\begin{aligned}2\left|\left(R_{N,1}^k,\tilde{u}_N^k\right)\right| &\leqslant 2\left|\left(\left((U^k-U^{k+1})\cdot\nabla\right)\tilde{u}_N^k,U^{k+1}\right)\right|+2\left|\left((U^k\cdot\nabla)\tilde{u}_N^k,U^k-U^{k+1}\right)\right|\\ &\leqslant \frac{1}{4}\nu\left|\tilde{u}_N^k\right|_1^2+\frac{c}{\nu}\left(\|U^k\|^2_{n/2+\delta}+\|U^{k+1}\|^2_{n/2+\delta}\right)\|U\|^2_{H^1(\Lambda_k;L_p^2(\Omega))}.\end{aligned} \tag{3.20}$$

Thanks to (3.15) and (3.2),

$$\begin{aligned}2\left|\left(R_{N,2}^k,\tilde{u}_N^k\right)\right| &\leqslant 2\left|\left(\left((U_N^k-U^k)\cdot\nabla\right)\tilde{u}_N^k,U_N^k\right)\right|+2\left|\left((U_N^k\cdot\nabla)\tilde{u}_N^k,U_N^k-U^k\right)\right|\\ &\leqslant \frac{1}{4}\nu\left|\tilde{u}_N^k\right|_1^2+\frac{c}{\nu}N^{-2r}\|U^k\|^2_{n/2+\delta}\|U^k\|_r^2.\end{aligned} \tag{3.21}$$

Similarly, we can derive that

$$\left|\xi\tau\left(R_{N,0}^k,\delta_t\tilde{u}_N^k\right)\right| \leqslant q\tau\left\|\delta_t\tilde{u}_N^k\right\|^2+c\tau^2\|U\|^2_{H^2(\Lambda_k;L_p^2(\Omega))}. \tag{3.22}$$

Furthermore, by (3.15) and an argument as before, we can show that

$$\begin{aligned}\left|\xi\tau\left(R_{N,1}^k,\delta_t\tilde{u}_N^k\right)\right| &\leqslant \frac{1}{6}q\nu\tau^2\left|\delta_t\tilde{u}_N^k\right|_1^2\\ &\quad+\frac{c\tau}{q\nu}\left(\|U^k\|^2_{n/2+\delta}+\|U^{k+1}\|^2_{n/2+\delta}\right)\|U\|^2_{H^1(\Lambda_k;L_p^2(\Omega))}.\end{aligned} \tag{3.23}$$

Using (3.15), (3.1) and (3.2) again yield

$$\left|\xi\tau\left(R_{N,2}^k,\delta_t\tilde{u}_N^k\right)\right| \leqslant \frac{1}{6}q\nu\tau^2\left|\delta_t\tilde{u}_N^k\right|_1^2+\frac{c}{q\nu}N^{-2r}\|U^k\|^2_{n/2+\delta}\|U^k\|_r^2. \tag{3.24}$$

Now, it follows by substituting (3.16)–(3.24) into (3.14) that

$$\begin{aligned}&\delta_t\left\|\tilde{u}_N^k\right\|^2+\tau\left\|\delta_t\tilde{u}_N^k\right\|^2+\left(\nu-c\tau N^n\left\|\tilde{u}_N^k\right\|^2\right)\left|\tilde{u}_N^k\right|_1^2+\nu\tau\delta_t\left|\tilde{u}_N^k\right|_1^2+\nu\tau^2\left|\delta_t\tilde{u}_N^k\right|_1^2\\ &\quad\leqslant c\left(\frac{1}{q\nu}\|U^k\|^2_{n/2+\delta}+\frac{1}{q\nu}\|U^{k+1}\|^2_{n/2+\delta}+1\right)\left\|\tilde{u}_N^k\right\|^2+G_N^k,\end{aligned} \tag{3.25}$$

where

$$\begin{aligned}G_N^k &= c\left(\|U^k\|^2_{n/2+\delta}+\|U^{k+1}\|^2_{n/2+\delta}\right)\|U\|^2_{H^1(\Lambda_k;L_p^2(\Omega))}\\ &\quad+c\tau\|U\|^2_{H^2(\Lambda_k;L_p^2(\Omega))}+cN^{-2r}\|U^k\|^2_{n/2+\delta}\|U^k\|_r^2.\end{aligned}$$

For the ease of describing the errors, we introduce the notations

$$E^k(w) = \|w^k\|^2 + \tau \sum_{j=0}^{k-1} \big(\nu |w^{j+1}|_1^2 + \tau \|\delta_t w^j\|^2 + \nu\tau^2 |\delta_t w^j|_1^2\big),$$

$$\rho_N^k = \tau \sum_{j=0}^{k-1} G_N^j. \tag{3.26}$$

Summing (3.25) with respect to k, we arrive at

$$E^k(\tilde{u}_N) - c\tau N^n E_N^{k-1}(\tilde{u}_N) E_N^k(\tilde{u}_N) \leqslant c_2 \tau \sum_{j=0}^{k-1} E^j(\tilde{u}_N) + \rho_N^k. \tag{3.27}$$

Clearly, $\rho_N^k \leqslant c_2(\tau^2 + N^{-2r})$. Let c_1 be a suitably small positive constant depending on c_2. Then if $\tau \leqslant c_1 N^{-n/3}$ and $r \geqslant n/3$, we have $\rho_N^k \leqslant c N^{-(2/3)n}$. Now, applying Lemma 3.1 to (3.27) with

$$E^k = E^k(\tilde{u}_N), \quad G^k = E^k(\tilde{u}_N) - c\tau N^n E^{k-1}(\tilde{u}_N) E^k(\tilde{u}_N), \quad F^k = E^k(\tilde{u}_N),$$
$$\rho^k = \rho_N^k, \quad d = \frac{1}{2c\tau N^n}, \quad \lambda = \frac{1}{2}, \quad b = c_2,$$

gives immediately that $E^k(\tilde{u}_N) \leqslant c_2(\tau^2 + N^{-2r})$. This with (3.2) completes the proof of Theorem 3.1. □

We now turn to the error estimate for p_N^{k+1}.

Theorem 3.2. *Assume that for some $r \geqslant n/2$ and $\delta > 0$, $P \in L^\infty(0, T; H_p^3(\Omega) \cap H_p^{r+1}(\Omega))$ and*

$$U \in L^\infty\big(0, T; H_p^{n/2+\delta+1}(\Omega) \cap H_p^{r+1}(\Omega)\big) \cap H^1\big(0, T; H_p^1(\Omega)\big)$$
$$\cap\, H^2\big(0, T; L_p^2(\Omega)\big).$$

Then there exist positive constants c_1 and c_2 depending only on ν, T, and the norms of U and P in the mentioned spaces, such that for all $\tau \leqslant c_1 N^{-n/2}$ and $k \leqslant M$,

$$\tau \sum_{j=0}^{k} \big\| P^j - p_N^j \big\|_1^2 \leqslant c_2(\tau^2 + N^{-2r}).$$

Proof. Taking the scalar product on the second equation of (3.9) with $\tilde{p}_N^{k+1}$, we obtain that

$$\big|\tilde{p}_N^{k+1}\big|_1^2 + \nu\tau \big|\tilde{p}_N^{k+1}\big|_2^2 - \nu\tau\big(\Delta\big(\nabla P_N^{k+1}\big), \nabla \tilde{p}_N^{k+1}\big) = -\big(A_N^k + R_{N,1}^k + R_{N,2}^k, \nabla \tilde{p}_N^{k+1}\big). \tag{3.28}$$

By the Poincaré inequality $\|\tilde{p}_N^{k+1}\| \leqslant c|\tilde{p}_N^{k+1}|_1$ and the Cauchy–Schwarz inequality, for any $\epsilon > 0$,

$$\begin{aligned}
\left|\left(A_N^k, \nabla \tilde{p}_N^{k+1}\right)\right| &\leqslant \epsilon\nu\left|\tilde{p}_N^{k+1}\right|_1^2 + \frac{c}{\epsilon\nu}\left(\|U^k\|_{n/2+\delta+1}^2\|\tilde{u}_N^k\|^2\right.\\
&\qquad \left. + \|U^k\|_{n/2+\delta}^2\left|\tilde{u}_N^k\right|_1^2 + N^n\|\tilde{u}_N^k\|^2\left|\tilde{u}_N^k\right|_1^2\right),\\
\left|\left(R_{N,1}^k, \nabla \tilde{p}_N^{k+1}\right)\right| &\leqslant \epsilon\nu\left|\tilde{p}_N^{k+1}\right|_1^2 + \frac{c\tau}{\epsilon\nu}\left(\|U^k\|_{n/2+\delta+1}^2\|U\|_{H^1(\Lambda_k; L_p^2(\Omega))}^2\right.\\
&\qquad \left. + \|U^k\|_{n/2+\delta}\|U\|_{H^1(\Lambda; H^1(\Omega))}^2\right),\\
\left|\left(R_{N,2}^k, \nabla \tilde{p}_N^{k+1}\right)\right| &\leqslant \epsilon\nu\left|\tilde{p}_N^{k+1}\right|_1^2 + \frac{c}{\epsilon\nu}N^{-2r}\left(\|U^k\|_{n/2+\delta}^2\|U^k\|_{r+1}^2\right.\\
&\qquad \left. + \|U^k\|_{n/2+\delta+1}^2\|U^k\|_r^2\right),\\
\nu\tau\left|\left(\Delta\left(\nabla P_N^{k+1}\right), \nabla \tilde{p}_N^{k+1}\right)\right| &\leqslant \epsilon\nu\left|\tilde{p}_N^{k+1}\right|_1^2 + c\nu\tau^2\|P^{k+1}\|_3^2.
\end{aligned}$$

