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Unique continuation from a generalized impedance edge-corner for Maxwell's system and applications to inverse problems

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Abstract

We consider the time-harmonic Maxwell system in a domain with a generalized impedance edge-corner, namely the presence of two generalized impedance planes that intersect at an edge. The impedance parameter can be 0, ∞ or a finite non-identically vanishing function. We establish an accurate relationship between the vanishing order of the solutions to the Maxwell system and the dihedral angle of the edge-corner. In particular, if the angle is irrational, the vanishing order is infinity, i.e. strong unique continuation holds from the edge-corner. The establishment of those new quantitative results involve a highly intricate and subtle algebraic argument. The unique continuation study is strongly motivated by our study of a longstanding inverse electromagnetic scattering problem. As a significant application, we derive several novel unique identifiability results in determining a polyhedral obstacle as well as its surface impedance by a single far-field measurement. We also discuss another potential and interesting application of our result in the inverse scattering theory related to the information encoding.

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Keywords: Maxwell's system, generalized impedance plane, edge-corner, vanishing order, inverse electromagnetic scattering, single far-field measurement

(Some figures may appear in colour only in the online journal)

1. Introduction

We are concerned with the unique continuation property (UCP) of the time-harmonic Maxwell system

$$\nabla \wedge \mathbf{E} - \mathbf{i}k\mathbf{H} = \mathbf{0}, \qquad \nabla \wedge \mathbf{H} + \mathbf{i}k\mathbf{E} = \mathbf{0} \quad \text{in } \Omega, \tag{1.1}$$

where $\mathbf{i} := \sqrt{-1}$ and $k \in \mathbb{R}_+$, in a particular scenario that is strongly motivated by our study of a long-standing problem in the inverse electromagnetic scattering theory (section 6). We start with the necessary mathematical setup. We consider the domain Ω to be an open set in $\mathbb{R}^3$, bounded or unbounded, and the solution $(\mathbf{E}, \mathbf{H})$ to the system (1.1) in the space $H_{\text{loc}}(\text{curl}, \Omega)$ defined by

$$H_{\text{loc}}(\text{curl}, \Omega) = \{U|_B \in H(\text{curl}, B); B \text{ is any bounded subdomain of } \Omega\},$$

$$H(\text{curl}, B) = \left\{U \in L^2(B)^3;\ \nabla \wedge U \in L^2(B)^3\right\}.$$

We will often use $B_\rho(\mathbf{x})$ for a ball of radius $\rho \in \mathbb{R}_+$ and centred at $\mathbf{x} \in \mathbb{R}^3$. For a set $K \subset \mathbb{R}^3$, $B_\rho(K) := \{\mathbf{x}; \mathbf{x} \in B_\rho(\mathbf{y}) \text{ for any } \mathbf{y} \in K\}$. Let Π_1 and Π_2 be two planes in $\mathbb{R}^3$ such that $\Pi_1 \cap \Pi_2 = \boldsymbol{L}$, where $\boldsymbol{L}$ is a straight line. We suppose that there exists an open line segment $\boldsymbol{l} \Subset \boldsymbol{L}$ and $\rho \in \mathbb{R}_+$ such that $B_\rho(\boldsymbol{l}) \Subset \Omega$. Let $\mathcal{W}(\Pi_1, \Pi_2)$ denote one of the wedge domains formed by Π_1 and Π_2, then $\partial\mathcal{W}(\Pi_1, \Pi_2) \cap B_\rho(\boldsymbol{l})$ is called an edge-corner associated with Π_1 and Π_2; see figure 1 for a schematic illustration. In the sequel, we let $\widetilde{\Pi}_j$, $j = 1, 2$, denote the two flat faces of the edge-corner lying on Π_j, respectively, and write the edge-corner as $\mathcal{E}(\widetilde{\Pi}_1, \widetilde{\Pi}_2, \boldsymbol{l})$. Any $\mathbf{x} \in \boldsymbol{l}$ is said to be an edge-corner point of $\mathcal{E}(\widetilde{\Pi}_1, \widetilde{\Pi}_2, \boldsymbol{l})$.

Let $\boldsymbol{\eta}_j$ denote a generalized impedance parameter on $\widetilde{\Pi}_j$, which can be one of the three cases:

$$\text{(i) } \boldsymbol{\eta}_j \equiv 0; \qquad \text{(ii) } \boldsymbol{\eta}_j \equiv \infty; \qquad \text{(iii) } \boldsymbol{\eta}_j \in L^\infty(\widetilde{\Pi}_j). \tag{1.2}$$

Let $\nu_j \in \mathbb{S}^2$ be the unit normal vector to Π_j, pointing to the exterior of $\mathcal{W}(\Pi_1, \Pi_2)$. We are interested in the *generalized impedance condition* on $\widetilde{\Pi}_j$ associated with $(\mathbf{E}, \mathbf{H})$ to the Maxwell system (1.1), i.e.,

$$\nu_j \wedge (\nabla \wedge \mathbf{E}) + \boldsymbol{\eta}_j (\nu_j \wedge \mathbf{E}) \wedge \nu_j |_{\widetilde{\Pi}_j} = 0. \tag{1.3}$$

In the case $\boldsymbol{\eta}_j \equiv \infty$, (1.3) is understood as

$$(\nu_j \wedge \mathbf{E}) \wedge \nu_j |_{\widetilde{\Pi}_j} = 0. \tag{1.4}$$

An edge-corner $\mathcal{E}(\widetilde{\Pi}_1, \widetilde{\Pi}_2, \boldsymbol{l})$ with the generalized impedance condition (1.3) imposed on $\widetilde{\Pi}_j$, $j = 1, 2$, is called a generalized impedance edge-corner associated with the Maxwell system (1.1). The main purpose of the current work is to first establish the UCP of the solution $(\mathbf{E}, \mathbf{H})$ to (1.1) with the presence of a generalized impedance edge-corner, then apply the result to a long-standing problem in the inverse electromagnetic scattering theory (in section 6).

The UCP for differential equations from a crack in a domain has been the subject of many existing studies in the literature; see, e.g., [2, 10, 11] and the references therein. However,

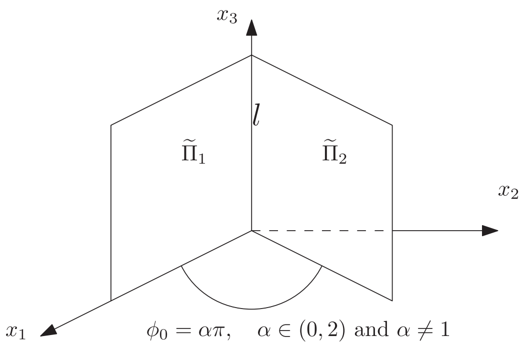

Figure 1. Schematic illustration of two intersecting planes with an edge-corner $\mathcal{E}(\widetilde{\Pi}_1, \widetilde{\Pi}_2, \boldsymbol{l})$ and the dihedral angle ϕ_0.

the corresponding study to the Maxwell system is rare. In addition, there are several important features that make our current study interestingly new and distinct from most existing UCP studies from cracks. First, the Maxwell system (1.1) is defined in the whole domain Ω, instead of the exterior of the crack, namely $\Omega\backslash\mathcal{E}(\widetilde{\Pi}_1, \widetilde{\Pi}_2, \boldsymbol{l})$. Usually, for a typical UCP problem from a crack, the differential equation is given over the exterior of the crack, and hence the solution inherits a certain singularity from the pathological geometry of the crack. But in our case, by the standard PDE theory, we know that $(\mathbf{E}, \mathbf{H})$ are real analytic in the interior of Ω, and in particular in $B_\rho(\boldsymbol{l})$ which is a neighbourhood of the edge-corner. This makes our UCP study seemingly rather 'artificial'. However, on the one hand, the UCP problem in this work is strongly motivated by our study of the inverse electromagnetic scattering problems. This shall become more evident in section 6, and the UCP results produce two significant applications that are of both theoretical and practical interest. On the other hand, it turns out that the analyticity of the solutions around the edge-corner is a key factor that helps us develop a highly intricate and subtle algebraic argument in achieving the desired UCP. Second, the edge-corner geometry enables us to establish an accurate relationship between the vanishing order of the solutions to the Maxwell system and the angle of the edge-corner. In particular, if the angle is irrational, then the vanishing order is infinity, i.e., strong unique continuation holds from the edge-corner. Third, it is remarked that in our UCP study, the Robin-type generalized impedance condition (1.3) is considered on the crack, namely the edge-corner, whereas in most of the existing studies of UCP from cracks, homogeneous Dirichlet-type or Neumann-type conditions are more concerned, which correspond to $\boldsymbol{\eta} \equiv 0$ or $\boldsymbol{\eta} \equiv \infty$, respectively.

As mentioned earlier, we shall consider two interesting and significant applications of the newly established UCP results to the study of inverse electromagnetic scattering problems. We postpone the mathematical formulation of the inverse problem to section 6, which is mainly concerned with the unique determination of an impenetrable obstacle as well as its boundary impedance by a single electromagnetic far-field measurement. This constitutes a longstanding problem in the inverse scattering theory (see [9]). The case $\boldsymbol{\eta} \equiv 0$ or $\boldsymbol{\eta} \equiv \infty$ was studied in [17, 19, 20], and it is shown that a single far-field measurement can uniquely determine an obstacle of the general polyhedral shape and the corresponding stability estimate was established in [18]. The proofs are mainly based on the path argument that was originated in [22] for the inverse acoustic problem as well as a certain reflection principle for the Maxwell system that was established in [19, 20]. However, the arguments developed therein cannot be extended to tackle the case that the impedance parameter $\boldsymbol{\eta}$ is finite and non-identically zero, especially when it is a variable function, which constitutes an open problem in the literature [9]. Using the UCP results derived in this paper, we are able to establish several novel unique identifiability results for this challenging problem in the general polyhedral case, especially in the case that $\boldsymbol{\eta}$ is a finite and non-identically zero variable function. Nevertheless, it is our intention to point out that we shall require certain mild but unobjectionable *a-priori* knowledge of the underlying polyhedral obstacle as well as its surface impedance. The other interesting application of our UCP results is about the 'information encoding' for the inverse electromagnetic scattering problems. Indeed, we shall regard our UCP results as generalizing the classical Holmgren's principle [8, 24] for the Maxwell equations. With this view, we can provide an alternative means of electromagnetic scattering measurements for inverse problems that might have some practical implications.

Finally, we briefly discuss the technical aspects of our work. In deriving the UCP results, we make essential use of the Fourier series representations of the solutions $\mathbf{E}$ and $\mathbf{H}$ to (1.1) in terms of the spherical waves locally around an edge-corner point. The homogeneous generalized impedance conditions (1.3) on the two faces of the edge-corner applied to the Fourier series shall generate certain recursive relations of the Fourier coefficients. The recursive relations are highly intricate and are hardly seen to connect to the vanishing order of the solutions, *a fortiori*, to connect to the dihedral angle of the edge-corner. Nevertheless, through subtle analysis and tedious calculations, we manage to decode the useful information from those algebraic recursive relations and establish the desired UCP results. Two remarks are in order. First, the analysis involves highly tedious calculations and in fact, when the impedance parameter $\boldsymbol{\eta}_j$ in (1.3) changes, say from 0 to ∞, the corresponding analysis requires completely new calculations. In what follows, in order to present a comprehensive study, we shall only present the major ingredients of the computations involved and skip most of the tedious details. Second, the algebraic arguments enable us to derive an accurate relationship between the vanishing order of the solutions to the Maxwell system and the dihedral angle of the edge-corner. Finally, in order to establish the unique determination results for the inverse scattering problem, we combine the newly established UCP results with the path argument mentioned before.

The rest of the paper is organized as follows. Section 2 is devoted to some preliminary knowledge and auxiliary results. In sections 3 and 4, we establish the UCP results from a generalized impedance edge-corner for the Maxwell equation (1.1) in two different scenarios. In section 6, we consider the inverse electromagnetic scattering problems and present two applications of the newly established UCP results.

2. Preliminaries and auxiliary lemmas

In this section, we collect some preliminary knowledge for the Maxwell system (1.1) as well as derive several auxiliary lemmas for our subsequent use.

First, we note that the Maxwell system (1.1) is invariant under rigid motions (see [3, 21]). Hence, throughout the rest of this paper and without loss of generality, we can assume that the edge-corner $\mathcal{E}(\widetilde{\Pi}_1, \widetilde{\Pi}_2, \boldsymbol{l}) \Subset \Omega$ of our interest is of the following form:

$$\boldsymbol{l} = \left\{ \mathbf{x} = (\mathbf{x}', x_3) \in \mathbb{R}^3; \mathbf{x}' := (x_1, x_2) = \mathbf{0},\ x_3 \in (-h, h) \right\} \Subset \Omega,$$

where $2h \in \mathbb{R}_+$ is the length of $\boldsymbol{l}$, and furthermore Π_1 coincides with the (x_1, x_3)-plane while Π_2 possesses a dihedral angle $\phi_0 = \alpha\pi$ away from Π_1 in the anti-clockwise direction; see figure 1 for a schematic illustration. Throughout the paper, we assume that

$$\alpha \in (0,2) \quad \text{but} \quad \alpha \neq 1. \tag{2.1}$$

It can be directly verified that the exterior unit normal vectors ν_j to Π_j, $j = 1, 2$ are given by

$$\nu_1 = (0, -1, 0)^\top, \qquad \nu_2 = (-\sin\phi_0, \cos\phi_0, 0)^\top. \tag{2.2}$$

As specified earlier, we have the generalized impedance condition (1.3) imposed on $\widetilde{\Pi}_j$, where the boundary impedance parameter $\boldsymbol{\eta}_j$ fulfils (1.2). In order to consider the unique continuation from the edge-corner as described above, we introduce the following definition.

Definition 2.1. Let $\mathbf{E} \in H_{\mathrm{loc}}(\mathrm{curl}, \Omega)$ be a solution to (1.1) and suppose there exists an edge-corner $\mathcal{E}(\widetilde{\Pi}_1, \widetilde{\Pi}_2, \boldsymbol{l}) \Subset \Omega$ as described above. For a given point $\mathbf{x}_0 \in \boldsymbol{l}$, if there exists a number $N \in \mathbb{N} \cup \{0\}$ such that

$$\lim_{\rho \to +0} \frac{1}{\rho^m} \int_{B_\rho(\mathbf{x}_0)} |\mathbf{E}(\mathbf{x})| \, \mathrm{d}\mathbf{x} = 0 \quad \text{for } m = 0, 1, \ldots, N+2, \tag{2.3}$$

we say that $\mathbf{E}$ vanishes at $\mathbf{x}_0$ up to the order N. The largest possible N such that (2.3) is fulfiled is called the vanishing order of $\mathbf{E}$ at $\mathbf{x}_0$, and we write

$$\mathrm{Vani}(\mathbf{E}; \mathbf{x}_0) = N.$$

If (2.3) holds for any $N \in \mathbb{N}$, we say that the vanishing order is infinity.

Since $\mathbf{E}$ is (real) analytic in Ω, we immediately see that if the vanishing order of $\mathbf{E}$ at any point $\mathbf{x}_0 \in \boldsymbol{l}$ is infinity, then $\mathbf{E} \equiv 0$ in Ω, namely the strong UCP holds. In what follows, it is sufficient to consider the UCP at the origin $\mathbf{0} \in \boldsymbol{l}$. Moreover, due to the symmetry role between $(\mathbf{E}, \mathbf{H})$ and $(-\mathbf{H}, \mathbf{E})$, namely both of them satisfy the same Maxwell system (1.1), we only consider the vanishing order of $\mathbf{E}$, and the same result equally holds for $\mathbf{H}$. It turns out that the vanishing order of $\mathbf{E}$ is related to the *rationality* of the edge-corner angle, i.e. $\alpha\pi$, and we shall make it more rigorous in the sequel.

In the subsequent analysis, we will often use the spherical coordinates of a point $\mathbf{x}$ in $\mathbb{R}^3$:

$$\mathbf{x} = (r\sin\theta\cos\phi, r\sin\theta\sin\phi, r\cos\theta) := (r, \theta, \phi), \ \ r \geqslant 0, \theta \in [0, \pi), \phi \in [0, 2\pi). \tag{2.4}$$

Let $\hat{\mathbf{x}} = (1, 0, 0)^\top, \hat{\mathbf{y}} = (0, 1, 0)^\top, \hat{\mathbf{z}} = (0, 0, 1)^\top$. Then we know

$$\begin{aligned} \hat{\boldsymbol{r}} &= \sin\theta\cos\phi \cdot \hat{\mathbf{x}} + \sin\theta\sin\phi \cdot \hat{\mathbf{y}} + \cos\theta \cdot \hat{\mathbf{z}} \\ \hat{\boldsymbol{\theta}} &= \cos\theta\cos\phi \cdot \hat{\mathbf{x}} + \cos\theta\sin\phi \cdot \hat{\mathbf{y}} - \sin\theta \cdot \hat{\mathbf{z}} \\ \hat{\boldsymbol{\phi}} &= -\sin\phi \cdot \hat{\mathbf{x}} + \cos\phi \cdot \hat{\mathbf{y}} \end{aligned} \tag{2.5}$$

constitutes an orthonormal basis in the spherical coordinate system.

Definition 2.2. Suppose that $\psi(r, \theta)$ is a complex-valued function for $(r, \theta) \in \Sigma := [0, r_0] \times [-\theta_0, \theta_0]$, where $r_0, \theta_0 \in \mathbb{R}_+$. ψ is said to belong to class $\mathcal{A}$ in Σ if it allows an absolutely convergent series representation as follows

$$\psi(r, \theta) = a_0 + \sum_{j=1}^{\infty} a_j(\theta) r^j, \tag{2.6}$$

where $a_0 \in \mathbb{C}\backslash\{0\}$ and $a_j(\theta) \in C[-\theta_0, \theta_0]$.

Here are two simple scenarios for $\psi(r,\theta)$ to belong to the class $\mathcal{A}$: first, ψ is a non-zero constant; second, $\psi(r,\theta)$ is real-analytic in Σ with r_0, θ_0 sufficiently small and $\psi(0,\theta)$ independent of θ. For an impedance parameter $\boldsymbol{\eta}_j$ in (1.3) in the third case, namely $\boldsymbol{\eta}_j \in L^\infty(\widetilde{\Pi}_j)$, we readily see that in the (r,θ,ϕ)-coordinate, $\phi|_{\widetilde{\Pi}_1} = 0$ and $\phi|_{\widetilde{\Pi}_2} = \phi_0$. In what follows, if for any $\mathbf{x}_0 \in \boldsymbol{l}$ there exists a neighbourhood $\Sigma_{\mathbf{x}_0}$ of $\mathbf{x}_0$ which is of the form in definition 2.2 and is contained in $\overline{\widetilde{\Pi}}_j$ such that $\psi_{\mathbf{x}_0}(r,\theta) := \boldsymbol{\eta}_j(\mathbf{x}-\mathbf{x}_0)$ belongs to the class $\mathcal{A}$ in $\Sigma_{\mathbf{x}_0}$, then we say that η_j belongs to the class $\mathcal{A}(\boldsymbol{l})$. It is emphasized that $\boldsymbol{\eta}_j$ belonging to the class $\mathcal{A}(\boldsymbol{l})$ is a local property, which is localized around a neighbourhood of $\boldsymbol{l}$ on $\widetilde{\Pi}_j$. In fact, our subsequent analysis of the UCP from the edge-corner $\mathcal{E}(\widetilde{\Pi}_1, \widetilde{\Pi}_2, \boldsymbol{l})$ is confined locally around a neighbourhood of $\boldsymbol{l}$, and indeed, around a neighbourhood of the origin $\mathbf{0}$ according to our earlier discussion.

Next, we consider the Fourier representations of the solutions to (1.1) in terms of the spherical waves. Throughout the rest of the paper, for a fixed $l \in \mathbb{N}$ we adopt the notation

$$[l]_0 := \{0, \pm 1, \ldots, \pm l\}, \qquad [l]_1 := \{\pm 1, \ldots, \pm l\}. \tag{2.7}$$

Recall that the spherical harmonics $Y_l^m(\theta,\phi)$ are given by

$$Y_l^m(\theta,\phi) = c_l^m P_l^{|m|}(\cos\theta)\mathrm{e}^{\mathbf{i}m\phi}, \quad c_l^m = \sqrt{\frac{2l+1}{4\pi}\frac{(l-|m|)!}{(l+|m|)!}}, \tag{2.8}$$

where $P_l^m(t)$ is the Legendre function. For simplicity, we shall often write the notation Y_l^m for $Y_l^m(\theta,\phi)$. For our subsequent use, the following lemma presents some important properties of the associated Legendre functions [1].

Lemma 2.3. *In the spherical coordinate system, the Legendre functions fulfil the following orthogonality conditions for any fixed $n \in \mathbb{N}$, and any two integers $m \geqslant 0$ and $l \leqslant n$:*

$$\int_{-\pi}^{\pi} \frac{P_n^m(\cos\theta) P_n^l(\cos\theta)}{\sin\theta}\,\mathrm{d}\theta = \begin{cases} 0 & \text{if } l \neq m, \\ \dfrac{(n+m)!}{m(n-m)!} & \text{if } l = m \neq 0. \end{cases} \tag{2.9}$$

Furthermore, the following recursive relationships hold for $l \in \mathbb{N}$ and $m \in [l]_0$,

$$\begin{aligned} \frac{\mathrm{d}P_l^{|m|}(\cos\theta)}{\mathrm{d}\theta} &= \frac{1}{2}\left[(l+|m|)(l-|m|+1)P_l^{|m|-1}(\cos\theta) - P_l^{|m|+1}(\cos\theta)\right], \\ \frac{|m|}{\sin\theta}P_l^{|m|}(\cos\theta) &= -\frac{1}{2}\left[P_{l-1}^{|m|+1}(\cos\theta) + (l+|m|-1)(l+|m|)P_{l-1}^{|m|-1}(\cos\theta)\right], \end{aligned} \tag{2.10}$$

If $P_l^m(\cos\theta)$ is evaluated at $\theta = 0$, for $l \in \mathbb{N}\cup\{0\}$ we have

$$P_l^m(1) = 0, \quad m \in [l]_1; \quad P_l^0(1) = 1. \tag{2.11}$$

For a fixed $n \in \mathbb{N}\cup\{0\}$ and $m \in \mathbb{N}$ with $m \leqslant n$, it holds that

$$P_n^{-m}(\cos\theta) = (-1)^m \frac{(n-m)!}{(n+m)!}P_n^m(\cos\theta). \tag{2.12}$$

Recall that the spherical Bessel function $j_\ell(t)$ of the order ℓ is defined by

$$j_\ell(t)=\frac{t^\ell}{(2\ell+1)!!}\left(1-\sum_{l=1}^{\infty}\frac{(-1)^l t^{2l}}{2^l l!(2\ell+3)\cdots(2\ell+2l+1)}\right)=\frac{t^\ell}{(2\ell+1)!!}+\mathcal{O}(t^{\ell+2}). \tag{2.13}$$

There holds the following recursive relationships [1]:

$$\frac{j_\ell(t)}{t}=\frac{j_{\ell-1}(t)+j_{\ell+1}(t)}{2\ell+1},\qquad j_\ell'(t)=\frac{\ell j_{\ell-1}(t)-(\ell+1)j_{\ell+1}(t)}{2\ell+1},\quad \ell\in\mathbb{N}. \tag{2.14}$$

Lemma 2.4. [6, *lemma 2.5*]. *Suppose that for* $t\in(0,h)$, $h\in\mathbb{R}_+$,

$$\sum_{n=0}^{\infty}\alpha_n j_n(t)=0,$$

where $j_n(t)$ *is the nth spherical Bessel function. Then we have* $\alpha_n=0, n=0,1,2,\ldots$

Lemma 2.5. [8]. *Recall that* $\hat{\boldsymbol{r}},\hat{\boldsymbol{\theta}}$ *and* $\hat{\boldsymbol{\phi}}$ *are defined in* (2.5). *Denote*

$$\mathbf{M}_l^m(\mathbf{x})=j_l(kr)\cdot\mathbf{X}_l^m,\qquad \mathbf{N}_l^m(\mathbf{x})=\mathbf{i}\left(\frac{j_l(kr)}{kr}+j_l'(kr)\right)\mathbf{Z}_l^m$$
$$-\frac{\sqrt{l(l+1)}}{kr}\cdot j_l(kr)Y_l^m\cdot\hat{\boldsymbol{r}}, \tag{2.15}$$

where $k\in\mathbb{R}_+$, $j_l'(kr)$ *is the derivative of* $j_l(kr)$ *with respect to* kr, *and*

$$\mathbf{X}_l^m=\frac{\mathbf{i}}{\sqrt{l(l+1)}}\left(\frac{\mathbf{i}\cdot m}{\sin\theta}Y_l^m\hat{\boldsymbol{\theta}}-\frac{\partial Y_l^m}{\partial\theta}\cdot\hat{\boldsymbol{\phi}}\right),$$
$$\mathbf{Z}_l^m=\frac{\mathbf{i}}{\sqrt{l(l+1)}}\left(\frac{\partial Y_l^m}{\partial\theta}\hat{\boldsymbol{\theta}}+\frac{\mathbf{i}\cdot m}{\sin\theta}Y_l^m\hat{\boldsymbol{\phi}}\right).$$

The solution $\mathbf{E}(\mathbf{x})$ *to* (1.1) *has the following Fourier expansion around* $\mathbf{0}$,

$$\mathbf{E}(\mathbf{x})=\sum_{l=1}^{\infty}\sum_{m=-l}^{l}\left(a_l^m\cdot\mathbf{M}_l^m(\mathbf{x})+b_l^m\cdot\mathbf{N}_l^m(\mathbf{x})\right),\quad a_l^m,b_l^m\in\mathbb{C},$$

which (along with its derivatives) converges uniformly in $B_{\rho_0}(\mathbf{0})$ *for a sufficiently small* $\rho_0\in\mathbb{R}_+$.

In this paper, we adopt the following notation

$$\sum_{l,m,N}=\sum_{l=N}^{+\infty}\sum_{m=-l}^{l},\quad N\in\mathbb{N}.$$

Using (2.14), from lemma 2.5, we can derive that

$$\mathbf{E}(\mathbf{x})=-\sum_{l,m,1}\frac{1}{\sqrt{l(l+1)}}\left\{b_l^m\cdot l(l+1)p_l(kr)\cdot Y_l^m\cdot\hat{\boldsymbol{r}}+\left[a_l^m\cdot j_l(kr)\frac{m}{\sin\theta}Y_l^m\right.\right.$$
$$\left.\left.+b_l^m\cdot q_l(kr)\cdot\frac{\partial Y_l^m}{\partial\theta}\right]\cdot\hat{\boldsymbol{\theta}}+\mathbf{i}\left[a_l^m\cdot j_l(kr)\frac{\partial Y_l^m}{\partial\theta}+b_l^m\cdot q_l(kr)\frac{m}{\sin\theta}Y_l^m\right]\cdot\hat{\boldsymbol{\phi}}\right\}, \tag{2.16}$$

where

$$p_l(kr) = \frac{j_{l-1}(kr) + j_{l+1}(kr)}{2l+1}, \qquad q_l(kr) = \frac{(l+1)j_{l-1}(kr) - l\,j_{l+1}(kr)}{2l+1}. \tag{2.17}$$

Remark 2.6. In view of (2.13), we know the lowest order terms of $p_l(kr)$ and $q_l(kr)$ with respect to the power of r are given, respectively, by

$$\frac{k^{l-1}}{(2l+1)(2l-1)!!}r^{l-1} \quad \text{and} \quad \frac{(l+1)k^{l-1}}{(2l+1)(2l-1)!!}r^{l-1}.$$

Lemma 2.7. *[16, proposition 2.1.7]. If the power series $\sum_\mu a_\mu \mathbf{x}^\mu$ converges at a point $\mathbf{x}_0$, then it converges uniformly and absolutely on compact subsets of $U(\mathbf{x}_0)$, where*

$$U(\mathbf{x}_0) = \{(r_1 x_{0,1}, \dots, r_n x_{0,n}) : -1 < r_j < 1, j = 1, \dots, n\},$$
$$\mathbf{x}_0 = (x_{0,1}, \dots, x_{0,n}) \in \mathbb{R}^n.$$

Using definition 2.1 and (2.16), we can obtain the following lemma.

Lemma 2.8. *Let $\mathbf{E}$ be a solution to (1.1). Recall that $\mathbf{E}$ has the radial wave expansion (2.16) in $B_{\rho_0}(\mathbf{0})$. For a fixed $N \in \mathbb{N}$, if*

$$a_l^m = b_l^m = 0, \quad m \in [l]_0, \quad l = 1, 2, \dots, N, \tag{2.18}$$

where $[l]_0$ is defined in (2.7), then

$$\mathrm{Vani}(\mathbf{E}; \mathbf{0}) \geqslant N. \tag{2.19}$$

Conversely, if there exists $N \in \mathbb{N}$ such that (2.19) holds, then we have (2.18).

The proof of the lemma 2.8 is based on straightforward but tedious calculations which we choose to skip in the present paper. Nevertheless, we refer interested readers to the arXiv version of this paper [14] for the relevant details. Similarly, we present the following two lemmas and refer to [14] for the detailed proofs.

Lemma 2.9. *Let $\mathbf{E}$ be a solution to (1.1). Recall that $\mathbf{E}$ has the radial wave expansion (2.16) in $B_{\rho_0}(\mathbf{0})$. Consider an edge-corner $\mathcal{E}(\widetilde{\Pi}_1, \widetilde{\Pi}_2, \boldsymbol{l}) \Subset \Omega$ associated with $\mathbf{E}$, with ν_i defined in (2.2) being the outward unit normal vectors to Π_i, $i = 1, 2$. Then it holds that*

$$\begin{aligned}
\nu_1 \wedge \mathbf{E}|_{\widetilde{\Pi}_1} &= \sum_{l,m,1} -\frac{1}{\sqrt{l(l+1)}}\Bigg\{ b_l^m l(l+1) p_l(kr) Y_l^m|_{\phi=0}\, \boldsymbol{e}_1(\theta, 0) \\
&\quad + \left(a_l^m j_l(kr) \frac{m}{\sin\theta} Y_l^m \bigg|_{\phi=0} + b_l^m \cdot q_l(kr) \frac{\partial Y_l^m}{\partial\theta}\bigg|_{\phi=0} \right) \boldsymbol{e}_2(\theta, 0) \Bigg\}, \\
\nu_2 \wedge \mathbf{E}|_{\widetilde{\Pi}_2} &= \sum_{l,m,1} -\frac{1}{\sqrt{l(l+1)}}\Bigg\{ b_l^m l(l+1) p_l(kr) Y_l^m|_{\phi=\phi_0}\, \boldsymbol{e}_1(\theta, \phi_0) \\
&\quad + \left(a_l^m j_l(kr) \frac{m}{\sin\theta} Y_l^m \bigg|_{\phi=\phi_0} + b_l^m \cdot q_l(kr) \frac{\partial Y_l^m}{\partial\theta}\bigg|_{\phi=\phi_0} \right) \boldsymbol{e}_2(\theta, \phi_0) \Bigg\},
\end{aligned} \tag{2.20}$$

where

$$\boldsymbol{e}_1(\theta, \phi) = [\cos\phi \cos\theta, \sin\phi \cos\theta, -\sin\theta]^\top,$$
$$\boldsymbol{e}_2(\theta, \phi) = -[\cos\phi \sin\theta, \sin\phi \sin\theta, \cos\theta]^\top, \tag{2.21}$$

are linearly independent for any θ and ϕ. Furthermore, we have

$$
\begin{aligned}
\nu_1\wedge(\nabla\wedge\mathbf{E}|_{\widetilde{\Pi}_1}) &= \mathbf{i}k\sum_{l,m,1}\frac{1}{\sqrt{l(l+1)}}\left\{a_l^m l(l+1)p_l(kr)Y_l^m|_{\phi=0}\cdot\boldsymbol{e_1}(\theta,0)\right.\\
&\quad+\left(-b_l^m j_l(kr)\cdot\frac{m}{\sin\,\theta}Y_l^m+a_l^m\cdot q_l(kr)\right.\\
&\quad\left.\left.\times\cdot\frac{\partial Y_l^m}{\partial\theta}\bigg|_{\phi=0}\right)\cdot\boldsymbol{e_2}(\theta,0)\right\},\\
\nu_2\wedge(\nabla\wedge\mathbf{E}|_{\widetilde{\Pi}_2}) &= \mathbf{i}k\sum_{l,m,1}\frac{1}{\sqrt{l(l+1)}}\left\{a_l^m l(l+1)p_l(kr)Y_l^m|_{\phi=\phi_0}\cdot\boldsymbol{e_1}(\theta,\phi_0)\right.\\
&\quad+\left(-b_l^m j_l(kr)\cdot\frac{m}{\sin\,\theta}Y_l^m\bigg|_{\phi=\phi_0}+a_l^m q_l(kr)\right.\\
&\quad\left.\left.\times\cdot\frac{\partial Y_l^m}{\partial\theta}\bigg|_{\phi=\phi_0}\right)\cdot\boldsymbol{e_2}(\theta,\phi_0)\right\}.
\end{aligned}
\tag{2.22}
$$