Substituting the above estimates into (3.30) and summing the resulting inequality with respect to k, we obtain by taking ϵ to be suitably small that

$$\tau\sum_{j=0}^{k}\left(\|\tilde{p}_N^j\|_1^2 + \nu\tau\|\tilde{p}_N^j\|_2^2\right) \leqslant c_2\tau\sum_{j=0}^{k}\left(\|\tilde{u}_N^j\|^2 + N^n\|\tilde{u}_N^j\|^2\left|\tilde{u}_N^j\right|_1^2\right) + c_2(\tau^2 + N^{-2r}).$$

This with Theorem 3.1 and (3.2) completes the proof Theorem 3.2. □

Remark 3.1. The errors $(p_N^k - P^k)$ and $\nabla(p_N^k - P^k)$ have the same order as $(u_N^k - U^k)$, provided that U and P are slightly more regular as in Theorem 3.2. This is a merit of the projection method, and is discovered here at the first time.

Remark 3.2. Since we used the equivalent formulation of the projection method, we derive the error estimates for U and P separately. So the projection method simplifies the calculation, and the equivalent formulation simplifies the analysis.

Below, we consider the error estimates in the maximum norms.

Theorem 3.3. *Assume that $n \leqslant 4$, $r \geqslant n/2$, $\delta > 0$, and*

$$\begin{aligned}
U \in\ & L^\infty\left(0, T; H_p^{n/2+\delta+2}(\Omega) \cap H_p^{r+2}(\Omega)\right) \cap H^2\left(0, T; H_p^2(\Omega)\right)\\
&\cap W^{1,\infty}\left(0, T; H_p^2(\Omega)\right).
\end{aligned}$$

Then for all $\tau \leqslant c N^{-n/2}$ and $k \leqslant M$, we have

$$\|U^k - u_N^k\|_2 \leqslant c(\tau + N^{-r}).$$

If, in addition, $P \in L^\infty(0, T; H_p^4(\Omega) \cap H_p^{r+2}(\Omega))$, then

$$\|P^k - p_N^k\|_2 \leqslant c(\tau + N^{-r}).$$

Proof. Taking the Laplacian on both sides of (3.9) gives

$$\begin{aligned}
&\delta_t\left(\Delta\tilde{u}_N^k\right) + \Pi_N\Delta A_N^k + \Delta\left(\nabla\tilde{p}_N^{k+1}\right) - \nu\tau\Delta^2\left(\nabla P_N^{k+1}\right) - \nu\Delta^2\left(\tilde{u}_N^k + \tau\delta_t\tilde{u}_N^k\right)\\
&\quad = -\Pi_N\Delta\left(R_{N,0}^k + R_{N,1}^k + R_{N,2}^k\right). \qquad (3.29)
\end{aligned}$$

Now multiplying both sides by $2\Delta\tilde{u}_N^k + \xi\tau\delta_t(\Delta u_N^k)$ and integrating over Ω and then following the same strategy as in Part 1 of the proof of Theorem 3.1, we derive

$$\delta_t\left|\tilde{u}_N^k\right|_2^2 + \tau(2q+1)\left|\delta_t\tilde{u}_N^k\right|_2^2 + 2\nu\left|\nabla\tilde{u}_N^k\right|_2^2 + \nu\tau(q+2)\delta_t\left|\nabla\tilde{u}_N^k\right|_2^2 + q\nu\tau^2\left|\delta_t\nabla\tilde{u}_N^k\right|_2^2 = J_N^k, \tag{3.30}$$

where

$$J_N^k = -\left(\Delta A_N^k + \Delta R_{N,0}^k + \Delta R_{N,1}^k + \Delta R_{N,2}^k, 2\Delta\tilde{u}_N^k + \xi\tau\delta_t\Delta\tilde{u}_N^k\right).$$

So it remains to estimate $|J_N^k|$. For this purpose, let $\tilde{u}_N^k = (\tilde{u}_N^{(1)k}, \ldots, \tilde{u}_N^{(n)k})^T$. It follows from $\nabla\cdot\tilde{u}_N^k = 0$ that

$$\left(\tilde{u}_N^k\cdot\nabla\right)U_N^k = \sum_{\alpha=1}^n \frac{\partial}{\partial x_l}\left(\tilde{u}_N^{(\alpha)k}U_N^k\right).$$

By integration by parts three times, we have

$$\begin{aligned}\left(\Delta A_{N,1}^k, \Delta\tilde{u}_N^k\right) &= \left(\left(\tilde{u}_N^k\cdot\nabla\right)U_N^k, \Delta^2\tilde{u}_N^k\right)\\ &= -\sum_{\alpha,\beta,\gamma=1}^n\left(\Delta\left(\tilde{u}_N^{(\alpha)k}U_N^{(\beta)k}\right), \frac{\partial^3}{\partial x_\alpha\partial x_\gamma^2}\tilde{u}_N^{(\beta)k}\right).\end{aligned}$$

Similar equalities are valid for $(\Delta A_{N,j}^k, \Delta\tilde{u}_N^k)$, $j = 2, 3$. Therefore, we obtain

$$2\left|\left(\Delta A_N^k, \Delta\tilde{u}_N^k\right)\right| \leqslant \frac{1}{3}\nu\left|\nabla\tilde{u}_N^k\right|_2^2 + \frac{c}{\nu}\left(\|U^k\|_{n/2+\delta+2}^2\left\|\tilde{u}_N^k\right\|_2^2 + N^n\left\|\tilde{u}_N^k\right\|_2^4\right).$$

Following the same lines as in Part 2 of the proof of Theorem 3.1 and the previous argument, it is not difficult to show that

$$\begin{aligned}\xi\tau\left|\left(\Delta A_N^k, \delta_t\Delta\tilde{u}_N^k\right)\right| &\leqslant q\tau\left|\delta_t\nabla\tilde{u}_N^k\right|_2^2 + \frac{c}{q}\|U^k\|_{n/2+\delta+2}^2\left\|\tilde{u}_N^k\right\|_2^2\\ &\quad + \frac{c\tau}{q}N^n\left\|\tilde{u}_N^k\right\|_2^2\left|\nabla\tilde{u}_N^k\right|_2^2.\end{aligned}$$

It is clear that

$$\begin{aligned}2\sum_{j=0}^2\left|\left(\Delta R_{N,j}^k, \Delta\tilde{u}_N^k\right)\right| &\leqslant \nu\left|\tilde{u}_N^k\right|_2^2 + c\tau\|U\|_{H^2(\Lambda_k;H^2(\Omega))}^2\\ &\quad + \frac{c\tau}{\nu}\left(\|U^k\|_{n/2+\delta+2}^2 + \|U^{k+1}\|_{n/2+\delta+2}^2\right)\|U\|_{H^1(\Lambda_k;H^2(\Omega))}^2\\ &\quad + \frac{c}{\nu}N^{-2r}\|U^k\|_{n/2+\delta+2}^2\|U^k\|_{r+2}^2.\end{aligned}$$

Moreover, as for (3.22)–(3.24), we can prove that

$$\begin{aligned}&\xi\tau\left|\left(\Delta R_{N,0}^k + \Delta R_{N,1}^k + \Delta R_{N,2}^k, \delta_t\Delta\tilde{u}_N^k\right)\right|\\ &\quad\leqslant q\tau\left|\delta_t\tilde{u}_N^k\right|_2^2 + \frac{c\tau}{q}\left(\|U^k\|_{n/2+\delta+2}^2 + \|U^{k+1}\|_{n/2+\delta+2}^2\right)\|U\|_{H^1(\Lambda_k;H^2(\Omega))}^2\\ &\qquad + \frac{c}{q\nu}N^{-2r}\|U^k\|_{n/2+\delta+2}^2\|U^k\|_{r+2}^2.\end{aligned}$$