Lemma 2.10. *Under the same notations and conditions as the ones in lemma 2.9, we further assume that $\boldsymbol{\eta}_1,\boldsymbol{\eta}_2$ belong to the class $\mathcal{A}(\boldsymbol{l})$. Then we have*

$$
\begin{aligned}
&\nu_1\wedge(\nabla\wedge\mathbf{E}|_{\widetilde{\Pi}_1})+\boldsymbol{\eta_1}(\nu_1\wedge\mathbf{E}|_{\widetilde{\Pi}_1})\wedge\nu_1=\sum_{l,m,1}\frac{1}{\sqrt{l(l+1)}}\\
&\quad\times\left\{\left(\mathbf{i}ka_l^m l(l+1)p_l(kr)Y_l^m-\boldsymbol{\eta}_1 a_l^m j_l(kr)\frac{m}{\sin\,\theta}Y_l^m\right.\right.\\
&\quad\left.-\boldsymbol{\eta}_1 b_l^m q_l(kr)\frac{\partial Y_l^m}{\partial\theta}\right)\boldsymbol{e_1}(\theta,0)+\left(-\mathbf{i}kb_l^m j_l(kr)\frac{m}{\sin\,\theta}Y_l^m\right.\\
&\quad\left.\left.+\mathbf{i}ka_l^m q_l(kr)\frac{\partial Y_l^m}{\partial\theta}+\boldsymbol{\eta}_1 b_l^m l(l+1)p_l(kr)Y_l^m\right)\cdot\boldsymbol{e_2}(\theta,0)\right\},
\end{aligned}
\tag{2.23}
$$

and

$$
\begin{aligned}
&\nu_2\wedge(\nabla\wedge\mathbf{E}|_{\widetilde{\Pi}_2})+\boldsymbol{\eta_2}(\nu_2\wedge\mathbf{E}|_{\widetilde{\Pi}_2})\wedge\nu_2=\sum_{l,m,1}\frac{1}{\sqrt{l(l+1)}}\\
&\quad\times\left\{\left(\mathbf{i}ka_l^m l(l+1)p_l(kr)Y_l^m-\boldsymbol{\eta_2} a_l^m j_l(kr)\frac{m}{\sin\,\theta}Y_l^m\right.\right.\\
&\quad\left.-\boldsymbol{\eta}_2 b_l^m q_l(kr)\frac{\partial Y_l^m}{\partial\theta}\right)\boldsymbol{e_1}(\theta,\phi_0)+\left(-\mathbf{i}kb_l^m j_l(kr)\frac{m}{\sin\,\theta}Y_l^m\right.\\
&\quad\left.\left.+\mathbf{i}ka_l^m q_l(kr)\frac{\partial Y_l^m}{\partial\theta}+\boldsymbol{\eta_2} b_l^m l(l+1)p_l(kr)Y_l^m\right)\cdot\boldsymbol{e_2}(\theta,\phi_0)\right\},
\end{aligned}
\tag{2.24}
$$

where $\boldsymbol{e_1}(\theta,0),\boldsymbol{e_2}(\theta,0)$ $\boldsymbol{e_1}(\theta,\phi_0)$ and $\boldsymbol{e_2}(\theta,\phi_0)$ are defined in (2.21).

3. Vanishing orders for an edge-corner $\mathcal{E}(\widetilde{\Pi}_1,\widetilde{\Pi}_2,\boldsymbol{l})$ with $\eta_j\in\mathcal{A}(\boldsymbol{l})$

In this section, we consider the case that $\mathcal{E}(\widetilde{\Pi}_1,\widetilde{\Pi}_2,\boldsymbol{l})$ is an edge-corner, with both $\boldsymbol{\eta}_1$ and $\boldsymbol{\eta}_2$ belonging to the class $\mathcal{A}(\boldsymbol{l})$. We shall derive the vanishing order of $\mathbf{E}$ to (1.1) at the origin

$\mathbf{0} \in \boldsymbol{l}$. The major idea is to make use of the radial wave expansion (2.16) of $\mathbf{E}$ in $B_{\rho_0}(\mathbf{0})$, and to investigate the relationships between $a_n^{\pm 1}$, a_n^0 and $b_n^{\pm 1}$, b_n^0. Henceforth, according to definition 2.2, we assume that $\boldsymbol{\eta}_j$, $j = 1, 2$, are given by the following absolutely convergent series at $\mathbf{0} \in \boldsymbol{l}$:

$$\boldsymbol{\eta}_1 = \eta_1 + \sum_{j=1}^{\infty} \eta_{1,j}(\theta) r^j, \qquad \boldsymbol{\eta}_2 = \eta_2 + \sum_{j=1}^{\infty} \eta_{2,j}(\theta) r^j \tag{3.1}$$

where $\eta_\ell \in \mathbb{C}\backslash\{0\}$, $\eta_{\ell,j}(\theta) \in C[-\pi, \pi]$ and $r \in [-h, h]$, $\ell = 1, 2$. Next, based on the above setting, we derive several critical lemmas, whose proofs are based on tedious calculations and can be found in the arXiv version of this paper [14].

Lemma 3.1. *Let $\mathbf{E}$ be a solution to (1.1), whose radial wave expansion in $B_{\rho_0}(\mathbf{0})$ is given by (2.16). Consider a generalized impedance edge-corner $\mathcal{E}(\widetilde{\Pi}_1, \widetilde{\Pi}_2, \boldsymbol{l}) \Subset \Omega$, with $\angle(\Pi_1, \Pi_2) = \phi_0 = \alpha\pi$ for $\alpha \in (0, 2)$ but $\alpha \neq 1$. Suppose that the generalized impedance parameters $\boldsymbol{\eta}_j$ on $\widetilde{\Pi}_j$, $j = 1, 2$, are given by (3.1), then it holds that*

$$0 = \frac{4\mathbf{i}kc_1^1 \sin^2\phi_0}{6\sqrt{2}}(a_1^1 + a_1^{-1}) - \frac{4kc_1^1 \sin\phi_0 \cos\phi_0}{6\sqrt{2}}(a_1^1 - a_1^{-1}) - \frac{(\eta_2 \cos\phi_0 + \eta_1)\sqrt{2}c_1^0}{3} b_1^0, \tag{3.2a}$$

$$0 = -\frac{4\mathbf{i}kc_1^1 \sin\phi_0 \cos\phi_0}{6\sqrt{2}}(a_1^1 + a_1^{-1}) - \frac{4kc_1^1 \sin^2\phi_0}{6\sqrt{2}}(a_1^1 - a_1^{-1}) - \frac{\eta_2\sqrt{2}c_1^0 \sin\phi_0}{3} b_1^0, \tag{3.2b}$$

$$0 = -\frac{4c_1^1(-\eta_1 + \eta_2\cos\phi_0)}{6\sqrt{2}}(b_1^1 + b_1^{-1}) + \frac{4\eta_2 c_1^1 \sin\phi_0 \mathbf{i}}{6\sqrt{2}}(b_1^1 - b_1^{-1}). \tag{3.2c}$$

Assume that there exists $n \in \mathbb{N}\backslash\{1\}$ such that

$$a_l^0 = b_l^0 = a_l^{\pm 1} = b_l^{\pm 1} = 0, \quad l = 1, \ldots, n-1. \tag{3.3}$$

Then it holds that

$$\begin{aligned}\frac{\eta_1\sqrt{n(n+1)}c_n^0}{2n+1} b_n^0 &= \frac{\mathbf{i}kn(n+1)^2 c_n^1 \sin^2\phi_0}{2(2n+1)\sqrt{n(n+1)}}(a_n^1 + a_n^{-1}) \\ &\quad - \frac{kn(n+1)^2 c_n^1 \sin\phi_0 \cos\phi_0}{2(2n+1)\sqrt{n(n+1)}}(a_n^1 - a_n^{-1}) - \frac{\eta_2\sqrt{n(n+1)}c_n^0 \cos\phi_0}{2n+1} b_n^0,\end{aligned} \tag{3.4a}$$

$$\begin{aligned}\frac{kn(n+1)^2 c_n^1}{2(2n+1)\sqrt{n(n+1)}}(a_n^1 - a_n^{-1}) &= -\frac{\mathbf{i}kn(n+1)^2 c_n^1 \sin\phi_0 \cos\phi_0}{2(2n+1)\sqrt{n(n+1)}}(a_n^1 + a_n^{-1}) \\ &\quad + \frac{kn(n+1)^2 c_n^1 \cos^2\phi_0}{2(2n+1)\sqrt{n(n+1)}}(a_n^1 - a_n^{-1}) - \frac{\eta_2\sqrt{n(n+1)}c_n^0 \sin\phi_0}{2n+1} b_n^0,\end{aligned} \tag{3.4b}$$

$$\begin{aligned}-\frac{\eta_1 n(n+1)^2 c_n^1}{2(2n+1)\sqrt{n(n+1)}}(b_n^1 + b_n^{-1}) &= \frac{\eta_2 n(n+1)^2 c_n^1 \cos\phi_0}{2(2n+1)\sqrt{n(n+1)}}(b_n^1 + b_n^{-1}) \\ &\quad - \frac{n(n+1)^2 \eta_2 \sin\phi_0 \mathbf{i}}{2(2n+1)\sqrt{n(n+1)}}(b_n^1 - b_n^{-1}).\end{aligned} \tag{3.4c}$$

Lemma 3.2. *Under the same setup in lemma 3.1, it holds that*

$$0=-\frac{4\eta_2 c_1^1\cos^2\phi_0}{6\sqrt{2}}(b_1^1+b_1^{-1})+\frac{4\mathbf{i}\eta_2 c_1^1\,\sin\,\phi_0\,\cos\,\phi_0}{6\sqrt{2}}(b_1^1-b_1^{-1})+\frac{\mathbf{i}k\sqrt{2}c_1^0\,\cos\,\phi_0}{3}a_1^0, \tag{3.5}$$

$$0=\frac{4\eta_2 c_1^1\,\sin\,\phi_0\,\cos\,\phi_0}{6\sqrt{2}}(b_1^1+b_1^{-1})+\frac{4\mathbf{i}\eta_2 c_1^1\sin^2\phi_0}{6\sqrt{2}}(b_1^1-b_1^{-1})+\frac{\mathbf{i}k\sqrt{2}c_1^0\,\sin\,\phi_0}{3}a_1^0, \tag{3.6}$$

$$0=\frac{4\mathbf{i}kc_1^1\,\cos\,\phi_0}{6\sqrt{2}}(a_1^1+a_1^{-1})+\frac{4kc_1^1\,\sin\,\phi_0}{6\sqrt{2}}(a_1^1-a_1^{-1})+\frac{\eta_2\sqrt{2}c_1^0}{3}b_1^0. \tag{3.7}$$

Furthermore, if we assume there exists $n\in\mathbb{N}\backslash\{1\}$ such that (3.3) is fulfiled, then it holds that

$$\begin{aligned}0=&\frac{\mathbf{i}k\sqrt{n(n+1)}c_n^0\,\cos\,\phi_0}{2n+1}a_n^0-\frac{\eta_2 n(n+1)^2c_n^1\cos^2\phi_0}{2(2n+1)\sqrt{n(n+1)}}(b_n^1+b_n^{-1})\\&+\frac{\mathbf{i}\eta_2 n(n+1)^2c_n^1\,\sin\,\phi_0\,\cos\,\phi_0}{2(2n+1)\sqrt{n(n+1)}}(b_n^1-b_n^{-1}),\end{aligned} \tag{3.8a}$$

$$\begin{aligned}0=&\frac{\mathbf{i}kc_n^0\sqrt{n(n+1)}\,\sin\,\phi_0}{2n+1}a_n^0+\frac{\eta_2 n(n+1)^2c_n^1\,\sin\,\phi_0\,\cos\,\phi_0}{2(2n+1)\sqrt{n(n+1)}}(b_n^1+b_n^{-1})\\&+\frac{\mathbf{i}\eta_2 n(n+1)^2c_n^1\sin^2\phi_0}{2(2n+1)\sqrt{n(n+1)}}(b_n^1-b_n^{-1}),\end{aligned} \tag{3.8b}$$

$$\begin{aligned}0=&\frac{\mathbf{i}kn(n+1)^2c_n^1\,\cos\,\phi_0}{2(2n+1)\sqrt{n(n+1)}}(a_n^1+a_n^{-1})+\frac{kn(n+1)^2c_n^1\,\sin\,\phi_0}{2(2n+1)\sqrt{n(n+1)}}(a_n^1-a_n^{-1})\\&+\frac{\eta_2\sqrt{n(n+1)}c_n^0}{2n+1}b_n^0.\end{aligned} \tag{3.8c}$$

Lemma 3.3. *Under the same setup in lemma 3.1, one has the following linear relations:*

$$\begin{cases}\beta_{11}^1(b_1^1+b_1^{-1})+\beta_{12}^1(b_1^1-b_1^{-1})+\beta_{13}^1a_1^0=0,\\ \beta_{21}^1(b_1^1+b_1^{-1})+\beta_{22}^1(b_1^1-b_1^{-1})+\beta_{23}^1a_1^0=0,\\ \beta_{31}^1(b_1^1+b_1^{-1})+\beta_{32}^1(b_1^1-b_1^{-1})+\beta_{33}^1a_1^0=0,\end{cases} \tag{3.9}$$

where

$$\beta_{11}^1=-\frac{4\eta_2c_1^1\cos^2\phi_0}{6\sqrt{2}},\qquad \beta_{12}^1=\frac{4\mathbf{i}\eta_2c_1^1\,\sin\,\phi_0\,\cos\,\phi_0}{6\sqrt{2}},\qquad \beta_{13}^1=\frac{\mathbf{i}k\sqrt{2}c_1^0\,\cos\,\phi_0}{3},$$

$$\beta_{21}^1=\frac{4\eta_2c_1^1\,\sin\,\phi_0\,\cos\,\phi_0}{6\sqrt{2}},\qquad \beta_{22}^1=\frac{4\mathbf{i}\eta_2c_1^1\sin^2\phi_0}{6\sqrt{2}},\qquad \beta_{23}^1=\frac{\mathbf{i}k\sqrt{2}c_1^0\,\sin\,\phi_0}{3},$$

$$\beta_{31}^1=-\frac{4c_1^1(-\eta_1+\eta_2\,\cos\,\phi_0)}{6\sqrt{2}},\qquad \beta_{32}^1=\frac{4\eta_2c_1^1\,\sin\,\phi_0\mathbf{i}}{6\sqrt{2}},\qquad \beta_{33}^1=0.$$

If we assume that there exists $n \in \mathbb{N}\backslash\{1\}$ such that (3.3) is fulfiled, then one has that

$$\begin{cases} \beta_{11}^n(b_n^1 + b_n^{-1}) + \beta_{12}^n(b_n^1 - b_n^{-1}) + \beta_{13}^n a_n^0 = 0, \\ \beta_{21}^n(b_n^1 + b_n^{-1}) + \beta_{22}^n(b_n^1 - b_n^{-1}) + \beta_{23}^n a_n^0 = 0, \\ \beta_{31}^n(b_n^1 + b_n^{-1}) + \beta_{32}^n(b_n^1 - b_n^{-1}) + \beta_{33}^n a_n^0 = 0, \end{cases} \tag{3.10}$$

where

$$\beta_{11}^n = -\frac{\eta_2 n(n+1)^2 c_n^1 \cos^2\phi_0}{2(2n+1)\sqrt{n(n+1)}}, \quad \beta_{12}^n = \frac{\mathbf{i}\eta_2 n(n+1)^2 c_n^1 \sin\phi_0 \cos\phi_0}{2(2n+1)\sqrt{n(n+1)}},$$

$$\beta_{13}^n = \frac{\mathbf{i}k\sqrt{n(n+1)}c_n^0 \cos\phi_0}{2n+1}, \beta_{21}^n = \frac{\eta_2 n(n+1)^2 c_n^1 \sin\phi_0 \cos\phi_0}{2(2n+1)\sqrt{n(n+1)}},$$

$$\beta_{22}^n = \frac{\mathbf{i}\eta_2 n(n+1)^2 c_n^1 \sin^2\phi_0}{2(2n+1)\sqrt{n(n+1)}}, \quad \beta_{23}^n = \frac{\mathbf{i}k\sqrt{n(n+1)}c_n^0 \sin\phi_0}{2n+1},$$

$$\beta_{31}^n = -\frac{n(n+1)^2 c_n^1(-\eta_1 + \eta_2 \cos\phi_0)}{2(2n+1)\sqrt{n(n+1)}}, \beta_{32}^n = \frac{\mathbf{i}\eta_2 n(n+1)^2 c_n^1 \sin\phi_0}{2(2n+1)\sqrt{n(n+1)}}, \qquad \beta_{33}^n = 0.$$

Furthermore, if $\alpha \neq \frac{1}{2}$ and $\alpha \neq \frac{3}{2}$, then it holds that

$$a_n^0 = b_n^{\pm 1} = 0. \tag{3.11}$$

The following two important lemmas reveal the recursive relationships for $a_n^{\pm m}$ and $b_n^{\pm m}$, $m = 0, 1, \ldots, n$, which will be used to characterize the vanishing order of $\mathbf{E}$ with respect to the corresponding dihedral angle of the edge-corner $\mathcal{E}(\widetilde{\Pi}_1, \widetilde{\Pi}_2, \boldsymbol{l}) \Subset \Omega$ in theorem 3.6.