Substituting the above estimates into (3.30), we obtain that

$$\delta_t\left|\tilde{u}_N^k\right|_2^2+\tau\left|\delta_t\tilde{u}_N^k\right|_2^2+\left(\nu-cN^n\left\|\tilde{u}_N^k\right\|_2^2\right)\left|\nabla\tilde{u}_N^k\right|_2^2+\nu\tau\delta_t\left|\nabla\tilde{u}_N^k\right|_2^2+\nu\tau^2\left|\delta_t\nabla\tilde{u}_N^k\right|_2^2 \leqslant c\left(\frac{1}{q\nu}\|U^k\|^2_{n/2+\delta+2}+1\right)\left|\tilde{u}_N^k\right|_2^2+\frac{c}{\nu}N^n\left\|\tilde{u}_N^k\right\|_2^4+\bar{G}_N^k, \tag{3.31}$$

where

$$\bar{G}_N^k=c\tau\left(\|U^k\|^2_{n/2+\delta+2}+\|U^{k+1}\|^2_{n/2+\delta+2}+1\right)\|U\|^2_{H^2(\Lambda_k;H^2(\Omega))} + cN^{-2r}\|U^k\|^2_{n/2+\delta+2}\|U^k\|^2_{r+2}.$$

Let

$$\bar{E}^k(\tilde{u}_N)=\left\|\tilde{u}_N^k\right\|_2^2+\nu\tau\sum_{j=1}^{k}\left(\left|\tilde{u}_N^j\right|_1^2+\left|\nabla\tilde{u}_N^j\right|_2^2\right),\qquad \bar{\rho}_N^k=\rho_N^k+\tau\sum_{j=1}^{k}\bar{G}_N^j,$$

where ρ_N^k is the same as in (3.25). Obviously, $\bar{\rho}_N^k\leqslant c_2(\tau^2+N^{-2r})$. Adding (3.25) to (3.31) and summing the result over k, we assert that

$$\bar{E}^k\left(\tilde{u}_N^k\right)-cN^n\bar{E}^{k-1}(\tilde{u}_N)\bar{E}^k(\tilde{u}_N)\leqslant c_2\tau\sum_{j=0}^{k-1}\left(\bar{E}^j(\tilde{u}_N)+\left(\bar{E}^j(\tilde{u}_N)\right)^2\right)+\bar{\rho}_N^k.$$

We now apply Lemma 3.1 to the above with

$$E^k=\bar{E}^k(\tilde{u}_N),\qquad G^k=\bar{E}^k(u_N)-cN^n\bar{E}^{k-1}(\tilde{u}_N)\bar{E}^k(\tilde{u}_N),$$
$$F^k=\bar{E}^k(u_N)+cN^n\left(\bar{E}^{k-1}(\tilde{u}_N)\right)^2,$$
$$\rho^k=\bar{\rho}_N^k,\quad d=\frac{1}{2cN^n},\quad \lambda=\frac{1}{2},\quad b=c_2.$$

Then the corresponding result with the imbedding theory leads to $\|\tilde{u}^k\|_2\leqslant c_2(\tau+N^{-r})$. This with (3.2) implies $\|U^k-u_N^k\|_2\leqslant c_2(\tau+N^{-r})$.

Next, we take the scalar product on the second equation of (3.9) with $\Delta\tilde{p}_N^{k+1}$. It is noted that

$$-\nu\tau\left(\Delta^2\tilde{p}_N^{k+1},\Delta\tilde{p}_N^{k+1}\right)=\nu\tau\left|\nabla\tilde{p}_N^{k+1}\right|_2^2.$$

Thus by the Poincaré inequality, we deduce that

$$\left\|\tilde{p}_N^{k+1}\right\|_2^2\leqslant c\left(\left\|\nabla\cdot A_N^k\right\|^2+\left\|\nabla\cdot R_{N,1}^k\right\|^2+\left\|\nabla\cdot R_{N,2}^k\right\|^2+\tau^2|P^{k+1}|_4^2\right).$$

Furthermore, using (3.2) and the imbedding theory again, yields

$$\begin{aligned}\left\|\nabla\cdot A_N^k\right\|^2&\leqslant c\left(\left\|U_N^k\right\|^2_{W^{2,4}}\left\|\tilde{u}_N^k\right\|^2_{W^{1,4}}+\left\|U_N^k\right\|^2_{W^{1,\infty}}\left\|\tilde{u}_N^k\right\|_2^2\right.\\&\qquad\left.+\left\|\tilde{u}_N^k\right\|^2_{L^\infty}\left\|\tilde{u}_N^k\right\|_2^2+\left\|\tilde{u}_N^k\right\|^4_{W^{1,4}}\right)\\&\leqslant c_2(\tau^2+N^{-2r}).\end{aligned}$$

It is easy to verify that

$$\begin{aligned}&\left\|\nabla\cdot R_{N,1}^k\right\|^2+\left\|\nabla\cdot R_{N,2}^k\right\|^2\\&\quad\leqslant c_2\tau^2\big(\|U^k\|_{n/2+\delta+2}^2+\|U^{k+1}\|_{n/2+\delta+2}^2\big)\|U\|_{W^{1,4}(\Lambda_k;H^2(\Omega))}^2\\&\qquad+c_2N^{-2r}\|U^k\|_{n/2+\delta+1}^2\|U^k\|_{r+2}^2.\end{aligned}$$

Now it follows immediately from the above three inequalities that $\|\tilde{p}_N^{k+1}\|_2\leqslant c_2(\tau+N^{-r})$. □

By Theorem 3.3 and the imbedding theory, we have

Corollary 3.1. *Under the same assumptions as in Theorem* 3.3*, we have*

$$\left\|U^k-u_N^k\right\|_{L^\infty}\leqslant c_2(\tau+N^{-r}),\qquad \left\|P^k-p_N^k\right\|_{L^\infty}\leqslant c_3(\tau+N^{-r}).$$

3.3. A modified first-order Fourier spectral projection method

In this subsection we consider the following modified first-order Fourier spectral projection method: Find $u_N^k, u_{N,*}^k\in V_N$ and $p_N^k\in V_{N,0}$, such that $u_N^0=\Pi_NU^0$, and

$$\begin{cases}\frac{1}{\tau}(u_{N,*}^{k+1}-u_N^k)+\Pi_N(u_N^k\cdot\nabla)u_{N,*}^{k+1}-\nu\Delta u_{N,*}^{k+1}=\Pi_Nf^{k+1}, & 0\leqslant k\leqslant M-1,\\ \frac{1}{\tau}(u_N^{k+1}-u_{N,*}^{k+1})+\nabla p_N^{k+1}=0, \quad 0\leqslant k\leqslant M-1,\\ \nabla\cdot u_N^k=0,\quad 0\leqslant k\leqslant M.\end{cases}\tag{3.32}$$

It is equivalent to the following system with $u_N^0=\Pi_NU^0$ and

$$\begin{aligned}&\delta_tu_N^k+\Pi_N\big(u_N^k\cdot\nabla\big)u_N^{k+1}+\tau\Pi_N\big(u_N^k\cdot\nabla\big)\nabla p_N^{k+1}\\&\quad+(1-\nu\tau\Delta)\nabla p_N^{k+1}-\nu\Delta u_N^{k+1}=\Pi_Nf^{k+1},\\&(1-\nu\tau\Delta)\Delta p_N^{k+1}+\Pi_N\nabla\cdot\big(\big(u_N^k\cdot\nabla\big)u_N^{k+1}\big)\\&\quad+\tau\Pi_N\nabla\cdot\big(\big(u_N^k\cdot\nabla\big)\nabla p_N^{k+1}\big)=\Pi_N\nabla\cdot f^{k+1}\end{aligned}$$

for $0\leqslant k\leqslant M-1$. The errors $\tilde{u}_N^k=u_N^k-U^k$ satisfy the following equations:

$$\begin{cases}\delta_t\tilde{u}_N^k+\Pi_N(\bar{A}_N^k+B_N^k)+\nabla\tilde{p}_N^{k+1}-\nu\tau\Delta(\nabla p_N^{k+1})-\nu\Delta\tilde{u}_N^{k+1}\\ \quad=-\Pi_N(\bar{R}_{N,0}^k+\bar{R}_{N,1}^k+\bar{R}_{N,2}^k+\bar{R}_{N,3}^k),\\ (1-\nu\tau\Delta)\Delta\tilde{p}_N^{k+1}+\Pi_N\nabla\cdot(\bar{A}_N^k+B_N^k)-\nu\tau\Delta^2P_N^{k+1}\\ \quad=-\Pi_N\nabla\cdot(R_{N,1}^k+R_{N,2}^k+R_{N,3}^k),\end{cases}\tag{3.33}$$

where

$$\begin{aligned}&R_{N,0}^k=\delta_tU_N^k-\frac{\partial}{\partial t}U_N^{k+1},\qquad R_{N,1}^k=\big((U^k-U^{k+1})\cdot\nabla\big)U^{k+1},\\&R_{N,2}^k=\big(\big(U_N^k-U^k\big)\cdot\nabla\big)U_N^{k+1}+(U^k\cdot\nabla)\big(U_N^{k+1}-U^{k+1}\big),\\&R_{N,3}^k=\tau\big(U_N^k\cdot\nabla\big)\nabla P_N^{k+1},\end{aligned}$$

and

$$\bar{A}_N^k=\bar{A}_{N,1}^k+\bar{A}_{N,2}^k+\bar{A}_{N,3}^k,\qquad B_N^k=B_{N,1}^k+B_{N,2}^k+B_{N,3}^k$$

with

$$\bar{A}^k_{N,1} = (\tilde{u}^k_N \cdot \nabla)U^{k+1}_N, \quad \bar{A}^k_{N,2} = (U^k_N \cdot \nabla)\tilde{u}^{k+1}_N, \quad \bar{A}^k_{N,3} = (\tilde{u}^k_N \cdot \nabla)\tilde{u}^{k+1}_N,$$
$$B^k_{N,1} = \tau(\tilde{u}^k_N \cdot \nabla)\nabla P^{k+1}_N, \quad B^k_{N,2} = \tau(U^k_N \cdot \nabla)\nabla \tilde{p}^{k+1}_N, \quad B^k_{N,3} = \tau(\tilde{u}^k_N \cdot \nabla)\nabla \tilde{p}^{k+1}_N.$$

Theorem 3.4. *Assume that for some $r \geqslant n/2$ and $\delta > 0$,*

$$U \in L^\infty\big(0, T; H^{n/2+\delta+1}_p(\Omega) \cap H^{r+1}_p(\Omega)\big) \cap H^2\big(0, T; L^2_p(\Omega)\big),$$
$$P \in L^\infty\big(0, T; H^2_p(\Omega) \cap H^{r+1}_p(\Omega)\big).$$

Then there exist positive constants c_1 and c_2, depending only on ν, T, and the norms of U and P in the mentioned spaces, such that for all $\tau \leqslant c_1 N^{-n/2}$ and $k \leqslant M$,

$$\|U^k - u^k_N\|^2 + \tau \sum_{j=0}^{k} |U^j - u^j_N|_1^2 \leqslant c_2(\tau^2 + N^{-2r}).$$

Proof. Taking the scalar product on the first equation of (3.33) with $2\tilde{u}^{k+1}_N$, and using (3.16), we obtain that

$$\delta_t \|\tilde{u}^k_N\|^2 + \tau \|\delta_t \tilde{u}^k_N\|^2 + 2\nu |\tilde{u}^{k+1}_N|_1^2 = J^k_N, \tag{3.34}$$

where

$$J^k_N = -2\big(\bar{A}^k_N + B^k_N + R^k_{N,0} + R^k_{N,1} + R^k_{N,2} + R^k_{N,3}, \tilde{u}^{k+1}_N\big).$$

Similarly to (3.16), we have for any $\epsilon > 0$,

$$2\big|\big(\bar{A}^k_N, \tilde{u}^{k+1}_N\big)\big| = 2\big|\big(\bar{A}^k_{N,1}, \tilde{u}^{k+1}_N\big)\big| \leqslant \epsilon\nu |\tilde{u}^{k+1}_N|_1^2 + \frac{c}{\epsilon\nu}\|U^{k+1}\|^2_{n/2+\delta} \|\tilde{u}^k_N\|^2. \tag{3.35}$$

Thanks to (3.1) and (3.2),

$$2\big|\big(B^k_N, \tilde{u}^{k+1}_N\big)\big| \leqslant \epsilon\nu\tau |\tilde{p}^{k+1}_N|_2^2 + \frac{c\tau}{\nu\epsilon}\|U^k\|^2_{n/2+\delta} \|\tilde{u}^{k+1}_N\|^2 + \frac{c}{\nu}\tau N^n \|\tilde{u}^k_N\|^2 \|\tilde{u}^{k+1}_N\|^2. \tag{3.36}$$

Using (3.1), (3.2), and (3.15) again, we deduce that

$$\begin{aligned} &2\big|\big(R^k_{N,0} + R^k_{N,1} + R^k_{N,2} + R^k_{N,3}, \tilde{u}^{k+1}_N\big)\big| \\ &\quad\leqslant \|\tilde{u}^{k+1}_N\|^2 + c\tau \|U\|^2_{H^2(\Lambda_k; L^2(\Omega))} \\ &\qquad + c\tau \|U^{k+1}\|^2_{n/2+\delta+1} \|U\|^2_{H^1(\Lambda_k; L^2_p(\Omega))} + c\tau^2 \|U^k\|^2_{n/2+\delta} |P^{k+1}|_2^2 \\ &\qquad + \frac{c}{\nu} N^{-2r}\big(\|U^{k+1}\|^2_{n/2+\delta+1}\|U^k\|^2_r + \|U^k\|^2_{n/2+\delta}\|U^{k+1}\|^2_{r+1}\big). \end{aligned} \tag{3.37}$$

Next, taking the scalar product on the second equation of (3.33) with $\tilde{p}^{k+1}_N$, we find that

$$\begin{aligned} \|\tilde{p}^{k+1}_N\|_1^2 + \nu\tau \|\tilde{p}^{k+1}_N\|_2^2 \leqslant{}& c\big|\big(\bar{A}^k_N + B^k_N + R^k_{N,1} + R^k_{N,2} + R^k_{N,3}, \nabla \tilde{p}^{k+1}_N\big)\big| \\ &+ c\nu\tau^2 \|P^{k+1}\|_3^2. \end{aligned} \tag{3.38}$$

By an argument as in the proof of Theorem 3.2, we know that for any $\epsilon > 0$,

$$\begin{aligned}\left|\left(\bar{A}_N^k, \nabla \tilde{p}_N^{k+1}\right)\right| \leqslant \epsilon \left\| \tilde{p}_N^{k+1} \right\|_1^2 &+ \frac{c}{\epsilon}\left(\|U^k\|_{n/2+\delta+1}^2 + \|U^{k+1}\|_{n/2+\delta}^2\right)\left(\left\|\tilde{u}_N^k\right\|^2 + \left\|\tilde{u}_N^{k+1}\right\|^2\right) \\ &+ \frac{c}{\epsilon} N^n \left\|\tilde{u}_N^k\right\|^2 \left|\tilde{u}_N^{k+1}\right|_1^2. \end{aligned} \tag{3.39}$$

On use of (3.15), we have $(B_{N,j}^k, \nabla \tilde{p}_N^{k+1}) = 0$ for $j = 2, 3$, and so

$$\left|\left(B_N^k, \nabla \tilde{p}_N^{k+1}\right)\right| \leqslant \epsilon\nu\tau \left|\tilde{p}_N^{k+1}\right|_2^2 + \frac{\tau}{\epsilon\nu}\|P^{k+1}\|_{n/2+\delta+1}^2 \left\|\tilde{u}_N^k\right\|^2. \tag{3.40}$$

Finally, we can further show that

$$\begin{aligned}&\left|\left(R_{N,1}^k + R_{N,2}^k + R_{N,3}^k, \nabla \tilde{p}_N^{k+1}\right)\right| \\ &\quad \leqslant \epsilon\nu \left|\tilde{p}_N^{k+1}\right|_1^2 + c\tau \|U^{k+1}\|_{n/2+\delta+1}^2 \|U\|_{H^1(\Lambda_k; L_p^2(\Omega))}^2 \\ &\qquad + \tau^2 \|U^k\|_{n/2+\delta}^2 \|P^{k+1}\|_2^2 \\ &\qquad + \frac{c}{\nu} N^{-2r}\left(\|U^{k+1}\|_{n/2+\delta+1}^2 \|U^k\|_r^2 + \|U^k\|_{n/2+\delta}^2 \|U^{k+1}\|_{r+1}^2\right). \end{aligned} \tag{3.41}$$

Now adding (3.38) to (3.34), substituting (3.35)–(3.37) and (3.39)–(3.41) into the resulting inequality, and then summing up the result over k, we arrive at

$$\begin{aligned}&\left\|\tilde{u}_N^k\right\|^2 + \tau \sum_{j=0}^{k}\Big(\left(\nu - cN^n \left\|\tilde{u}_N^{j-1}\right\|^2\right)\left|\tilde{u}_N^j\right|_1^2 + \tau \left\|\delta_t \tilde{u}_N^j\right\|^2 \\ &\qquad\qquad + \left(1 - c\tau N^n \left\|\tilde{u}_N^{j-1}\right\|^2\right)\left\|\tilde{p}_N^j\right\|_1^2 + \nu\tau \left\|\tilde{p}_N^j\right\|_2^2\Big) \\ &\quad \leqslant c_2 \tau \sum_{j=0}^{k-1}\left(\left\|\tilde{u}_N^j\right\|^2 + \tau N^n \left\|\tilde{u}_N^j\right\|^4\right) + c_2(\tau^2 + N^{-2r}). \end{aligned}$$