Lemma 3.4. *Under the same setup as in lemma 3.1, we assume further an $n \in \mathbb{N}\backslash\{1\}$ such that*

$$a_l^m = b_l^m = 0, \quad l = 1, \ldots, n-1, \text{ and } m \in [l]_0. \tag{3.12}$$

Then we have the following recursive linear equations:

$$\begin{cases} 0 = \mathbf{i}k\frac{\sqrt{n(n+1)}}{2n+1}c_n^0 a_n^0 - \frac{\eta_1(n+1)}{2(2n+1)}\frac{c_n^1(n+1)n}{\sqrt{n(n+1)}}(b_n^1 + b_n^{-1}), \\ 0 = \mathbf{i}k\frac{\sqrt{n(n+1)}}{2n+1}c_n^1(a_n^1 + a_n^{-1}) - \frac{\eta_1(n+1)}{2(2n+1)}\frac{c_n^2}{\sqrt{n(n+1)}}(n+2)(n-1) \\ \quad \times (b_n^2 + b_n^{-2}) + \frac{\eta_1(n+1)}{2n+1}\frac{c_n^0}{\sqrt{n(n+1)}}b_n^0, \\ 0 = \mathbf{i}k\frac{\sqrt{n(n+1)}}{2n+1}c_n^m(a_n^m + a_n^{-m}) - \frac{\eta_1(n+1)}{2(2n+1)}\frac{c_n^{m+1}}{\sqrt{n(n+1)}}(n+m+1)(n-m) \\ \quad \times (b_n^{m+1} + b_n^{-(m+1)}) + \frac{\eta_1(n+1)}{2(2n+1)}\frac{c_n^{m-1}}{\sqrt{n(n+1)}}(b_n^{m-1} + b_n^{-(m-1)}), \\ \quad \times m = 2, 3, \ldots, n-1, \\ 0 = \mathbf{i}k\frac{\sqrt{n(n+1)}}{2n+1}c_n^n(a_n^n + a_n^{-n}) + \frac{\eta_1(n+1)}{2(2n+1)}\frac{c_n^{n-1}}{\sqrt{n(n+1)}}(b_n^{n-1} + b_n^{-(n-1)}), \end{cases} \tag{3.13}$$

and

$$
\begin{cases}
0 = \mathbf{i}k\dfrac{n+1}{2(2n+1)}\dfrac{c_n^1}{\sqrt{n(n+1)}}(n+1)n(a_n^1+a_n^{-1}) + \eta_1\dfrac{c_n^0\sqrt{n(n+1)}}{2n+1}b_n^0, \\
0 = \mathbf{i}k\dfrac{n+1}{2(2n+1)}\dfrac{c_n^2}{\sqrt{n(n+1)}}(n+2)(n-1)(a_n^2+a_n^{-2}) - \mathbf{i}k\dfrac{n+1}{2n+1}\dfrac{c_n^0}{\sqrt{n(n+1)}}a_n^0 \\
\quad + \eta_1\dfrac{c_n^1\sqrt{n(n+1)}}{2n+1}(b_n^1+b_n^{-1}), \\
0 = \mathbf{i}k\dfrac{n+1}{2(2n+1)}\dfrac{c_n^m}{\sqrt{n(n+1)}}(n+m)(n-m+1)(a_n^m+a_n^{-m}) \\
\quad - \mathbf{i}k\cdot\dfrac{n+1}{2(2n+1)}\dfrac{c_n^{m-2}}{\sqrt{n(n+1)}}(a_n^{m-2}+a_n^{-(m-2)}) + \eta_1\dfrac{c_n^{m-1}\sqrt{n(n+1)}}{2n+1} \\
\quad \times (b_n^{m-1}+b_n^{-(m-1)}), \quad m=3,4,\ldots,n, \\
0 = -\mathbf{i}k\dfrac{n+1}{2(2n+1)}\dfrac{c_n^{n-1}}{\sqrt{n(n+1)}}(a_n^{n-1}+a_n^{-(n-1)}) \\
\quad + \eta_1\dfrac{c_n^n\sqrt{n(n+1)}}{2n+1}(b_n^n+b_n^{-n}) = 0.
\end{cases}
\tag{3.14}
$$

Lemma 3.5. *Under the same setup to lemma 3.4 and assuming that there exists $n \in \mathbb{N}\backslash\{1\}$ such that (3.12) is fulfiled, we have the following recursive linear equations:*

$$
\begin{aligned}
0 &= \mathbf{i}k\frac{\sqrt{n(n+1)}}{2n+1}c_n^0 a_n^0 - \frac{\eta_2(n+1)}{2(2n+1)}\frac{c_n^1(n+1)n}{\sqrt{n(n+1)}}(b_n^1\,\mathrm{e}^{\mathbf{i}\alpha\cdot\pi}+b_n^{-1}\,\mathrm{e}^{-\mathbf{i}\alpha\cdot\pi}), \\
0 &= \mathbf{i}k\frac{\sqrt{n(n+1)}}{2n+1}c_n^1(a_n^1\,\mathrm{e}^{\mathbf{i}\alpha\cdot\pi}+a_n^{-1}\,\mathrm{e}^{-\mathbf{i}\alpha\cdot\pi}) - \frac{\eta_2(n+1)}{2(2n+1)}\frac{c_n^2(n+2)(n-1)}{\sqrt{n(n+1)}} \\
&\quad \times (b_n^2\,\mathrm{e}^{\mathbf{i}2\alpha\cdot\pi}+b_n^{-2}\,\mathrm{e}^{-\mathbf{i}2\alpha\cdot\pi}) + \frac{\eta_2(n+1)}{2n+1}\frac{c_n^0}{\sqrt{n(n+1)}}b_n^0, \\
0 &= \mathbf{i}k\frac{\sqrt{n(n+1)}}{2n+1}c_n^m(a_n^m\,\mathrm{e}^{\mathbf{i}m\alpha\cdot\pi}+a_n^{-m}\,\mathrm{e}^{-\mathbf{i}m\alpha\cdot\pi}) - \frac{\eta_2(n+1)}{2(2n+1)}\frac{c_n^{m+1}}{\sqrt{n(n+1)}}(n+m+1) \\
&\quad \times (n-m)(b_n^{m+1}\,\mathrm{e}^{\mathbf{i}(m+1)\alpha\cdot\pi}+b_n^{-(m+1)}\,\mathrm{e}^{-\mathbf{i}(m+1)\alpha\cdot\pi}) + \frac{\eta_2(n+1)}{2(2n+1)}\frac{c_n^{m-1}}{\sqrt{n(n+1)}} \\
&\quad \times (b_n^{m-1}\,\mathrm{e}^{\mathbf{i}(m-1)\alpha\cdot\pi}+b_n^{-(m-1)}\,\mathrm{e}^{-\mathbf{i}(m-1)\alpha\cdot\pi}), \quad m=2,3,\ldots,n-1, \\
0 &= \mathbf{i}k\frac{\sqrt{n(n+1)}}{2n+1}c_n^n(a_n^n\,\mathrm{e}^{\mathbf{i}n\alpha\cdot\pi}+a_n^{-n}\,\mathrm{e}^{-\mathbf{i}n\alpha\cdot\pi}) + \frac{\eta_2(n+1)}{2(2n+1)}\frac{c_n^{n-1}}{\sqrt{n(n+1)}} \\
&\quad \times (b_n^{n-1}\,\mathrm{e}^{\mathbf{i}(n-1)\alpha\pi}+b_n^{-(n-1)}\,\mathrm{e}^{-\mathbf{i}(n-1)\alpha\pi}),
\end{aligned}
\tag{3.15}
$$

and

$$
\begin{aligned}
0 &= \mathbf{i}k\frac{n+1}{2(2n+1)}\frac{c_n^1}{\sqrt{n(n+1)}}(n+1)n(a_n^1\,\mathrm{e}^{\mathbf{i}\alpha\cdot\pi}+a_n^{-1}\,\mathrm{e}^{-\mathbf{i}\alpha\cdot\pi})+\eta_2\frac{c_n^0\sqrt{n(n+1)}}{2n+1}b_n^0,\\
0 &= \mathbf{i}k\frac{n+1}{2(2n+1)}\frac{c_n^2}{\sqrt{n(n+1)}}(n+2)(n-1)(a_n^2\,\mathrm{e}^{2\mathbf{i}\alpha\cdot\pi}+a_n^{-2}\,\mathrm{e}^{-2\mathbf{i}\alpha\cdot\pi})-\mathbf{i}k\frac{n+1}{2n+1}\\
&\quad\times\frac{c_n^0}{\sqrt{n(n+1)}}a_n^0+\eta_2\frac{c_n^1\sqrt{n(n+1)}}{2n+1}(b_n^1\,\mathrm{e}^{\mathbf{i}\alpha\cdot\pi}+b_n^{-1}\,\mathrm{e}^{-\mathbf{i}\alpha\cdot\pi}),\\
0 &= \mathbf{i}k\frac{n+1}{2(2n+1)}\frac{c_n^m(n+m)(n-m+1)}{\sqrt{n(n+1)}}(a_n^m\,\mathrm{e}^{\mathbf{i}m\alpha\cdot\pi}+a_n^{-m}\,\mathrm{e}^{-\mathbf{i}m\alpha\cdot\pi})-\mathbf{i}k\cdot\frac{n+1}{2(2n+1)}\\
&\quad\times\frac{c_n^{m-2}}{\sqrt{n(n+1)}}(a_n^{m-2}\,\mathrm{e}^{\mathbf{i}(m-2)\alpha\cdot\pi}+a_n^{-(m-2)}\,\mathrm{e}^{-\mathbf{i}(m-2)\alpha\cdot\pi})+\eta_2\frac{c_n^{m-1}\sqrt{n(n+1)}}{2n+1}\\
&\quad\times(b_n^{m-1}\,\mathrm{e}^{\mathbf{i}(m-1)\alpha\cdot\pi}+b_n^{-(m-1)}\,\mathrm{e}^{-\mathbf{i}(m-1)\alpha\cdot\pi}),\quad m=3,4,\ldots,n,\\
0 &= -\mathbf{i}k\frac{n+1}{2(2n+1)}\frac{c_n^{n-1}}{\sqrt{n(n+1)}}(a_n^{n-1}\,\mathrm{e}^{\mathbf{i}(n-1)\alpha\cdot\pi}+a_n^{-(n-1)}\,\mathrm{e}^{-\mathbf{i}(n-1)\alpha\cdot\pi})\\
&\quad+\eta_2\frac{c_n^n\sqrt{n(n+1)}}{2n+1}(b_n^n\,\mathrm{e}^{\mathbf{i}n\alpha\cdot\pi}+b_n^{-n}\,\mathrm{e}^{-\mathbf{i}n\alpha\cdot\pi})=0.
\end{aligned}\tag{3.16}
$$

The next theorem characterises the vanishing order of $\mathbf{E}$ to (1.1) at $\mathbf{0}\in\mathcal{E}(\widetilde{\Pi}_1,\widetilde{\Pi}_2,\boldsymbol{l})$ with $\boldsymbol{\eta}_j\in\mathcal{A}(\boldsymbol{l})$.

Theorem 3.6. *Let $\mathbf{E}$ be a solution to (1.1), whose radial wave expansion in $B_{\rho_0}(\mathbf{0})$ is given by (2.16). Consider a generalized impedance edge-corner $\mathcal{E}(\widetilde{\Pi}_1,\widetilde{\Pi}_2,\boldsymbol{l})\Subset\Omega$, with $\angle(\Pi_1,\Pi_2)=\phi_0=\alpha\pi$ for $\alpha\in(0,2)$ but $\alpha\neq 1$. Suppose that the generalized impedance parameters $\boldsymbol{\eta}_j$ on $\widetilde{\Pi}_j$, $j=1,2$, are given by (3.1). Then it holds that $\mathbf{E}$ vanishes up to the order N at $\mathbf{0}$:*

$$
\mathrm{Vani}(\mathbf{E};\mathbf{0})\geqslant\begin{cases}1, & \text{if }\ \alpha\neq\dfrac{1}{2}\text{ and }\alpha\neq\dfrac{3}{2},\\[2mm] N\in\mathbb{N}\backslash\{1\}, & \text{if }\ \alpha\neq\dfrac{q}{p},\ p=1,\ldots,N,\text{ and for a fixed }p, q=1,\ldots,2p-1.\end{cases}
$$

Proof. We prove this theorem by induction. Assume that $\alpha\neq\frac{1}{2}$ and $\alpha\neq\frac{3}{2}$. Since the generalized impedance condition (1.3) associated with $\boldsymbol{\eta}_1$ is imposed on $\widetilde{\Pi}_1$, by virtue of (2.10) and (2.23), we derive that

$$
\begin{aligned}
\mathbf{0} &= \sum_{l,m,1}\frac{1}{\sqrt{l(l+1)}}\Bigg\{\Bigg(\mathbf{i}ka_l^m l(l+1)p_l(kr)c_l^m P_l^m+\boldsymbol{\eta}_1 a_l^m j_l(kr)c_l^m\\
&\quad\times\frac{\mathrm{sgn}(m)}{2}\left[P_{l-1}^{|m|+1}(\cos\theta)+(l+|m|-1)(l+|m|)P_{l-1}^{|m|-1}(\cos\theta)\right]\\
&\quad-\boldsymbol{\eta}_1 b_l^m q_l(kr)\frac{c_l^m}{2}\left[(l+|m|)(l-|m|+1)\times P_l^{|m|-1}(\cos\theta)-P_l^{|m|+1}(\cos\theta)\right]\Bigg)\\
&\quad\times\boldsymbol{e_1}(\theta,0)+\Bigg(\mathbf{i}kb_l^m j_l(kr)c_l^m\frac{\mathrm{sgn}(m)}{2}\left[P_{l-1}^{|m|+1}(\cos\theta)+(l+|m|-1)\right.\\
&\quad\left.\times(l+|m|)P_{l-1}^{|m|-1}(\cos\theta)\right]+\mathbf{i}ka_l^m q_l(kr)\frac{c_l^m}{2}\left[(l+|m|)(l-|m|+1)\right.\\
&\quad\left.\times P_l^{|m|-1}(\cos\theta)-P_l^{|m|+1}(\cos\theta)\right]+\boldsymbol{\eta}_1 b_l^m l(l+1)p_l(kr)c_l^m P_l^m\Bigg)\boldsymbol{e_2}(\theta,0)\Bigg\},
\end{aligned}\tag{3.17}
$$

where $\boldsymbol{e_1}(\theta,0)$, $\boldsymbol{e_2}(\theta,0)$ $\boldsymbol{e_1}(\theta,\phi_0)$ and $\boldsymbol{e_2}(\theta,\phi_0)$ are defined in (2.21). Comparing the coefficient of r^0 associated with $\boldsymbol{e_2}(\theta,0)$ on both sides of (3.17), in view of (3.1), we can obtain from lemma 2.3 that

$$0=\mathbf{i}k\frac{4c_1^1}{6\sqrt{2}}(a_1^1+a_1^{-1})+\eta_1\frac{\sqrt{2}c_1^0}{3}b_1^0. \tag{3.18}$$

Combining (3.18) with (3.2a) and (3.2b) from lemma 3.3, we derive that

$$\mathcal{A}_1\begin{bmatrix}a_1^1+a_1^{-1}\\ a_1^1-a_1^{-1}\\ b_1^0\end{bmatrix}=\mathbf{0},\quad \mathcal{A}_1=\left(\alpha_{ij}^1\right)_{i,j=1}^3, \tag{3.19}$$

where

$$\alpha_{11}=\frac{4\mathbf{i}kc_1^1\sin^2\phi_0}{6\sqrt{2}},\qquad \alpha_{12}=-\frac{4kc_1^1\,\sin\,\phi_0\,\cos\,\phi_0}{6\sqrt{2}},\qquad \alpha_{13}=\frac{(-\eta_2\,\cos\,\phi_0-\eta_1)\sqrt{2}c_1^0}{3}$$

$$\alpha_{21}=-\frac{4\mathbf{i}kc_1^1\,\sin\,\phi_0\,\cos\,\phi_0}{6\sqrt{2}},\qquad \alpha_{22}=-\frac{4kc_1^1\sin^2\phi_0}{6\sqrt{2}},\qquad \alpha_{23}=-\frac{\eta_2\sqrt{2}c_1^0\,\sin\,\phi_0}{3}$$

$$\alpha_{31}=\frac{4\mathbf{i}kc_1^1}{6\sqrt{2}},\qquad \alpha_{32}=0,\qquad \alpha_{33}=\frac{\sqrt{2}c_1^0\eta_1}{3}.$$

By direct calculations, it yields that

$$|\mathcal{A}_1|=-\mathbf{i}k^2\eta_1\left(\frac{2}{3}\right)^3\frac{\sqrt{2}}{2}\left(c_1^1\right)^2c_1^0\sin^2(\alpha\pi).$$

Since $\alpha\neq\frac{1}{2}$, $\alpha\neq\frac{3}{2}$ and $\eta_1\neq 0$, by virtue of (3.19) and $k\in\mathbb{R}_+$, it can be derived that $a_1^{\pm1}=b_1^0=0$. Recall that (3.9) is given by lemma 3.3. In view of $\alpha\neq\frac{1}{2}$ and $\alpha\neq\frac{3}{2}$, $\alpha\in(0,1)$, $k\in\mathbb{R}_+$ and $\eta_2\neq 0$, using the fact that

$$|\mathcal{B}_1|=-k\eta_2^2\left(\frac{2}{3}\right)^3\frac{\sqrt{2}}{2}\left(c_1^1\right)^2c_1^0\sin^2(\alpha\pi)\cos^2(\alpha\pi)\neq 0,$$

where $\mathcal{B}_1$ is defined in (3.9), we can obtain that $b_1^{\pm1}=a_1^0=0$. Therefore, from lemma 2.8, we prove that $\mathrm{Vani}(\mathbf{E};\mathbf{0})\geqslant 1$ under conditions $\alpha\neq\frac{1}{2}$, $\alpha\neq\frac{3}{2}$ and $\eta_\ell\neq 0$, $\ell=1,2$.

By mathematical induction, suppose that $\alpha\neq\frac{q}{p}$, $p=1,\ldots,n-1$ and $q=1,2,\ldots,2p-1$ for a fixed p, then we have

$$\mathrm{Vani}(\mathbf{E};\mathbf{0})\geqslant n-1.$$

From lemma 2.8, we know that

$$a_l^m=b_l^m=0,\quad m\in[l]_0,\quad l=1,2,\ldots,n. \tag{3.20}$$

Therefore we see that (3.13)–(3.16) hold from lemmas 3.4 and 3.5. In the following under the assumption

$$\eta_\ell\neq 0\quad \text{for } \ell=1,2 \text{ and } \alpha\neq\frac{q}{p},\ p=1,\ldots,n, \tag{3.21}$$

where $q=1,2,\ldots,2p-1$ for a fixed p, we shall show that

$$a_n^m = b_n^m = 0, \quad \forall\, m \in [n]_0 \tag{3.22}$$

by utilizing the recursive equations of (3.13)–(3.16). Indeed, following a similar argument as we did for (3.19), we come from (3.2a) and (3.2b) to derive (3.24) as follows

$$\mathcal{A}_n \begin{bmatrix} a_n^1 + a_n^{-1} \\ a_n^1 - a_n^{-1} \\ b_n^0 \end{bmatrix} = \mathbf{0}, \quad \mathcal{A}_n = \left(\alpha_{ij}^n\right)_{i,j=1}^3, \tag{3.23}$$

where

$$\alpha_{11}^n = \frac{\mathbf{i}kn(n+1)^2 c_n^1 \sin^2\phi_0}{2(2n+1)\sqrt{n(n+1)}}, \qquad \alpha_{12}^n = -\frac{kn(n+1)^2 c_n^1 \sin\phi_0 \cos\phi_0}{2(2n+1)\sqrt{n(n+1)}},$$

$$\alpha_{13}^n = \frac{(-\eta_2 \cos\phi_0 - \eta_1)\sqrt{n(n+1)}c_n^0}{2n+1}, \qquad \alpha_{21}^n = -\frac{\mathbf{i}kn(n+1)^2 c_n^1 \sin\phi_0 \cos\phi_0}{2(2n+1)\sqrt{n(n+1)}},$$

$$\alpha_{22}^n = -\frac{kn(n+1)^2 c_n^1 \sin^2\phi_0}{2(2n+1)\sqrt{n(n+1)}}, \qquad \alpha_{23}^n = -\frac{\eta_2\sqrt{n(n+1)}c_n^0 \sin\phi_0}{2n+1},$$

$$\alpha_{31}^n = \mathbf{i}k\frac{n(n+1)^2 c_n^1}{2(2n+1)\sqrt{n(n+1)}}, \qquad \alpha_{32}^n = 0, \qquad \alpha_{33}^n = \eta_1 \frac{\sqrt{n(n+1)}c_n^0}{2n+1}.$$

It can be derived that

$$|\mathcal{A}_n| = -\mathbf{i}k^2\eta_1 \left(\frac{n+1}{2n+1}\right)^3 \frac{n\sqrt{n(n+1)}}{2} \left(c_n^1\right)^2 c_n^0 \sin^2(\alpha\pi). \tag{3.24}$$

Since $\alpha \in (0,2)$, $\alpha \neq 1$, $\alpha \neq \frac{1}{2}$, $\alpha \neq \frac{3}{2}$ and $\eta_\ell \neq 0$, $\ell = 1,2$, by virtue of (3.23), (3.24) and lemma 3.3, we have

$$a_n^{\pm 1} = a_n^0 = b_n^{\pm 1} = b_n^0 = 0. \tag{3.25}$$

Substituting (3.25) into the second equation of (3.13)–(3.16), since $k \in \mathbb{R}_+$, $\eta_\ell \neq 0$ for $\ell = 1,2$ and $c_n^2 \neq 0$, we obtain that

$$\begin{cases} a_n^2 + a_n^{-2} = 0, \\ a_n^2\, \mathrm{e}^{2\mathbf{i}\alpha\cdot\pi} + a_n^{-2}\, \mathrm{e}^{-2\mathbf{i}\alpha\cdot\pi} = 0, \end{cases} \qquad \begin{cases} b_n^2 + b_n^{-2} = 0, \\ b_n^2\, \mathrm{e}^{2\mathbf{i}\alpha\cdot\pi} + b_n^{-2}\, \mathrm{e}^{-2\mathbf{i}\alpha\cdot\pi} = 0, \end{cases}$$

which can be shown to prove that $a_n^{\pm 2} = b_n^{\pm 2} = 0$, since

$$\begin{vmatrix} 1 & 1 \\ \mathrm{e}^{\mathbf{i}2\alpha\cdot\pi} & \mathrm{e}^{-\mathbf{i}2\alpha\cdot\pi} \end{vmatrix} = -2\mathbf{i}\sin(2\alpha\pi) \neq 0$$

under (3.21). Substituting

$$a_n^{\pm 1} = b_n^{\pm 1} = a_n^{\pm 2} = b_n^{\pm 2} = 0$$

into the third equation of (3.13)–(3.16), since $k \in \mathbb{R}_+$, $\eta_\ell \neq 0$ for $\ell = 1,2$ and $c_n^3 \neq 0$, we get that

$$\begin{cases} a_n^3 + a_n^{-3} = 0, \\ a_n^3\, \mathrm{e}^{3\mathbf{i}\alpha\cdot\pi} + a_n^{-3}\, \mathrm{e}^{-3\mathbf{i}\alpha\cdot\pi} = 0, \end{cases} \qquad \begin{cases} b_n^3 + b_n^{-3} = 0, \\ b_n^3\, \mathrm{e}^{3\mathbf{i}\alpha\cdot\pi} + b_n^{-3}\, \mathrm{e}^{-3\mathbf{i}\alpha\cdot\pi} = 0, \end{cases}$$

which can be shown to prove that $a_n^{\pm 3} = b_n^{\pm 3} = 0$, since

$$\begin{vmatrix} 1 & 1 \\ \mathrm{e}^{\mathbf{i}3\alpha\cdot\pi} & \mathrm{e}^{-\mathbf{i}3\alpha\cdot\pi} \end{vmatrix} = -2\mathbf{i}\,\sin(3\alpha\pi) \neq 0$$

under (3.21). Repeating the above procedures step by step, utilizing the recursive property of (3.13)–(3.16), we can prove (3.22). Generally, assume that we have proved that

$$a_n^{\pm m} = b_n^{\pm m} = 0 \quad \text{for } m = 0, 1, \ldots, \ell.$$

Substituting $a_n^{\pm(\ell-1)} = b_n^{\pm(\ell-2)} = 0$ into the ℓth equation of (3.13) and (3.15), we can obtain that

$$\begin{cases} b_n^{\ell} + b_n^{-\ell} = 0, \\ b_n^{\ell}\,\mathrm{e}^{\mathbf{i}\ell\alpha\cdot\pi} + b_n^{-\ell}\,\mathrm{e}^{-\mathbf{i}\ell\alpha\cdot\pi} = 0, \end{cases} \tag{3.26}$$

under the assumption $\eta_1 \neq 0$ and $\eta_2 \neq 0$. Substituting $a_n^{\pm(\ell-2)} = b_n^{\pm(\ell-1)} = 0$ into the ℓth equation of (3.14) and (3.16), we can get that

$$\begin{cases} a_n^{\ell} + a_n^{-\ell} = 0, \\ a_n^{\ell}\,\mathrm{e}^{\mathbf{i}\ell\alpha\cdot\pi} + a_n^{-2}\,\mathrm{e}^{-\mathbf{i}\ell\alpha\cdot\pi} = 0, \end{cases} \tag{3.27}$$

Hence from (3.26) and (3.27), under (3.21) it yields that $a_n^{\pm\ell} = b_n^{\pm\ell} = 0$.

Now we can directly see from (3.22) and lemma 2.8 that

$$\mathrm{Vani}(\mathbf{E}; \mathbf{0}) \geqslant n,$$

which completes the proof of theorem 3.6. □

4. Vanishing orders for an edge-corner $\mathcal{E}(\widetilde{\Pi}_1, \widetilde{\Pi}_2, \boldsymbol{l})$ with $\boldsymbol{\eta}_j \in \mathcal{A}(\boldsymbol{l})$ or $\boldsymbol{\eta}_j = 0, \infty$

In this section, we investigate the vanishing order of the solution $\mathbf{E}$ to (1.1) at an edge-corner point $\mathbf{0} \in \mathcal{E}(\widetilde{\Pi}_1, \widetilde{\Pi}_2, \boldsymbol{l}) \Subset \Omega$, with $\angle(\Pi_1, \Pi_2) = \phi_0 = \alpha\pi$, $\alpha \in (0, 2)$ and $\alpha \neq 1$, and two different generalized impedance conditions (1.3) on $\widetilde{\Pi}_1$ and $\widetilde{\Pi}_2$. More specifically, we shall first consider the associated generalized impedance parameters of the generalized impedance edge-corner $\mathcal{E}(\Pi_1, \Pi_2, \boldsymbol{l})$ (theorem 4.2) to be respectively $\boldsymbol{\eta}_1 \equiv \infty$ and $\boldsymbol{\eta}_2 \equiv 0$, and utilize lemma 4.1 to reveal the vanishing order of $\mathbf{E}$ at $\mathbf{0}$. Then we will consider (theorems 4.4 and 4.5) the case that $\boldsymbol{\eta}_2 \in \mathcal{A}(\boldsymbol{l})$ has the expansion (3.1) whereas $\boldsymbol{\eta}_1$ could be either ∞ or 0. The reflection principle established in [19, 20] are adopted to transform the corresponding generalized impedance edge-corner to be the generalized impedance edge-corner intersected by two plane cells with the generalized impedance condition (1.3) and two associated generalized impedance parameters belonging to $\mathcal{A}(\boldsymbol{l})$.

Lemma 4.1. *Let $\mathbf{E}$ be a solution to (1.1), whose radial wave expansion in $B_{\rho_0}(\mathbf{0})$ is given by (2.16). Consider a generalized impedance edge-corner $\mathcal{E}(\widetilde{\Pi}_1, \widetilde{\Pi}_2, \boldsymbol{l}) \Subset \Omega$, with $\angle(\Pi_1, \Pi_2) = \phi_0 = \alpha\pi$ for $\alpha \in (0, 2)$ but $\alpha \neq 1$. Suppose that the generalized impedance parameters $\boldsymbol{\eta}_1$ on $\widetilde{\Pi}_1$ and $\boldsymbol{\eta}_2$ on $\widetilde{\Pi}_2$ satisfy (ii) and (i) in (1.2) respectively. Then it holds that*

$$b_1^1 + b_1^{-1} = 0, \qquad b_1^0 = 0, \qquad a_1^1 - a_1^{-1} = 0, \tag{4.1a}$$

$$b_2^m + b_2^{-m} = 0, \quad m = 1, 2, \textit{ and } b_2^0 = 0, \tag{4.1b}$$

and

$$a_1^1 \,\mathrm{e}^{\mathbf{i}\alpha\cdot\pi} + a_1^{-1}\,\mathrm{e}^{-\mathbf{i}\alpha\cdot\pi} = 0, \qquad a_1^0 = 0, \qquad b_1^1\,\mathrm{e}^{\mathbf{i}\alpha\cdot\pi} - b_1^{-1}\,\mathrm{e}^{-\mathbf{i}\alpha\cdot\pi} = 0, \tag{4.2a}$$

$$a_2^m\,\mathrm{e}^{\mathbf{i}m\alpha\cdot\pi} + a_2^{-m}\,\mathrm{e}^{-\mathbf{i}m\alpha\cdot\pi} = 0, \quad m = 1, 2, \textit{ and } a_2^0 = 0. \tag{4.2b}$$

Assume that there exists an $n \in \mathbb{N}$ *such that*

$$a_l^m = b_l^m = 0, \quad l = 1, 2, \ldots, n-1, \ m \in [l]_0, \tag{4.3}$$

then we have

$$b_n^m + b_n^{-m} = 0, \quad m = 1, \ldots, n, \textit{ and } b_n^0 = 0, \tag{4.4a}$$

$$a_n^m\,\mathrm{e}^{\mathbf{i}m\alpha\cdot\pi} + a_n^{-m}\,\mathrm{e}^{-\mathbf{i}m\alpha\cdot\pi} = 0, \quad m = 1, \ldots, n, \textit{ and } a_n^0 = 0, \tag{4.4b}$$

and

$$\sum_{m=1}^{n} m c_n^m (a_n^m - a_n^{-m}) \frac{P_n^m(\cos\theta)}{\sin\theta} + \sum_{m=-(n+1)}^{n+1} \frac{c_{n+1}^m (n+2)}{2n+3} b_{n+1}^m \frac{\partial Y_{n+1}^m}{\partial\theta}\Big|_{\phi=0} = 0, \tag{4.5a}$$

$$\sum_{m=1}^{n} m c_n^m (b_n^m\,\mathrm{e}^{\mathbf{i}m\alpha\cdot\pi} - b_n^{-m}\,\mathrm{e}^{-\mathbf{i}m\alpha\cdot\pi}) \frac{P_n^m(\cos\theta)}{\sin\theta} + \sum_{m=-(n+1)}^{n+1} \frac{c_{n+1}^m (n+2)}{2n+3} a_{n+1}^m \frac{\partial Y_{n+1}^m}{\partial\theta}\Big|_{\phi=\phi_0} = 0, \tag{4.5b}$$

where c_n^m *are nonzero constants defined in* (2.8) *for* $\mathrm{m} = 0, 1, \ldots, n$. *Furthermore, we have*

$$b_{n+1}^m + b_{n+1}^{-m} = 0, \quad m = 1, \ldots, n+1, \textit{ and } b_{n+1}^0 = 0, \tag{4.6a}$$

$$a_{n+1}^m\,\mathrm{e}^{\mathbf{i}m\alpha\cdot\pi} + a_{n+1}^{-m}\,\mathrm{e}^{-\mathbf{i}m\alpha\cdot\pi} = 0, \quad m = 1, \ldots, n+1, \textit{ and } a_{n+1}^0 = 0, \tag{4.6b}$$

where c_{n+1}^m *are nonzero constants defined in* (2.8) *for* $\mathrm{m} = 0, 1, \ldots, n+1$.