Now the desired result follows directly from Lemma 3.1 and (3.2). □

4. Second-order Fourier spectral projection method

In this section, we focus on the second-order Fourier spectral projection method by Kim and Moin [14]. The scheme aims to find $u_N^k, u_{N,*}^k \in V_N$ and $p_N^{k+1/2} \in V_{N,0}$, such that $u_N^0 = \Pi_N U^0$, $\nabla \cdot u_N^M = 0$, and for $1 \leqslant k \leqslant M-1$,

$$\begin{cases} \frac{1}{\tau}(u_{N,*}^{k+1} - u_N^k) + \Pi_N\left(\frac{3}{2}(u_N^k \cdot \nabla)u_N^k - \frac{1}{2}(u_N^{k-1} \cdot \nabla)u_N^{k-1}\right) - \frac{1}{2}\nu\Delta(u_N^k + u_{N,*}^{k+1}) \\ \quad = \Pi_N\left(\frac{3}{2}f^k - \frac{1}{2}f^{k-1}\right), \\ \frac{1}{\tau}(u_N^{k+1} - u_{N,*}^{k+1}) + \nabla p_N^{k+1/2} = 0, \\ \nabla \cdot u_N^k = 0. \end{cases} \tag{4.1}$$

In actual computations, one may use (3.3) to evaluate the starting value u_N^1.

By the first two equations of (4.1), we can deduce that

$$\delta_t u_N^k + \Pi_N\left(\frac{3}{2}\left(u_N^k\cdot\nabla\right)u_N^k - \frac{1}{2}\left(u_N^{k-1}\cdot\nabla\right)u_N^{k-1}\right) + \nabla p_N^{k+1/2} - \frac{1}{2}\nu\tau\Delta\left(\nabla p_N^{k+1/2}\right) - \frac{1}{2}\nu\Delta\left(u_N^k + u_N^{k+1}\right) = \Pi_N\left(\frac{3}{2}f^k - \frac{1}{2}f^{k-1}\right), \quad 1\leqslant k\leqslant M-1. \tag{4.2}$$

Taking the divergence on both sides of the above equation gives

$$\left(1-\frac{1}{2}\nu\tau\Delta\right)\Delta p_N^{k+1/2} + \Pi_N\nabla\cdot\left(\frac{3}{2}\left(u_N^k\cdot\nabla\right)u_N^k - \frac{1}{2}\left(u_N^{k-1}\cdot\nabla\right)u_N^{k-1}\right) = \Pi_N\nabla\cdot\left(\frac{3}{2}f^k - \frac{1}{2}f^{k-1}\right). \tag{4.3}$$

One can show as before that (4.1) is equivalent to system (4.2)–(4.3) with $u_N^0 = \Pi_N U^0$.

It is important to note that one can improve the accuracy of the numerical pressure $p_N^{k+1/2}$ by correcting it as follows:

$$p_{N,c}^{k+1/2} = \left(1-\frac{1}{2}\nu\tau\Delta\right)p_N^{k+1/2}. \tag{4.4}$$

For the subsequent error estimates, we introduce a Sobolev space $W_m^*(I;\Omega)$ equipped with the norm

$$\|v\|_{W_m^*(I;\Omega)} = \left(\sum_{q=0}^{2}\int_0^T\left\|\frac{\partial^q}{\partial t^q}v(s)\right\|_{W^{m,4}(\Omega)}^4 ds\right)^{1/4}.$$

The main results of this section are stated in Theorems 4.1–4.3.

Theorem 4.1. *Assume that for some $r\geqslant n/2$ and $\delta>0$, we have $f\in H^2(0,T;L_p^2(\Omega))$ and*

$$U\in L^\infty\left(0,T;H_p^{n/2+\delta}(\Omega)\cap H_p^r(\Omega)\right)\cap H^2\left(0,T;H_p^2(\Omega)\right) \cap H^3\left(0,T;L_p^2(\Omega)\right)\cap W_1^*(0,T;\Omega).$$

Then there exist positive constants c_1 and c_2 depending only on ν, T, and the norms U and f in the mentioned spaces, such that for all $\tau\leqslant c_1N^{-n/4}$ and $k\leqslant M$,

$$\left\|U^k - u_N^k\right\|^2 \leqslant c_2\left(\tau^4 + N^{-2r}\right).$$

If, in addition, $U\in L^2(0,T;H_p^{r+1}(\Omega))$, then

$$\tau\sum_{j=0}^{k}\left|U^j + U^{j+1} - u_N^j - u_N^{j+1}\right|_1^2 \leqslant c_3\left(\tau^4 + N^{-2r}\right),$$

where c_3 depends on c_2 and $\|U\|_{L^2(0,T;H_p^{r+1}(\Omega))}$. Moreover, if

$$U \in L^\infty\big(0,T;H^{n/2+\delta+1}(\Omega)\cap H_p^{r+1}(\Omega)\big)\cap H^1\big(0,T;H_p^r(\Omega)\big),$$

then

$$\|U^k-u_N^k\|_1^2+\tau\sum_{j=0}^k\big\|\delta_t\big(U^j-u_N^j\big)\big\|^2\leqslant c_4(\tau^4+N^{-2r}),$$

where c_4 depends on c_2 and $\|U\|_{L^\infty(0,T;H^{n/2+\delta+1}(\Omega)\cap H^{r+1}(\Omega))\cap H^1(0,T;H_p^r(\Omega))}$.

Proof. We use the same notations U_N^k, P_N^k, $\tilde u_N^k$, $\tilde p_N^{k+1/2}$, and A_N^k, $A_{N,j}^k$ as in the last section, and $P_N^{k+1/2}=\Pi_N P^{k+1/2}$. Taking the $L^2(\Omega)$-orthogonal projection on (2.5) at $t=k\tau+\tau/2$, we have

$$\begin{cases}\delta_t U_N^k+\Pi_N\big(\frac32(U_N^k\cdot\nabla)U_N^k-\frac12(U_N^{k-1}\cdot\nabla)U_N^{k-1}\big)\\ \qquad+\nabla P_N^{k+1/2}-\frac12\nu\Delta(U_N^k+U_N^{k+1})\\ \quad=\Pi_N\big(\frac32 f^k-\frac12 f^{k-1}+R_{N,0}^k+R_{N,1}^k+R_{N,2}^k+R_{N,3}^k\big),\\ \Delta P_N^{k+1/2}+\Pi_N\nabla\cdot\big(\frac32(U_N^k\cdot\nabla)U_N^k-\frac12(U_N^{k-1}\cdot\nabla)U_N^{k-1}\big)\\ \quad=\Pi_N\nabla\cdot\big(\frac32 f^k-\frac12 f^{k-1}+R_{N,1}^k+R_{N,2}^k+R_{N,3}^k\big),\end{cases}\tag{4.5}$$

where

$$\begin{aligned}R_{N,0}^k&=\delta_t U_N^k-\frac{\partial}{\partial t}U_N^{k+1/2}+\frac12\nu\Delta\big(2U_N^{k+1/2}-U_N^k-U_N^{k+1}\big),\\ R_{N,1}^k&=\frac32(U^k\cdot\nabla)U^k-\frac12(U^{k-1}\cdot\nabla)U^{k-1}-(U^{k+1/2}\cdot\nabla)U^{k+1/2},\\ R_{N,2}^k&=\frac32\big(\big(U_N^k-U^k\big)\cdot\nabla\big)U_N^k+\frac32(U^k\cdot\nabla)\big(U_N^k-U^k\big)\\ &\quad-\frac12\big(\big(U_N^{k-1}-U^{k-1}\big)\cdot\nabla\big)U_N^{k-1}-\frac12(U^{k-1}\cdot\nabla)\big(U_N^{k-1}-U^{k-1}\big),\\ R_{N,3}^k&=f^{k+1/2}-\frac32 f^k+\frac12 f^{k-1}.\end{aligned}$$

Then subtracting (4.5) from (4.2) gives

$$\begin{cases}\delta_t\tilde u_N^k+\Pi_N\big(\frac32 A_N^k-\frac12 A_N^{k-1}\big)+\nabla\tilde p_N^{k+1/2}-\frac12\nu\tau\Delta(\nabla p_N^{k+1/2})-\frac12\nu\Delta(\tilde u_N^k+\tilde u_N^{k+1})\\ \quad=-\Pi_N(R_{N,0}^k+R_{N,1}^k+R_{N,2}^k+R_{N,3}^k),\\ (1-\frac12\nu\tau\Delta)\Delta\tilde p_N^{k+1/2}+\Pi_N\nabla\cdot\big(\frac32 A_N^k-\frac12 A_N^{k-1}\big)-\frac12\nu\tau\Delta^2 P_N^{k+1/2}\\ \quad=-\Pi_N\nabla\cdot(R_{N,1}^k+R_{N,2}^k+R_{N,3}^k).\end{cases}\tag{4.6}$$