The proof of lemma 4.1 can be found in the arXiv version of this paper [14].

Theorem 4.2. *Under the same setup in lemma 4.1, we have that*

$$\mathrm{Vani}(\mathbf{E}; \mathbf{0}) \geqslant N, \quad \textit{if } \alpha \neq \frac{q}{2p}, \ p = 1, \ldots, N,$$

where $N \in \mathbb{N}$ *and* $q = 1, 2, \ldots, 4p-1$ *for a fixed* p.

Proof. We prove this theorem by induction. We first assume that

$$\alpha \neq \frac{1}{2} \quad \text{and} \quad \alpha \neq \frac{3}{2}, \tag{4.7}$$

and prove that $\mathrm{Vani}(\mathbf{E}; \mathbf{0}) \geqslant 1$. Since the generalized impedance condition (1.3) associated with $\boldsymbol{\eta}_1$ is imposed on $\widetilde{\Pi}_1$ with $\boldsymbol{\eta}_1 \equiv \infty$, we know from lemma 4.1 that (4.1a) holds. Similarly, since the generalized impedance condition (1.3) associated with $\boldsymbol{\eta}_2$ is imposed on $\widetilde{\Pi}_2$ with $\boldsymbol{\eta}_2 \equiv 0$, we know (4.2a) from lemma 4.1.

Combining (4.1a) and (4.2a), it yields that

$$\begin{cases} a_1^1 - a_1^{-1} = 0, \\ a_1^1\, e^{\mathbf{i}\alpha\cdot\pi} + a_1^{-1}\, e^{-\mathbf{i}\alpha\cdot\pi} = 0, \end{cases} \qquad \begin{cases} b_1^1 + b_1^{-1} = 0, \\ b_1^1\, e^{\mathbf{i}\alpha\cdot\pi} - b_1^{-1}\, e^{-\mathbf{i}\alpha\cdot\pi} = 0. \end{cases} \tag{4.8}$$

Under (4.7) we have

$$\begin{vmatrix} 1 & -1 \\ e^{\mathbf{i}\alpha\cdot\pi} & e^{-\mathbf{i}\alpha\cdot\pi} \end{vmatrix} = 2\cos(\alpha\cdot\pi) \neq 0,$$

which implies that $a_1^{\pm 1} = b_1^{\pm 1} = 0$ from (4.8). Since $a_1^0 = b_1^0 = 0$, we have proved $\mathrm{Vani}(\mathbf{E};\mathbf{0}) \geqslant 1$ from lemma 2.8 under the assumption (4.7).

Now, by the induction we assume that $\mathrm{Vani}(\mathbf{E};\mathbf{0}) \geqslant n-1$, under the condition that

$$\alpha \neq \frac{2q+1}{2p}, \quad p = 1, \ldots, n, \text{and } q = 0, 1, \ldots, 2p-1 \text{ for a fixed } p. \tag{4.9}$$

Then we know from lemma 2.8 that

$$a_l^m = 0 \quad \text{for } l = 1, \ldots, n-1 \text{ and } m \in [l]_0. \tag{4.10}$$

Due to (4.10) and the fact that the generalized impedance condition (1.3) associated with $\boldsymbol{\eta}_1$ is imposed on $\widetilde{\Pi}_1$ with $\boldsymbol{\eta}_1 \equiv \infty$, we have from lemma 4.1, (4.4a) and (4.6a) that

$$b_n^m + b_n^{-m} = 0, \quad m = 1, \ldots, n, \text{ and } b_n^0 = 0, \tag{4.11}$$

$$b_{n+1}^m + b_{n+1}^{-m} = 0, \quad m = 1, \ldots, n+1, \text{ and } b_{n+1}^0 = 0. \tag{4.12}$$

Substituting (4.12) into the first equation of (4.5a) yields that

$$a_2^m - a_2^{-m} = 0, \quad m = 1, \ldots, n, \tag{4.13}$$

by noting that $c_{n+1}^m = c_{n+1}^{-m} \neq 0$ for $m = 1, \ldots, n$, where c_{n+1}^m and c_{n+1}^{-m} are defined in (2.8).

Similarly, due to (4.10) and the fact that the generalized impedance condition (1.3) associated with $\boldsymbol{\eta}_2$ is imposed on $\widetilde{\Pi}_2$ with $\boldsymbol{\eta}_2 \equiv 0$, using lemma 4.1, we get that

$$a_n^m\, e^{\mathbf{i}m\alpha\pi} + a_n^{-m}\, e^{-\mathbf{i}m\alpha\pi} = 0, \quad m = 1, \ldots, n, \text{ and } a_n^0 = 0 \tag{4.14}$$

by (4.4b) and

$$a_{n+1}^m\, e^{\mathbf{i}m\alpha\pi} + a_{n+1}^{-m}\, e^{-\mathbf{i}m\alpha\pi} = 0, \quad m = 1, \ldots, n+1, \text{ and } a_{n+1}^0 = 0 \tag{4.15}$$

by (4.6b). Substituting (4.15) into the second equation of (4.5b) yields that

$$b_n^m\, e^{\mathbf{i}m\alpha\cdot\pi} - b_n^{-m}\, e^{-\mathbf{i}m\alpha\cdot\pi} = 0, \quad m = 1, \ldots, n \tag{4.16}$$

by using the fact that $c_{n+1}^m = c_{n+1}^{-m} \neq 0$ for $m = 1, \ldots, n$ and the definition of $Y_{n+1}^m(\theta,\phi)$, where c_{n+1}^m and c_{n+1}^{-m} are defined in (2.8).

Combing (4.11), (4.13) and (4.14) with (4.16), we obtain that

$$\begin{cases} a_n^m - a_n^{-m} = 0, \\ a_n^m \,\mathrm{e}^{\mathbf{i}m\alpha\cdot\pi} + a_n^{-m}\,\mathrm{e}^{-\mathbf{i}m\alpha\cdot\pi} = 0, \end{cases} \qquad \begin{cases} b_n^m + b_n^{-m} = 0, \\ b_n^m \,\mathrm{e}^{\mathbf{i}m\alpha\cdot\pi} - b_n^{-m}\,\mathrm{e}^{-\mathbf{i}m\alpha\cdot\pi} = 0, \end{cases} \qquad m = 1, \ldots, n. \tag{4.17}$$

Under the assumption (4.9) it is not difficult to see that

$$\begin{vmatrix} 1 & -1 \\ \mathrm{e}^{\mathbf{i}m\alpha\cdot\pi} & \mathrm{e}^{-\mathbf{i}m\alpha\cdot\pi} \end{vmatrix} = 2\cos(m\alpha\cdot\pi) \neq 0,$$

which imply that $a_n^{\pm m} = b_n^{\pm m} = 0$ in view of (4.17). Due to (4.11) and (4.14), we have $a_n^0 = b_n^0 = 0$, hence we have proved $\mathrm{Vani}(\mathbf{E}; \mathbf{0}) \geqslant n$ from lemma 2.8 under the assumption (4.9). The proof is now completed. □

In the following two theorems, we study the generalized impedance edge-corner $\mathcal{E}(\widetilde{\Pi}_1, \widetilde{\Pi}_2, \boldsymbol{l})$ where the generalized impedance parameter $\boldsymbol{\eta}_2$ on $\widetilde{\Pi}_2$ satisfies (iii) in (1.2) and has the expansion (3.1), whereas the generalized impedance parameter $\boldsymbol{\eta}_1$ on $\widetilde{\Pi}_1$ satisfies either (i) or (ii) in (1.2). For this purpose, we will make use of the reflection principles for the Maxwell equations established in [19, 20], as stated below in lemma 4.3, where for a two-dimensional plane $\Pi \in \mathbb{R}^3$, ν_Π and $\mathcal{R}_\Pi$ are respectively the unit normal to Π and the reflection with respect to Π.

Lemma 4.3. [20, *theorems 2.1–2.2*]. *Consider a generalized impedance edge-corner* $\mathcal{E}(\widetilde{\Pi}_1, \widetilde{\Pi}_2, \boldsymbol{l}) \Subset \Omega$, *with* $\angle(\Pi_1, \Pi_2) = \phi_0 = \alpha\pi$ *for* $\alpha \in (0, 1)$. *Assume that the generalized impedance parameter* $\boldsymbol{\eta}_2$ *on* $\widetilde{\Pi}_2$ *satisfies (iii) in* (1.2) *and has the expansion* (3.1) *while the generalized impedance parameter* $\boldsymbol{\eta}_1$ *on* $\widetilde{\Pi}_1$ *satisfies (ii) in* (1.2) *(i.e.,* $\boldsymbol{\eta}_1 \equiv \infty$*). Recall that* Π_1 *be a plane containing* $\widetilde{\Pi}_1$. *Let* $\widetilde{\Pi}_2{}' = \mathcal{R}_{\Pi_1}(\widetilde{\Pi}_2)$. *Then*

$$\nu_{\widetilde{\Pi}_2'} \wedge (\nabla \wedge \mathbf{E}) + \widetilde{\boldsymbol{\eta}}_2 (\nu_{\widetilde{\Pi}_2'} \wedge \mathbf{E}) \wedge \nu_{\widetilde{\Pi}_2'} = \mathbf{0} \quad \text{on } \widetilde{\Pi}_2', \tag{4.18}$$

where $\nu_{\Pi_2'}$ *points to the interior of* $\mathcal{E}(\widetilde{\Pi}_1, \widetilde{\Pi}_2', \boldsymbol{l})$ *and* $\widetilde{\boldsymbol{\eta}}_2(\mathbf{x}) = \boldsymbol{\eta}_2(\mathcal{R}_{\Pi_1}(\mathbf{x}))$ *for* $\mathbf{x} \in \widetilde{\Pi}_2'$.

Similarly, consider a generalized impedance edge-corner $\mathcal{E}(\widetilde{\Pi}_1, \widetilde{\Pi}_2, \boldsymbol{l}) \Subset \Omega$, *with* $\angle(\Pi_1, \Pi_2) = \phi_0 = \alpha\pi$ *for* $\alpha \in (0, 1)$. *Assume that the generalized impedance parameter* $\boldsymbol{\eta}_2$ *on* $\widetilde{\Pi}_2$ *satisfies (iii) in* (1.2) *and has the expansion* (3.1) *while the generalized impedance parameter* $\boldsymbol{\eta}_1$ *on* $\widetilde{\Pi}_1$ *satisfies (i) in* (1.2) *(i.e.,* $\boldsymbol{\eta}_1 \equiv 0$*). Then*

$$\nu_{\widetilde{\Pi}_2'} \wedge (\nabla \wedge \mathbf{E}) + \widetilde{\boldsymbol{\eta}}_2 (\nu_{\widetilde{\Pi}_2'} \wedge \mathbf{E}) \wedge \nu_{\widetilde{\Pi}_2'} = \mathbf{0} \quad \text{on } \widetilde{\Pi}_2',$$

where $\nu_{\Pi_2'}$ *points to the interior of* $\mathcal{E}(\widetilde{\Pi}_1, \widetilde{\Pi}_2', \boldsymbol{l})$ *and* $\widetilde{\boldsymbol{\eta}}_2(\mathbf{x}) = \boldsymbol{\eta}_2(\mathcal{R}_{\Pi_1}(\mathbf{x}))$ *for* $\mathbf{x} \in \widetilde{\Pi}_2'$.

Theorem 4.4. *Let* $\mathbf{E}$ *be a solution to* (1.1). *Consider a generalized impedance edge-corner* $\mathcal{E}(\widetilde{\Pi}_1, \widetilde{\Pi}_2, \boldsymbol{l}) \Subset \Omega$, *with* $\angle(\Pi_1, \Pi_2) = \phi_0 = \alpha\pi$ *for* $\alpha \in (0, 2)$ *but* $\alpha \neq 1$. *Assume that the generalized impedance parameter* $\boldsymbol{\eta}_2$ *on* $\widetilde{\Pi}_2$ *satisfies (iii) in* (1.2) *and has the expansion* (3.1) *while the generalized impedance parameter* $\boldsymbol{\eta}_1$ *on* $\widetilde{\Pi}_1$ *satisfies (ii) in* (1.2) *(i.e.,* $\boldsymbol{\eta}_1 \equiv \infty$*). Then*

$$\mathrm{Vani}(\mathbf{E}; \mathbf{0}) \geqslant N, \quad \text{if } \alpha \neq \frac{q}{2p}, \ p = 1, \ldots, N, \tag{4.19}$$

where $N \in \mathbb{N}$ *and* $q = 1, 2, \ldots, 4p - 1$ *for a fixed* p.

Proof. Let $\widetilde{\Pi}_2' = \mathcal{R}_{\Pi_1}(\widetilde{\Pi}_2)$, where Π_1 is a plane containing Π_1. With the help of lemma 4.3, we know that $\mathbf{E}$ satisfies the generalized impedance boundary condition (4.18) on $\widetilde{\Pi}_2'$. Since $\mathbf{x} \in \widetilde{\Pi}_2$, we have the spherical coordinate of $\mathbf{x} = (r, \theta, \phi_0)$, where $0 \leqslant r \leqslant h$, $\theta \in [-\pi, \pi]$ and $\phi_0 = \alpha\pi$. It is clear that the spherical coordinate of $\mathcal{R}_{\Pi_1}(\mathbf{x}))$ for $\mathbf{x} \in \widetilde{\Pi}_2$ is given by

$$(r, \theta, \phi_1), \quad \text{with} \quad \phi_1 = 2 - \alpha \in (0, 2).$$

Recall that $\boldsymbol{\eta}_2$ has the expansion (3.1). Although $\mathbf{x} \in \widetilde{\Pi}_2$ and $\mathcal{R}_{\Pi_1}(\mathbf{x})) \in \widetilde{\Pi}_2'$ have different azimuthal angles but they have the same polar angle θ, hence we know from definition 2.2 that $\widetilde{\boldsymbol{\eta}}_2$ has the same expansion (3.1) as $\boldsymbol{\eta}_2$.

We can easily see the following cases for the dihedral angle between $\widetilde{\Pi}_2$ and $\widetilde{\Pi}_2'$:

$$\angle(\widetilde{\Pi}_2, \widetilde{\Pi}_2') = \begin{cases} 2\alpha\pi \in (0, \pi], & \alpha \in (0, 1/2), \\ 2(1-\alpha)\pi \in (0, \pi], & \alpha \in [1/2, 1), \\ 2(\alpha - 1)\pi \in (0, \pi], & \alpha \in (1, 3/2), \\ 2(2-\alpha)\pi \in (0, \pi], & \alpha \in [3/2, 2). \end{cases}$$

Based on these, we divide our proof into four cases. As we stated earlier, due to the invariance of the Maxwell system (1.1) under rigid motions, we may assume that the generalized impedance edge-corner $\mathcal{E}(\widetilde{\Pi}_2, \widetilde{\Pi}_2', \boldsymbol{l}) \Subset \Omega$ is placed as in figure 1.

Case 1. If $\alpha \in (0, 1/2)$, then $2\alpha \in (0, 1)$. By virtue of theorem 3.6, if

$$2\alpha \neq \frac{q}{p}, \quad p = 1, \ldots, N, \text{ and } q = 1, \ldots, p-1 \text{ for a fixed } p, \tag{4.20}$$

we have $\mathrm{Vani}(\mathbf{E}; \mathbf{0}) \geqslant N$. It is easy to see that (4.20) is equivalent to

$$\alpha \neq \frac{q}{2p}, \quad p = 1, \ldots, N, \text{ and } q = 1, \ldots, p-1 \text{ for a fixed } p. \tag{4.21}$$

Case 2. If $\alpha \in [1/2, 1)$, then $2(1-\alpha) \in (0, 1]$. By virtue of theorem 3.6, if

$$2(1-\alpha) \neq \frac{q}{p}, \quad p = 1, \ldots, N, \text{ and } q = 1, \ldots, p \text{ for a fixed } p, \tag{4.22}$$

we have $\mathrm{Vani}(\mathbf{E}; \mathbf{0}) \geqslant N$. It is easy to see that (4.22) is equivalent to

$$\alpha \neq \frac{q}{2p}, \quad p = 1, \ldots, N, \text{ and } q = p, \ldots, 2p-1 \text{ for a fixed } p. \tag{4.23}$$

Case 3. If $\alpha \in (1, 3/2)$, then $2(\alpha - 1) \in (0, 1)$. By virtue of theorem 3.6, if

$$2(\alpha - 1) \neq \frac{q}{p}, \quad p = 1, \ldots, N, \text{ and } q = 1, \ldots, p-1 \text{ for a fixed } p, \tag{4.24}$$

we have $\mathrm{Vani}(\mathbf{E}; \mathbf{0}) \geqslant N$. It is easy to see that (4.24) is equivalent to

$$\alpha \neq \frac{q}{2p}, \quad p = 1, \ldots, N, \text{ and } q = 2p+1, \ldots, 3p-1 \text{ for a fixed } p. \tag{4.25}$$

Case 4. If $\alpha \in [3/2, 2)$, then $2(2-\alpha) \in (0, 1]$. By virtue of theorem 3.6, if

$$2(2-\alpha) \neq \frac{q}{p}, \quad p = 1, \ldots, N, \text{ and } q = 1, \ldots, p-1 \text{ for a fixed } p, \tag{4.26}$$

we have $\mathrm{Vani}(\mathbf{E};\mathbf{0}) \geqslant N$. It is easy to see that (4.26) is equivalent to

$$\alpha \neq \frac{q}{2p}, \quad p = 1, \ldots, N, \text{ and } q = 3p, \ldots, 4p-1 \text{ for a fixed } p. \tag{4.27}$$

Now theorem4.4 is a direct consequence of (4.21), (4.23), (4.25) and (4.27). □

With the help of lemma 4.3 and similar argument to the one for theorem 4.4, we can demonstrate the following theorem, whose detailed proof is omitted.

Theorem 4.5. *Let* $\mathbf{E}$ *be a solution to* (1.1). *Consider a generalized impedance edge-corner* $\mathcal{E}(\widetilde{\Pi}_1, \widetilde{\Pi}_2, \boldsymbol{l}) \Subset \Omega$, *with* $\angle(\Pi_1, \Pi_2) = \phi_0 = \alpha\pi$ *for* $\alpha \in (0, 1)$. *Assume that the generalized impedance parameter* $\boldsymbol{\eta}_2$ *on* $\widetilde{\Pi}_2$ *satisfies* (*iii*) *in* (1.2) *and has the expansion* (3.1) *while the generalized impedance parameter* $\boldsymbol{\eta}_1$ *on* $\widetilde{\Pi}_1$ *satisfies* (*i*) *in* (1.2) (*i.e.,* $\boldsymbol{\eta}_1 \equiv 0$). *Then*

$$\mathrm{Vani}(\mathbf{E};\mathbf{0}) \geqslant N, \quad \text{if } \alpha \neq \frac{q}{2p}, \ p = 1, \ldots, N,$$

where $N \in \mathbb{N}$ *and* $q = 1, 2, \ldots, 4p-1$ *for a fixed* p.

5. Irrational intersections and infinite vanishing orders

From the results we established in sections 3 and 4, we see that the vanishing order of the eigenfunction $\mathbf{E}$ at a generalized impedance edge-corner relies on the degree of the dihedral angle of the underlying corner. Next, we introduce the irrational and rational edge-corners, and based on the results in sections 3 and 4, we then show that the vanishing order of the eigenfunction at an irrational edge-corner is generically infinity and hence it vanishes identically in Ω, leading to the strong uniqueness continuation principle in such cases.

Definition 5.1. Let $\mathcal{E}(\widetilde{\Pi}_1, \widetilde{\Pi}_2, \boldsymbol{l})$ be an edge-corner defined in section 1, with the corresponding dihedral angle of $\widetilde{\Pi}_1$ and $\widetilde{\Pi}_2$ given by $\phi_0 = \alpha\pi$ for $\alpha \in (0, 2)$ but $\alpha \neq 1$. If α is an irrational number, then the edge-corner is called irrational. If α is a rational number of the form q/p with $p, q \in \mathbb{N}$ being irreducible, the edge-corner is called rational and p is referred to as its rational degree.

We readily have the following conclusions from theorems 3.6, 4.2, 4.4 and 4.5.

Theorem 5.2. *Let* $\mathbf{E}$ *be a solution to* (1.1). *Consider an irrational generalized impedance edge-corner* $\mathcal{E}(\widetilde{\Pi}_1, \widetilde{\Pi}_2, \boldsymbol{l}) \Subset \Omega$, *with* $\angle(\Pi_1, \Pi_2) = \phi_0 = \alpha\pi$ *for* $\alpha \in (0, 2)$ *but* $\alpha \neq 1$. *Under the same requirements on* $\boldsymbol{\eta}_1$ *and* $\boldsymbol{\eta}_2$ *as the ones for either of theorems* 3.6, 4.2, 4.4 *and* 4.5, *it holds that*

$$\mathrm{Vani}(\mathbf{E};\mathbf{0}) = +\infty, \quad \mathbf{0} \in \boldsymbol{l}.$$

Remark 5.3. As remarked at the beginning of section 2, the statement of theorem 5.2 actually holds for any edge-corner point rather than just the origin.

6. Applications to inverse electromagnetic scattering problems

In this section, we consider two applications of the UCP results established in the previous sections to the inverse electromagnetic scattering problems. In what follows, we first present the mathematical formulation of the inverse problem of determining an impenetrable obstacle from its associated electromagnetic far-field measurement. It is a prototypical model problem for many real applications including radar/sonar, non-destructive testing and medical imaging.

6.1. Unique identifiability results for inverse obstacle scattering problems

Let Ω be a bounded Lipschitz domain in $\mathbb{R}^3$ such that $\mathbb{R}^3\backslash\bar{\Omega}$ is connected, and the incident electric and magnetic fields be of the form

$$\mathbf{E}^{\mathrm{i}}(\mathbf{x}) := \mathbf{p}\,\mathrm{e}^{\mathbf{i}k\mathbf{x}\cdot\mathbf{d}}, \quad \mathbf{H}^{\mathrm{i}}(\mathbf{x}) := \frac{1}{\mathbf{i}k}\nabla\wedge\mathbf{p}\,\mathrm{e}^{\mathbf{i}k\mathbf{x}\cdot\mathbf{d}} = \mathbf{d}\wedge\mathbf{p}\,\mathrm{e}^{\mathbf{i}k\mathbf{x}\cdot\mathbf{d}}, \tag{6.1}$$

which are known as the time-harmonic electromagnetic plane waves, with $\mathbf{p}\in\mathbb{R}^3\backslash\{\mathbf{0}\}$, $k\in\mathbb{R}_+$ and $\mathbf{d}\in\mathbb{S}^2 := \left\{\mathbf{x}\in\mathbb{R}^3; |\mathbf{x}| = 1\right\}$ representing respectively the polarization, wave number and direction of propagation. And it holds that $\mathbf{p}\perp\mathbf{d}$. The associated forward scattering problem can be described by the following the time-harmonic Maxwell equations (see [8]):

$$\begin{cases} \nabla\wedge\mathbf{E} - \mathbf{i}k\mathbf{H} = 0 & \text{in } \mathbb{R}^3\backslash\overline{\Omega}, \\ \nabla\wedge\mathbf{H} + \mathbf{i}k\mathbf{E} = 0 & \text{in } \mathbb{R}^3\backslash\overline{\Omega}, \\ \mathbf{E}(\mathbf{x}) = \mathbf{E}^{\mathrm{i}}(\mathbf{x}) + \mathbf{E}^{\mathrm{s}}(\mathbf{x}), & \\ \mathbf{H}(\mathbf{x}) = \mathbf{H}^{\mathrm{i}}(\mathbf{x}) + \mathbf{H}^{\mathrm{s}}(\mathbf{x}), & \\ \mathscr{B}(\mathbf{E}) = \mathbf{0} & \text{on } \partial\Omega, \\ \lim\limits_{|\mathbf{x}|\to\infty}\left(\mathbf{H}^{\mathrm{s}}\wedge\mathbf{x} - |\mathbf{x}|\mathbf{E}^{\mathrm{s}}\right) = \mathbf{0}, & \end{cases} \tag{6.2}$$

where $\mathbf{E} = (E_1, E_2, E_3)$ and $\mathbf{H} = (H_1, H_2, H_3)$ are respectively the total electric and magnetic fields formed by the incident fields $\mathbf{E}^{\mathrm{i}}(\mathbf{x})$ and $\mathbf{H}^{\mathrm{i}}(\mathbf{x})$ and scattered fields $\mathbf{E}^{\mathrm{s}}(\mathbf{x})$ and $\mathbf{H}^{\mathrm{s}}(\mathbf{x})$. The last equation of (6.2) is the Silver–Müller radiation condition. The boundary condition $\mathscr{B}(\mathbf{E})$ on $\partial\Omega$ could be either of the following three conditions:

(a) The Dirichlet condition (i.e., Ω is a perfectly electric conducting (PEC) obstacle):

$$\mathscr{B}(\mathbf{E}) = \nu\wedge\mathbf{E};$$

(b) The Neumann condition (i.e., Ω is a perfectly magnetic conducting (PMC) obstacle):

$$\mathscr{B}(\mathbf{E}) = \nu\wedge(\nabla\wedge\mathbf{E});$$

(c) The impedance condition (i.e., Ω is an impedance obstacle):

$$\mathscr{B}(\mathbf{E}) = \nu\wedge(\nabla\wedge\mathbf{E}) + \boldsymbol{\eta}(\nu\wedge\mathbf{E})\wedge\nu, \qquad \Re(\boldsymbol{\eta})\geqslant 0 \quad \text{and} \quad \Im(\boldsymbol{\eta}) < 0,$$

where ν denotes the exterior unit normal vector to $\partial\Omega$ and $\boldsymbol{\eta}\in L^\infty(\Omega)$. The conditions $\Re(\boldsymbol{\eta})\geqslant 0$ and $\Im(\boldsymbol{\eta}) < 0$ are the physical requirements.

To ease the exposition, similar to our notation in (1.2)–(1.3), we unify the aforementioned three types of boundary conditions as

$$\mathscr{B}(\mathbf{E}) = \nu\wedge(\nabla\wedge\mathbf{E}) + \boldsymbol{\eta}(\nu\wedge\mathbf{E})\wedge\nu \quad \text{on } \partial\Omega, \tag{6.3}$$

which yields the Dirichlet and Neumann boundary conditions respectively with $\boldsymbol{\eta} = \infty$ and 0.

For the forward scattering problem (6.2), it is known that there exists a unique pair of solutions $(\mathbf{E}, \mathbf{H}) \in H_{\mathrm{loc}}(\mathrm{curl}, \mathbb{R}^3\backslash\overline{\Omega}) \times H_{\mathrm{loc}}(\mathrm{curl}, \mathbb{R}^3\backslash\overline{\Omega})$ (see [23]). Furthermore, the radiating fields $\mathbf{E}^{\mathrm{s}}$ and $\mathbf{H}^{\mathrm{s}}$ to (6.2) possess the following asymptotic expansions

$$\begin{aligned}
\mathbf{E}^{s}(\mathbf{x};\Omega,k,\mathbf{d},\mathbf{p}) &= \frac{e^{ik\mathbf{x}\cdot\mathbf{d}}}{|\mathbf{x}|}\left\{\mathbf{E}_{\infty}(\hat{\mathbf{x}};\Omega,k,\mathbf{d},\mathbf{p})+\mathcal{O}\left(\frac{1}{|\mathbf{x}|}\right)\right\} \quad \text{as } |\mathbf{x}|\to\infty,\\
\mathbf{H}^{s}(\mathbf{x};\Omega,k,\mathbf{d},\mathbf{p}) &= \frac{e^{ik\mathbf{x}\cdot\mathbf{d}}}{|\mathbf{x}|}\left\{\mathbf{H}_{\infty}(\hat{\mathbf{x}};\Omega,k,\mathbf{d},\mathbf{p})+\mathcal{O}\left(\frac{1}{|\mathbf{x}|}\right)\right\} \quad \text{as } |\mathbf{x}|\to\infty,
\end{aligned} \tag{6.4}$$

which hold uniformly in the angular variable $\hat{\mathbf{x}} = \mathbf{x}/|\mathbf{x}| \in \mathbb{S}^2$. The functions $\mathbf{E}_\infty(\hat{\mathbf{x}})$ and $\mathbf{H}_\infty(\hat{\mathbf{x}})$ in (6.4) are called, respectively, the electric and magnetic far field patterns, and both are analytic on the entire unit sphere $\mathbb{S}^2$. As above and also in what follows, the notation $\mathbf{U}(\mathbf{x};\Omega,\mathbf{p},k,\mathbf{d})$ will be frequently used to specify the dependence of a given function $\mathbf{U}$ on the scatterer Ω, the polarization $\mathbf{p}$, the wave number k and the incident direction $\mathbf{d}$.

The inverse electromagnetic obstacle scattering problem corresponding to (6.2) is to recover the scatterer Ω (and $\boldsymbol{\eta}$ as well in the impedance case) by the knowledge of the far-field pattern $\mathbf{E}_\infty(\hat{\mathbf{x}};\Omega,\mathbf{p},k,\mathbf{d})$ (or equivalently $\mathbf{H}_\infty(\hat{\mathbf{x}};\Omega,\mathbf{p},k,\mathbf{d})$). By introducing an operator $\mathcal{F}$ which sends the obstacle to the corresponding far-field pattern, defined by the forward scattering system (6.2), the aforementioned inverse problem can be formulated as

$$\mathcal{F}(\Omega,\boldsymbol{\eta}) = \mathbf{E}_\infty(\hat{\mathbf{x}};\Omega,k,\mathbf{d},\mathbf{p}). \tag{6.5}$$

This inverse problem (6.5) is highly nonlinear, and severely ill-posed (see [8]). It is a long-standing problem that one can establish the one-to-one correspondence for (6.5) by a single far-field pattern or a finite number of far-field patterns (namely with a fixed triplet of k, $\mathbf{d}$ and $\mathbf{p}$ or a finite number of triplets of k, $\mathbf{d}$ and $\mathbf{p}$); see the recent survey paper [9] by Colton and Kress for more discussions about the historical developments of this fundamental problem.

Under the assumption that Ω is a polyhedral obstacle associated with $\boldsymbol{\eta} \equiv 0$ or $\boldsymbol{\eta} \equiv \infty$, the unique correspondence, a.k.a unique identifiability, for the inverse problem (6.5) by at most a few far-field measurements was established in the literature; see [4, 7, 12, 13, 17–22]. However, it is still unclear whether one can establish the unique identifiability for an impedance obstacle of the non-convex polyhedral shape, even for the case that $\boldsymbol{\eta}$ is a nonzero constant, and *a fortiori* $\boldsymbol{\eta}$ is a generalised impedance parameter which can be 0, ∞ or a variable function in our case. To be more specific about the generalised impedance obstacle, we introduce the following definition.

Definition 6.1. Let Ω be an open and bounded polyhedron in $\mathbb{R}^3$. Hence, $\partial\Omega$ possesses finitely many edge-corners that are formed by the intersections of any two adjacent faces of $\partial\Omega$. Ω is said to be irrational if all of its edge-corners are irrational, otherwise it is called rational, and the smallest degree among the rational degrees of all of its rational corners is referred to the degree of the polyhedron.

Definition 6.2. $(\Omega,\boldsymbol{\eta})$ is said to be an admissible polyhedral obstacle if Ω is an open bounded polyhedron and $\boldsymbol{\eta}$ fulfils the following requirements:

(a) For each face of $\partial\Omega$, say $\widetilde{\Pi}$, and each edge of $\widetilde{\Pi}$, say $\boldsymbol{l}$, there exists a neighbourhood $\Sigma_{\boldsymbol{l}} := B_\rho(\boldsymbol{l}) \cap \widetilde{\Pi}$ with $\rho \in \mathbb{R}_+$ and $B_\rho(\boldsymbol{l}) := \{\mathbf{x} \in \mathbb{R}^3;\, |\mathbf{x}-\mathbf{x}'| < \rho, \forall\ \mathbf{x}' \in \boldsymbol{l}\}$, such that either $\boldsymbol{\eta}|_{\Sigma_{\boldsymbol{l}}} = 0$, or $\boldsymbol{\eta}|_{\Sigma_{\boldsymbol{l}}} = \infty$, or $\eta|_{\Sigma_{\boldsymbol{l}}} \in \mathcal{A}(\boldsymbol{l})$.

(b) On any open subset of the other part of $\partial\Omega$ other than the neighbourhood of each edge of $\partial\Omega$ introduced in (a), $\boldsymbol{\eta}$ can be 0, or ∞ or $\eta \in L^\infty$.

(c) In the case $\boldsymbol{\eta} \in L^\infty$, one has that $\Re(\boldsymbol{\eta}) \geqslant 0$ and $\Im(\boldsymbol{\eta}) < 0$.