Now, we take the scalar product on the first equation of (4.6) with $\tilde u_N^k+\tilde u_N^{k+1}$ to obtain

$$\begin{aligned}&\delta_t\|\tilde u_N^k\|^2+\frac12\nu\big|\tilde u_N^k+\tilde u_N^{k+1}\big|_1^2\\ &\quad=-\bigg(\frac32 A_N^k-\frac12 A_N^{k-1}+R_{N,0}^k+R_{N,1}^k+R_{N,2}^k+R_{N,3}^k,\tilde u_N^k+\tilde u_N^{k+1}\bigg).\end{aligned}\tag{4.7}$$

By (3.15), we have

$$\frac{3}{2}\left|\left(A_{N,1}^k, \tilde{u}_N^k + \tilde{u}_N^{k+1}\right)\right| = \frac{3}{2}\left|\left(\left(\tilde{u}_N^k \cdot \nabla\right)\left(\tilde{u}_N^k + \tilde{u}_N^{k+1}\right), U_N^k\right)\right| \leqslant \frac{1}{16}\nu\left|\tilde{u}_N^k + \tilde{u}_N^{k+1}\right|_1^2 + \frac{c}{\nu}\|U^k\|_{n/2+\delta}^2\left\|\tilde{u}_N^k\right\|^2.$$

We can estimate the other terms in A_N^k and A_N^{k-1} similarly, and obtain

$$\left|\left(\frac{3}{2}A_N^k - \frac{1}{2}A_N^{k-1}, \tilde{u}_N^k + \tilde{u}_N^{k+1}\right)\right| \leqslant \frac{1}{16}\nu\left|\tilde{u}_N^k + \tilde{u}_N^{k+1}\right|_1^2 + \frac{c}{\nu}\left(\|U^k\|_{n/2+\delta}^2\left\|\tilde{u}_N^k\right\|^2 + \|U^{k-1}\|_{n/2+\delta}^2\left\|\tilde{u}_N^{k-1}\right\|^2\right) + \frac{c}{\nu}N^n\left(\left\|\tilde{u}_N^k\right\|^4 + \left\|\tilde{u}_N^{k-1}\right\|^4\right).$$

By the property of Bochner integral and the Cauchy–Schwarz inequality,

$$\sum_{j=0,1,3}\left|\left(R_{N,j}^k, \tilde{u}_N^k + \tilde{u}_N^{k+1}\right)\right| \leqslant \left\|\tilde{u}_N^k\right\|^2 + \left\|\tilde{u}_N^{k+1}\right\|^2 + c\tau^3\left\|(U\cdot\nabla)U\right\|_{H^2(\Lambda_k;L^2(\Omega))} + c\tau^3\|U\|_{W_1^*(\Lambda_k;\Omega)}^2 + \frac{c\tau^3}{\nu}\|f\|_{H^2(\Lambda_k;L^2(\Omega))}^2.$$

By (3.2), (3.15), and a similar argument as in Part 2 of the proof of Theorem 3.1,

$$\left|\left(R_{N,2}^k, \tilde{u}_N^k + \tilde{u}_N^{k+1}\right)\right| \leqslant \frac{1}{16}\nu\left|\tilde{u}_N^k + \tilde{u}_N^{k+1}\right|_1^2 + \frac{c}{\nu}N^{-2r}\left(\|U^k\|_{n/2+\delta}^2\|U^k\|_r^2 + \|U^{k-1}\|_{n/2+\delta}^2\|U^{k-1}\|_r^2\right).$$

Substituting the above estimates into (4.7), and summing up the result over k, we obtain

$$\left\|\tilde{u}_N^k\right\|^2 + \nu\tau\sum_{j=1}^{k}\left|\tilde{u}_N^j + \tilde{u}_N^{j-1}\right|_1^2 \leqslant c\tau\sum_{j=0}^{k-1}\left(\left(\|U^j\|_{n/2+\delta}^2 + 1\right)\left\|\tilde{u}_N^j\right\|^2 + N^n\left\|\tilde{u}_N^j\right\|^4\right) + \rho_N^k, \tag{4.8}$$

where $\rho_N^k \leqslant c_2(\tau^4 + N^{-2r})$. In particular, for $\tau \leqslant c_1 N^{-n/4}$ and $r \geqslant n/2$, we have $\rho_N^k \leqslant c_2 N^{-n}$. Now, the first two conclusions of Theorem 4.1 follows directly from this result, (3.2), and Lemma 3.1.

Next, we take the scalar product on the first equation of (4.6) with $\delta_t\tilde{u}_N^k$ to obtain that

$$\left\|\delta_t\tilde{u}_N^k\right\|^2 + \frac{1}{2}\nu\delta_t\left|\tilde{u}_N^k\right|_1^2 = -\left(\frac{3}{2}A_N^k - \frac{1}{2}A_N^{k-1} + R_{N,0}^k + R_{N,1}^k + R_{N,2}^k + R_{N,3}^k, \delta_t\tilde{u}_N^k\right). \tag{4.9}$$

It can be verified that

$$\begin{aligned}
&\left|\left(\frac{3}{2}A_N^k-\frac{1}{2}A_N^{k-1},\delta_t\tilde{u}_N^k\right)\right| \\
&\quad\leqslant \frac{1}{2}\left\|\delta_t\tilde{u}_N^k\right\|^2+c\Big(\|U^k\|^2_{n/2+\delta+1}\left\|\tilde{u}_N^k\right\|^2+\|U^{k-1}\|^2_{n/2+\delta+1}\left\|\tilde{u}_N^{k-1}\right\|^2 \\
&\qquad+\|U^k\|^2_{n/2+\delta}\left|\tilde{u}_N^k\right|_1^2+\|U^{k-1}\|^2_{n/2+\delta}\left|\tilde{u}_N^{k-1}\right|_1^2 \\
&\qquad+N^n\left\|\tilde{u}_N^k\right\|^2\left|\tilde{u}_N^k\right|_1^2+N^n\left\|\tilde{u}_N^{k-1}\right\|^2\left|\tilde{u}_N^{k-1}\right|_1^2\Big).
\end{aligned}$$

We can estimate $|(R_{N,j}^k,\delta_t\tilde{u}_N^k)|$, $j=0,1,3$, as before. Moreover

$$\begin{aligned}
\left|\left(R_{N,2}^k,\delta_t\tilde{u}_N^k\right)\right| \leqslant \frac{1}{2}\left\|\delta_t\tilde{u}_N^k\right\|^2+\frac{c}{\nu}N^{-2r}\Big(&\|U^k\|^2_{n/2+\delta+1}\|U^k\|_r^2 \\
&+\|U^{k-1}\|_{n/2+\delta+1}\|U^{k-1}\|_r^2 \\
&+\|U^k\|^2_{n/2+\delta}\|U^k\|^2_{r+1}+\|U^{k-1}\|^2_{n/2+\delta}\|U^{k-1}\|^2_{r+1}\Big).
\end{aligned}$$

Thus substituting the above estimates into (4.9), summing up it over k and adding the result to (4.8), we obtain

$$\left\|\tilde{u}_N^k\right\|_1^2+\tau\sum_{j=0}^{k-1}\left\|\delta_t\tilde{u}_N^j\right\|^2\leqslant c\tau\sum_{j=0}^{k-1}\left(\|U^j\|^2_{n/2+\delta+1}\left\|\tilde{u}_N^j\right\|_1^2+N^n\left\|\tilde{u}_N^j\right\|^2\left|\tilde{u}_N^j\right|_1^2\right)+\rho_N^k,$$

which, with Lemma 3.1, leads to the third conclusion of Theorem 4.1. □

Remark 4.1. One can see from Theorem 4.1 that the second-order projection method not only improves the accuracy, but also relaxes the restriction on τ. For example, if $n=2$, $\tau\ll N^{-1/2}$ and $r\geqslant 1$, we have $\|U^k-u_N^k\|=\mathcal{O}(\tau^2+N^{-r})$.

For the accuracy of the numerical pressure, we have the following result.

Theorem 4.2. *Assume that for some $r\geqslant n/2$ and $\delta>0$,*

$$\begin{aligned}
&U\in L^\infty\big(0,T;H_p^{n/2+\delta+1}(\Omega)\cap H_p^{r+1}(\Omega)\big)\cap H^2\big(0,T;H_p^2(\Omega)\big) \\
&\qquad\cap H^3\big(0,T;L_p^2(\Omega)\big)\cap W_1^*(0,T;\Omega), \\
&P\in L^\infty\big(0,T;H_p^3(\Omega)\cap H_p^{r+1}(\Omega)\big),\qquad f\in H^2\big(0,T;L_p^2(\Omega)\big).
\end{aligned}$$

Then there exist positive constants c_1 and c_2 depending only on ν, T, and the norms of U, P, and f in the mentioned spaces, such that for all $\tau\leqslant c_1N^{-n/4}$ and $k\leqslant M$,

$$\tau\sum_{j=0}^{k}\left\|P^{j+1/2}-p_N^{j+1/2}\right\|_1^2\leqslant c_2(\tau^2+N^{-2r}).$$

Moreover, for the corrected numerical pressure, we have

$$\tau\sum_{j=0}^{k}\left\|P^{j+1/2}-p_{N,c}^{j+1/2}\right\|_1^2\leqslant c_3(\tau^4+N^{-2r}),$$

where c_3 is independent of $\|P\|_{L^\infty(0,T;H_p^3(\Omega))}$.

Proof. The periodicity implies $\|\Delta \tilde{p}_N^{k+1/2}\| = |\tilde{p}_N^{k+1/2}|_2$, and $p_N^{k+1/2} \in L^2_{p,0}(\Omega)$ implies $\|p_N^{k+1/2}\| \leqslant c|p_N^{k+1/2}|_1$. So taking the scalar product on the second equation of (4.6) with $\tilde{p}_N^{k+1/2}$ gives

$$\begin{aligned}
&\|\tilde{p}_N^{k+1/2}\|_1^2 + \nu\tau\|\tilde{p}_N^{k+1/2}\|_2^2 \\
&\quad \leqslant c\left|\left(\frac{3}{2}A_N^k - \frac{1}{2}A_N^{k-1} + R_{N,1}^k + R_{N,2}^k + R_{N,3}^k, \nabla\tilde{p}_N^{k+1/2}\right)\right| \\
&\qquad + c\nu\tau^2\|P^{k+1}\|_3^2.
\end{aligned} \tag{4.10}$$

The right-hand side of (4.10) can be estimated in the same manner as we did in the proof of Theorem 3.2. Accordingly, we get from (4.10) that

$$\begin{aligned}
\tau\sum_{j=0}^{k}\big(\|\tilde{p}_N^{j+1/2}\|_1^2 + \nu\tau\|\tilde{p}_N^{j+1/2}\|_2^2\big) &\leqslant c_2\tau\sum_{j=0}^{k}\big(\|\tilde{u}_N^j\|_1^2 + N^n\|\tilde{u}_N^j\|^2|\tilde{u}_N^j|_1^2\big) \\
&\quad + c_2(\tau^2 + N^{-2r}).
\end{aligned} \tag{4.11}$$

Then the first conclusion of Theorem 4.2 follows from (4.11), Theorem 4.1, and (3.2).

Next, we rewrite (4.3) as follows:

$$\begin{aligned}
&\Delta p_{N,c}^{k+1/2} + \Pi_N\nabla\cdot\left(\frac{3}{2}\big(u_N^k\cdot\nabla\big)u_N^k - \frac{1}{2}\big(u_N^{k-1}\cdot\nabla\big)u_N^{k-1}\right) \\
&\quad = \Pi_N\nabla\cdot\left(\frac{3}{2}f^k - \frac{1}{2}f^{k-1}\right).
\end{aligned}$$

Following the same line as in the first part of the proof, and noting that the dominant error term $c\nu\tau^2\|P^{k+1}\|_3^2$ in (4.10) does not appear any more, we arrive at the second conclusion. □

By the same procedure as in the proof of Theorem 3.3, we can prove the following result.

Theorem 4.3. *For some $n \leqslant 4$, $r \geqslant n/2$, and $\delta > 0$, we assume that $f \in H^2(0,T;H_p^2(\Omega))$ and*

$$\begin{aligned}
U \in\ & L^\infty\big(0,T;H_p^{n/2+\delta+2}(\Omega)\cap H_p^{r+2}(\Omega)\big)\cap H^2\big(0,T;H_p^4(\Omega)\big) \\
&\cap H^3\big(0,T;H_p^2(\Omega)\big)\cap W_3^*(0,T;\Omega).
\end{aligned}$$

Then there exist positive constants c_1 and c_2 depending only on ν, T, and the norms of U and f in the mentioned spaces, such that for all $\tau \leqslant c_1N^{-n/4}$ and $k \leqslant M$,

$$\big\|U^k - u_N^k\big\|_2 \leqslant c_2(\tau^2 + N^{-r}).$$

If, in addition, $P \in L^\infty(0,T;H_p^{r+2}(\Omega))$, then

$$\big\|P^{k+1/2} - p_{N,c}^{k+1/2}\big\|_2 \leqslant c_3(\tau^2 + N^{-r}),$$

where c_3 depends on c_2 and $\|P\|_{L^\infty(0,T;H_p^{r+2}(\Omega))}$. Furthermore, if $P \in L^\infty(0,T;H_p^3(\Omega) \cap H_p^{r+2}(\Omega))$, then

$$\big\|P^{k+1/2} - p_N^{k+1/2}\big\|_2 \leqslant c_4(\tau + N^{-r}),$$

where c_4 depends on c_3 and $\|P\|_{L^\infty(0,T;H_p^4(\Omega))}$.

Theorem 4.3 leads directly to the following error estimates in the maximum norm.

Corollary 4.1. *Under the same assumptions as in Theorem* 4.3*, we have*

$$\big\|U^k - u_N^k\big\|_{L^\infty} \leqslant c\,(\tau^2 + N^{-r}), \qquad \big\|P^{k+1/2} - p_{N,c}^{k+1/2}\big\|_{L^\infty} \leqslant c\,(\tau^2 + N^{-r}),$$
$$\big\|P^{k+1/2} - p_N^{k+1/2}\big\|_{L^\infty} \leqslant c(\tau + N^{-r}).$$

Remark 4.4. In scheme (4.1), one may approximate the nonlinear term $(U \cdot \nabla)U$ by

$$\left(\left(\frac{3}{2}u_N^k - \frac{1}{2}u_N^{k-1}\right)\cdot\nabla\right)\left(\frac{3}{2}u_N^k - \frac{1}{2}u_N^{k-1}\right).$$

In this case, all the results in Theorems 4.1–4.3 are still valid, but the regularity with $U \in W_m^*(0,T;\Omega)$, $m = 1, 3$, in Theorems 4.1–4.3 are now replaced by the weaker regularity with $U \in H^2(0,T;H_p^m(\Omega))$.

5. Second-order Fourier spectral projection method based on pressure increment formulation

In this section, we investigate the second-order Fourier spectral projection method based on pressure increment formulation. Its higher accuracy has been observed; see [2,3,21].

The second-order Fourier spectral projection method based on pressure increment formulation is to find $u_N^k, u_{N,*}^k \in V_N$ and $p_N^{k+1/2} \in V_{N,0}$, such that $u^0 = \Pi_N U^0$, and

$$\begin{cases} \frac{1}{\tau}(u_{N,*}^{k+1} - u_N^k) + \Pi_N\big(\frac{3}{2}(u_N^k\cdot\nabla)u_N^k - \frac{1}{2}(u_N^{k-1}\cdot\nabla)u_N^{k-1}\big) \\ \qquad + \nabla p_N^{k-1/2} - \frac{1}{2}\nu\Delta(u_N^k + u_{N,*}^{k+1}) \\ \quad = \Pi_N\big(\frac{3}{2}f^k - \frac{1}{2}f^{k-1}\big), \quad 1 \leqslant k \leqslant M-1, \\ \frac{1}{\tau}(u_N^{k+1} - u_{N,*}^{k+1}) + \nabla(p_N^{k+1/2} - p_N^{k-1/2}) = 0, \quad 1 \leqslant k \leqslant M-1, \\ \nabla\cdot u_N^k = 0, \quad 0 \leqslant k \leqslant M. \end{cases} \tag{5.1}$$

It is equivalent to the following system:

$$\delta_t u_N^k + \Pi_N\left(\frac{3}{2}\big(u_N^k\cdot\nabla\big)u_N^k - \frac{1}{2}\big(u_N^{k-1}\cdot\nabla\big)u_N^{k-1}\right) + \nabla p_N^{k+1/2} - \frac{1}{2}\nu\tau^2\delta_t\Delta\big(\nabla p_N^{k-1/2}\big)$$
$$- \frac{1}{2}\nu\Delta\big(u_N^k + u_N^{k+1}\big) = \Pi_N\left(\frac{3}{2}f^k - \frac{1}{2}f^{k-1}\right), \tag{5.2}$$
$$\Delta p_N^{k+1/2} + \Pi_N\nabla\cdot\left(\frac{3}{2}\big(u_N^k\cdot\nabla\big)u_N^k - \frac{1}{2}\big(u_N^{k-1}\cdot\nabla\big)u_N^{k-1}\right) - \frac{1}{2}\nu\tau^2\delta_t\Delta^2 p_N^{k-1/2}$$

$$= \Pi_N \nabla \cdot \left(\frac{3}{2} f^k - \frac{1}{2} f^{k-1}\right), \tag{5.3}$$

where $1 \leqslant k \leqslant M-1$ and $u^0 = \Pi_N U^0$. In the actual computations, one may use (3.3) to evaluate u_N^1, and use certain method to evaluate $p_N^{1/2}$ in advance. Suppose that for some $\alpha \geqslant 1$ and $r \geqslant 1$,

$$\tau \left| P^{1/2} - p_N^{1/2} \right|_2 \leqslant c(\tau^\alpha + N^{-r}). \tag{5.4}$$

We define the corrected numerical pressure by

$$p_{N,c}^{k+1/2} = p_N^{k+1/2} - \frac{1}{2}\nu\tau^2 \delta_t \Delta p_N^{k-1/2} = \left(1 - \frac{1}{2}\nu\tau\Delta\right) p_N^{k+1/2} + \frac{1}{2}\nu\tau\Delta p_N^{k-1/2}. \tag{5.5}$$

Theorem 5.1. *Under the same assumptions as in Theorem* 4.1*, we have the same convergence results as in Theorem* 4.1 *for scheme* (5.1)*.*

Proof. We will use the same notations as in the last section, such as A_N^k, $R_{N,0}^k$, $R_{N,1}^K$, and $R_{N,2}^k$. As we did in the proof of Theorem 4.1, we have from (2.5) and (5.2) that

$$\begin{cases} \delta_t \tilde{u}_N^k + \Pi_N\left(\frac{3}{2}A_N^k - \frac{1}{2}A_N^{k-1}\right) + \nabla \tilde{p}_N^{k+1/2} \\ \quad -\frac{1}{2}\nu\tau^2\delta_t \Delta(\nabla p_N^{k-1/2}) - \frac{1}{2}\nu\Delta(\tilde{u}^k + \tilde{u}^{k+1}) \\ \quad = -(R_{N,0}^k + R_{N,1}^k + R_{N,2}^k + R_{N,3}^k), \quad 1 \leqslant k \leqslant M-1, \\ \Delta \tilde{p}_N^{k+1/2} + \Pi_N \nabla \cdot \left(\frac{3}{2}A_N^k - \frac{1}{2}A_N^{k-1}\right) - \frac{1}{2}\nu\tau^2\delta_t\Delta^2 \tilde{p}_N^{k-1/2} \\ \quad - \frac{1}{2}\nu\tau^2\delta_t\Delta^2 P_N^{k-1/2} \\ \quad = -\Pi_N\nabla\cdot(R_{N,1}^k + R_{N,2}^k + R_{N,3}^k), \quad 1 \leqslant k \leqslant M-1. \end{cases} \tag{5.6}$$

It is clear that

$$\int_\Omega \Delta\left(\nabla p_N^{k+1/2}\right)\left(\tilde{u}_N^k + \tilde{u}_N^{k+1}\right) dx = 0.$$

By taking the scalar product on the first equation of (5.6) with $\tilde{u}_N^k + \tilde{u}_N^{k+1}$, we obtain a basic energy equality which is exactly the same as (4.7). Then the conclusion follows directly. □

Theorem 5.2. *Assume that the conditions in Theorem* 4.2 *are satisfied and* $P \in H^1(0,T; H_p^3(\Omega))$*. Then there exist positive constants* c_1 *and* c_2 *depending on* ν, T*, and the norms of* U, P*, and* f *in the mentioned spaces, such that for all* $\tau \leqslant c_1 N^{-n/4}$ *and* $k \leqslant M$*,*

$$\tau \sum_{j=0}^{k} \left\| P^{j+1/2} - p_N^{j+1/2} \right\|_1^2 \leqslant c_2(\tau^4 + \tau^{2\alpha} + N^{-2r}),$$

$$\tau \sum_{j=0}^{k} \left\| P^{j+1/2} - p_{N,c}^{j+1/2} \right\|_1^2 \leqslant c_3(\tau^4 + N^{-2r}),$$

where c_3 *is independent of* $\|P\|_{H^1(0,T;H_p^3(\Omega))}$*.*

Proof. Taking the scalar product on both sides of the second equation of (5.6) with $\tilde{p}_N^{k+1/2}$ gives

$$\left|\tilde{p}_N^{k+1/2}\right|_1^2 + \frac{1}{4}\nu\tau^2\delta_t\left|\tilde{p}_N^{k-1/2}\right|_2^2 + \frac{1}{4}\nu\tau^3\left|\delta_t\tilde{p}_N^{k-1/2}\right|_2^2$$
$$\leqslant \left|\left(\frac{3}{2}A_N^k - \frac{1}{2}A_N^k + R_{N,1}^k + R_{N,2}^k + R_{N,3}^k, \nabla\cdot p_N^{k+1/2}\right)\right| + c\nu^2\tau^4\|\delta_t P^{k-1/2}\|_3^2.$$

It is easy to see that

$$\tau\sum_{j=1}^{k}\|\delta_t P^{j-1/2}\|_3^2 \leqslant c\|P\|_{H^1(0,T;H_p^3(\Omega))}^2.$$

Then the first conclusion of Theorem 5.2 follows immediately from (5.4) and a similar proof to Theorem 4.2.

Next, by (5.5) and the second equation of (5.6), we have

$$\Delta\tilde{p}_{N,c}^{k+1/2} + \Pi_N\nabla\cdot\left(\frac{3}{2}A_N^k - \frac{1}{2}A_N^{k-1}\right) = -\Pi_N\nabla\cdot\left(R_{N,1}^k + R_{N,2}^k + R_{N,3}^k\right).$$

This, along with the same argument as used in the second part of the proof of Theorem 4.3, leads to the second conclusion of Theorem 5.2. □

Remark 5.1. One can see from Theorems 5.1 and 5.2, the choice of $p_N^{1/2}$ does not affect the accuracy of u_N^k. On the other hand, $p_N^{k+1/2}$ and $p_{N,c}^{k+1/2}$ have the same accuracy if $\alpha \geqslant 2$. In other words, $p_N^{k+1/2}$ is corrected for $k \geqslant 1$ automatically.

Like Theorem 4.3, we have the following result for $n \leqslant 4$.

Theorem 5.3. *If $n \leqslant 4$ and the regularity of Theorem 4.3 is fulfilled, then for all $\tau \leqslant c_1 N^{-n/4}$ and $k \leqslant M$,*

$$\left\|U^k - u_N^k\right\|_2 \leqslant c_2(\tau^2 + N^{-r}),$$

where c_1 and c_2 are the same as in Theorem 4.3. Moreover, if $P \in L^\infty(0,T;H_p^{r+2}(\Omega)) \cap H^1(0,T;H_p^3(\Omega))$, then for all $k \leqslant M$,

$$\left\|P^{k+1/2} - p_N^{k+1/2}\right\|_2 \leqslant c_3(\tau^2 + \tau^\alpha + N^{-r}),$$

where c_3 depends on c_2 and $\|P\|_{L^\infty(0,T;H_p^{r+2}(\Omega))\cap H^3(0,T;H_p^4(\Omega))}$. Furthermore,

$$\left\|P^{k+1/2} - p_{N,c}^{k+1/2}\right\|_2 \leqslant c_4(\tau^2 + N^{-r}),$$

where c_4 depends on c_2 and $\|P\|_{L^\infty(0,T;H_p^{r+2}(\Omega))}$.

Theorem 5.3 implies the following error estimates in the maximum norm.

Corollary 5.1. *Under the same assumptions as in Theorem* 5.3, *we have*

$$\|U^k - u_N^k\|_{L^\infty}, \qquad \big\|P^{k+1/2} - p_{N,c}^{k+1/2}\big\|_{L^\infty} \leqslant C(\tau^2 + N^{-r}),$$
$$\big\|P^{k+1/2} - p_N^{k+1/2}\big\|_{L^\infty} \leqslant C(\tau^2 + \tau^\alpha + N^{-r}).$$

Remark 5.2. We notice from (5.5) that

$$\tilde{p}_N^{k+1/2} = \tilde{p}_{N,c}^{k+1/2} + \frac{1}{2}\nu\tau^2\delta_t \Delta p_N^{k-1/2}.$$

If $\alpha \geqslant 1$, then $\delta_t \Delta p_N^{k+1/2}$ is bounded uniformly for all x, k, N, and $\tau \leqslant c_1 N^{-n/4}$. Consequently, we have

$$\big\|P^{k+1/2} - p_N^{k+1/2}\big\|_{L^\infty} \leqslant c_3(\tau^2 + N^{-r}).$$

Thus $p_N^{k+1/2}$ has the same accuracy as u_N^k. This gives a new important feature of the projection method based on the pressure increment formulation. This seems to be the first time to observe such a nice feature.

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