Definition 6.3. Ω is said to be an admissible complex polyhedral obstacle if it consists of finitely many admissible polyhedral obstacles, which are pairwise disjoint. That is,

$$(\Omega, \boldsymbol{\eta}) = \bigcup_{j=1}^{l} (\Omega_j, \boldsymbol{\eta}_j),$$

where $l \in \mathbb{N}$ and each $(\Omega_j, \boldsymbol{\eta}_j)$ is an admissible polyhedral obstacle. Here, we define

$$\boldsymbol{\eta} = \sum_{j=1}^{l} \boldsymbol{\eta}_j \chi_{\partial\Omega_j}.$$

Moreover, Ω is said to be irrational if all of its component polyhedral obstacles are irrational, otherwise it is said to be rational. For the latter case, the smallest degree among all the degrees of its rational components is defined to be the degree of the complex obstacle Ω.

Next, we first derive a local unique identifiability result in determining an admissible complex irrational polyhedral obstacle by a single far-field pattern.

Theorem 6.4. *Consider a fixed triplet of* $k \in \mathbb{R}_+$, $\mathbf{d} \in \mathbb{S}^2$ *and* $\mathbf{p} \in \mathbb{R}^3 \backslash \{\mathbf{0}\}$. *Let* (Ω, η) *and* $(\widetilde{\Omega}, \widetilde{\eta})$ *be two admissible complex irrational obstacles, with* $\mathbf{E}_\infty$ *and* $\widetilde{\mathbf{E}}_\infty$ *being their corresponding far-field patterns and* $\mathbf{G}$ *being the unbounded connected component of* $\mathbb{R}^3 \backslash \overline{(\Omega \cup \widetilde{\Omega})}$. *If* $\mathbf{E}_\infty$ *and* $\widetilde{\mathbf{E}}_\infty$ *are the same in the sense that*

$$\mathbf{E}_\infty(\hat{\mathbf{x}}; \Omega, k, \mathbf{d}, \mathbf{p}) = \widetilde{\mathbf{E}}_\infty(\hat{\mathbf{x}}; \widetilde{\Omega}, k, \mathbf{d}, \mathbf{p}), \quad \text{for all } \hat{\mathbf{x}} \in \mathbb{S}^2, \tag{6.6}$$

then $(\partial\Omega \backslash \partial\overline{\widetilde{\Omega}}) \bigcup (\partial\widetilde{\Omega} \backslash \partial\overline{\Omega})$ *cannot possess an edge-corner on* $\partial\mathbf{G}$. *Moreover,*

$$\boldsymbol{\eta} = \widetilde{\boldsymbol{\eta}} \quad \text{on } \partial\Omega \cap \partial\widetilde{\Omega} \cap \partial\mathbf{G}. \tag{6.7}$$

Proof. We prove the theorem by contradiction. Assume that $(\partial\Omega \backslash \partial\overline{\widetilde{\Omega}}) \bigcup (\partial\widetilde{\Omega} \backslash \partial\overline{\Omega})$ has an edge corner $\mathbf{x}_c$ on $\partial\mathbf{G}$. Then, $\mathbf{x}_c$ is either located at $\partial\Omega$ or $\partial\widetilde{\Omega}$. Without loss of generality, we assume that $\mathbf{x}_c$ is an edge corner of $\partial\widetilde{\Omega}$, which also indicates that $\mathbf{x}_c$ lies outside Ω. Let $h \in \mathbb{R}_+$ be sufficiently small such that $B_h(\mathbf{x}_c) \Subset \mathbb{R}^3 \backslash \overline{\Omega}$, then we have

$$B_h(\mathbf{x}_c) \cap \partial\widetilde{\Omega} = \widetilde{\Pi}_\ell, \quad \ell = 1, 2,$$

where $\widetilde{\Pi}_\ell$ are two flat subsets lying on the faces of $\widetilde{\Omega}$ that intersect at $\mathbf{x}_c$. Moreover, for the subsequent use, we let h be smaller than ρ, where ρ is the parameter in definition 6.2. Hence we have an edge-corner $\mathcal{E}(\widetilde{\Pi}_1, \widetilde{\Pi}_2, \boldsymbol{l}) \in \partial\mathbf{G}$ with $\mathbf{x}_c \in \boldsymbol{l}$, where $\mathbf{G}$ is the unbounded connected component of $\mathbb{R}^3 \backslash \overline{(\Omega \cup \widetilde{\Omega})}$. By (6.6) and the Rellich theorem (see [8]), we know that

$$\mathbf{E}(\mathbf{x}; k, \mathbf{d}, \mathbf{p}) = \widetilde{\mathbf{E}}(\mathbf{x}; k, \mathbf{d}, \mathbf{p}), \quad \mathbf{x} \in \mathbf{G}. \tag{6.8}$$

Since $\widetilde{\Pi}_\ell \subset \partial\mathbf{G}$, $\ell = 1, 2$, combining (6.8) with the generalized boundary condition (6.3) on $\partial\widetilde{\Omega}$, it is easy to obtain that

$$\nu_\ell \wedge (\nabla \wedge \mathbf{E}) + \widetilde{\boldsymbol{\eta}}(\nu_\ell \wedge \mathbf{E}) \wedge \nu_\ell = \nu_\ell \wedge (\nabla \wedge \widetilde{\mathbf{E}}) + \widetilde{\boldsymbol{\eta}}(\nu_\ell \wedge \widetilde{\mathbf{E}}) \wedge \nu_\ell = \mathbf{0} \quad \text{on } \widetilde{\Pi}_\ell. \tag{6.9}$$

We consider the following two separate cases, depending on the values of $\widetilde{\boldsymbol{\eta}}$ on $\widetilde{\Pi}_\ell$ associated with the edge-corner $\mathcal{E}(\widetilde{\Pi}_1, \widetilde{\Pi}_2, \boldsymbol{l})$

Case 1. $\widetilde{\boldsymbol{\eta}}|_{\widetilde{\Pi}_\ell} = 0$ or $\widetilde{\boldsymbol{\eta}}|_{\widetilde{\Pi}_\ell} = \infty$, $\ell = 1, 2$. We only consider the case $\widetilde{\boldsymbol{\eta}}|_{\widetilde{\Pi}_\ell} = \infty$ and the other case can be treated in a similar manner. First, we note that one has from (6.9),

$$(\nu_\ell \wedge \mathbf{E}) \wedge \nu_\ell = 0 \quad \text{on } \widetilde{\Pi}_\ell,\ \ell = 1, 2. \tag{6.10}$$

Let $\widehat{\Pi}_\ell$ denote the full flat extension of $\widetilde{\Pi}_\ell$ within $\mathbb{R}^3\backslash\overline{\Omega}$. We claim that at least one of $\widehat{\Pi}_\ell$ is bounded. In fact, if on the contrary both $\widehat{\Pi}_1$ and $\widehat{\Pi}_2$ are unbounded, then one has from analytic continuation (noting that $\mathbf{E}$ is real analytic in $\mathbb{R}^3\backslash\overline{\Omega}$) and (6.10) that

$$\lim_{|\mathbf{x}|\to\infty, \mathbf{x}\in\widehat{\Pi}_\ell} |(\nu_\ell \wedge \mathbf{E}) \wedge \nu_\ell| = 0, \quad \ell = 1, 2. \tag{6.11}$$

Using (6.4), we note that $\mathbf{E}^s(\mathbf{x}) \to \mathbf{0}$ as $|\mathbf{x}| \to \infty$, and hence we further have from (6.11) that

$$\lim_{|\mathbf{x}|\to\infty, \mathbf{x}\in\widehat{\Pi}_\ell} \left|(\nu_\ell \wedge \mathbf{E}^i) \wedge \nu_\ell\right| = 0, \quad \ell = 1, 2, \tag{6.12}$$

which together with (6.1) readily implies that $|(\nu_\ell \wedge \mathbf{E}) \wedge \nu_\ell| = 0$, $\ell = 1, 2$. But this is impossible since ν_1 and ν_2 are linearly independent. Without loss of generality, we can assume that $\widehat{\Pi}_1$ is bounded. Clearly, $\widehat{\Pi}_1$ and part of $\partial\Omega$ form a bounded domain in $\mathbb{R}^3\backslash\overline{\Omega}$, and we denote it as Ω_1. It is noted from (6.9) that one has

$$\nu \wedge (\nabla \wedge \mathbf{E}) + \widetilde{\boldsymbol{\eta}}(\nu \wedge \mathbf{E}) \wedge \nu = \mathbf{0} \quad \text{on } \partial\Omega_1\backslash\widehat{\Pi}_1 \quad \text{and} \quad \nu \wedge (\nabla \wedge \mathbf{E}) = 0 \quad \text{on } \widehat{\Pi}_1. \tag{6.13}$$

We next show that $\widetilde{\boldsymbol{\eta}}$ can only take 0 or ∞ on $\partial\Omega_1\backslash\widehat{\Pi}_1$. Indeed, we assume on the contrary that there exists a nonempty open subset $\Lambda_1 \subset \partial\Omega_1\backslash\widehat{\Pi}_1$ such that $\widetilde{\boldsymbol{\eta}} \in L^\infty(\Lambda_1)$ with $\Re(\widetilde{\boldsymbol{\eta}}) \geqslant 0$ and $\Im(\widetilde{\boldsymbol{\eta}}) < 0$, and on $(\partial\Omega_1\backslash\widehat{\Pi}_1)\backslash\overline{\Lambda_1}$, $\widetilde{\boldsymbol{\eta}}$ takes either 0 or ∞. Noting that the Maxwell equations, namely the first two equations in (6.2) are satisfied in Ω_1, we have from Green's formula that

$$\begin{aligned} \mathrm{i}k\int_{\Omega_1} |\mathbf{H}|^2 &= \int_{\Omega_1} (\nabla \wedge \mathbf{E}) \cdot \overline{\mathbf{H}} = \int_{\Omega_1} \mathbf{E} \cdot (\nabla \wedge \overline{\mathbf{H}}) + \int_{\partial\Omega_1} (\overline{\mathbf{H}} \wedge \nu) \cdot \mathbf{E} \\ &= \mathrm{i}k\int_{\Omega_1} |\mathbf{E}|^2 + \int_{\partial\Omega_1} (\overline{\mathbf{H}} \wedge \nu) \cdot \mathbf{E} = \mathrm{i}k\int_{\Omega_1} |\mathbf{E}|^2 + \int_{\Lambda_1} (\overline{\mathbf{H}} \wedge \nu) \cdot \mathbf{E}, \end{aligned} \tag{6.14}$$

where in deriving the last equality, we have made use of the fact that $(\overline{\mathbf{H}} \wedge \nu) \cdot \mathbf{E} = 0$ on $\partial\Omega_1\backslash\overline{\Lambda_1}$. Using the fact that $\Im(\widetilde{\boldsymbol{\eta}}) < 0$ on Λ_1, one can readily infer from (6.14) that $\nu \wedge \mathbf{E}|_{\Lambda_1} = \mathbf{0}$, which together with (6.13) further implies that $\nu \wedge \mathbf{H}|_{\Lambda_1} = 0$. Hence, by the Holmgren's uniqueness principle (see [8]), one has that

$$\mathbf{E}(\mathbf{x}; k, \mathbf{d}, \mathbf{p}) = \mathbf{0} \quad \text{in } \mathbb{R}^3\backslash\overline{\Omega}, \tag{6.15}$$

which in particular yields that

$$\lim_{|\mathbf{x}|\to\infty} |\mathbf{E}(\mathbf{x}; k, \mathbf{d}, \mathbf{p})| = \mathbf{0}. \tag{6.16}$$

But this contradicts to the fact that follows from (6.4):

$$\lim_{|\mathbf{x}|\to\infty} |\mathbf{E}(\mathbf{x}; k, \mathbf{d}, \mathbf{p})| = \lim_{|\mathbf{x}|\to\infty} \left|\mathbf{p}\, \mathrm{e}^{\mathrm{i}k\mathbf{x}\cdot\mathbf{d}} + \mathbf{E}^s(\mathbf{x}; k, \mathbf{d}, \mathbf{p})\right| = |\mathbf{p}| \neq 0. \tag{6.17}$$

Hence, we actually can find a polyhedral domain $\Omega_1 \subset \mathbb{R}^3\backslash\Omega$ such that one has on $\partial\Omega_1$, either $\nu \wedge \mathbf{E} = 0$ or $\nu \wedge \mathbf{H} = 0$. The situation is reduced to that was considered in [17, 20]. It is noted that in [20], two far-field patterns are used to handle the above situation. However, the pair of incident fields $(\mathbf{E}^{\mathrm{i}}, \mathbf{H}^{\mathrm{i}})$ in (6.1) in our current case is chosen slightly different from that in [20], which enables one to apply the path-argument from [17] to arrive at a contradiction by starting from Ω_1.

Case 2. $\widetilde{\boldsymbol{\eta}}|_{\widetilde{\Pi}_\ell} \in \mathcal{A}(\boldsymbol{l})$, $\ell = 1, 2$; or one of $\widetilde{\boldsymbol{\eta}}|_{\widetilde{\Pi}_\ell}$ belongs to $\mathcal{A}(\boldsymbol{l})$, and the other one takes 0 or ∞; or one of $\widetilde{\boldsymbol{\eta}}|_{\widetilde{\Pi}_\ell}$ is 0 and the other one is ∞. This falls exactly to the situation considered in theorem 5.2. By the irrationality of the edge-corner as well as the strong uniqueness continuation principle in theorem 5.2, we readily have (6.15), which again leads to the contradiction (6.17).

It remains to prove (6.7), and we establish it by contradiction. Let $\Gamma \subset \partial\Omega \cap \partial\widetilde{\Omega} \cap \partial\mathbf{G}$ be an open subset such that $\boldsymbol{\eta} \neq \widetilde{\boldsymbol{\eta}}$ on Γ. By taking a smaller subset of Γ if necessary, we can assume that η (respectively $\widetilde{\eta}$) is either L^∞ or 0 or ∞ on Γ. Clearly, $\mathbf{E} = \widetilde{\mathbf{E}}$ in $\mathbf{G}$. Hence it holds that

$$(\nu \wedge \mathbf{E}) \wedge \nu = (\nu \wedge \widetilde{\mathbf{E}}) \wedge \nu \quad \text{and} \quad \nu \wedge (\nabla \wedge \mathbf{E}) = \nu \wedge (\nabla \wedge \widetilde{\mathbf{E}}) \quad \text{on } \Gamma.$$

and

$$\nu \wedge (\nabla \wedge \mathbf{E}) + \boldsymbol{\eta}(\nu \wedge \mathbf{E}) \wedge \nu = \mathbf{0}, \qquad \nu \wedge (\nabla \wedge \widetilde{\mathbf{E}}) + \widetilde{\boldsymbol{\eta}}(\nu \wedge \widetilde{\mathbf{E}}) \wedge \nu = \mathbf{0} \quad \text{on } \Gamma.$$

Combining with the assumption that $\boldsymbol{\eta} \neq \widetilde{\boldsymbol{\eta}}$ on $\mathcal{E}$, we can directly deduce that

$$\nu \wedge \mathbf{E} = \nu \wedge \mathbf{H} = 0 \quad \text{on } \Gamma,$$

which in turn yields by the Holmgren's uniqueness principle (see [8]) that $\mathbf{E} = \mathbf{0}$ in $\mathbb{R}^3 \backslash \Omega$. Therefore, we arrive at the same contradiction as that in (6.16) and (6.17), which readily proves (6.7).

The proof is now completed. □

It is recalled that the convex hull of Ω, denoted by $\mathcal{CH}(\Omega)$, is the smallest convex set that contains Ω. As a direct consequence of theorem 6.4, we show in the next corollary that the convex hull of a complex irrational obstacle can be uniquely determined by one far-field measurement, and the boundary impedance parameter $\boldsymbol{\eta}$ can be partially identified as well. From this conclusion, we know if the underlying polyhedral obstacle is convex, both the obstacle and its boundary impedance are uniquely determined by a single far-field pattern.

Corollary 6.5. *Consider a fixed triplet of $k \in \mathbb{R}_+$, $\mathbf{d} \in \mathbb{S}^2$ and $\mathbf{p} \in \mathbb{R}^3 \backslash \{\mathbf{0}\}$. Let (Ω, η) and $(\widetilde{\Omega}, \widetilde{\eta})$ be two admissible complex irrational obstacles, with $\mathbf{E}_\infty$ and $\widetilde{\mathbf{E}}_\infty$ being their corresponding far-field patterns. If $\mathbf{E}_\infty$ and $\widetilde{\mathbf{E}}_\infty$ satisfy (6.6), then one has that*

$$\mathcal{CH}(\Omega) = \mathcal{CH}(\widetilde{\Omega}) := \Sigma, \qquad \boldsymbol{\eta} = \widetilde{\boldsymbol{\eta}} \quad \text{on } \partial\Omega \cap \partial\widetilde{\Omega} \cap \partial\Sigma.$$

As a further application of the UCP results established in this work, we consider the unique determination of a rather general class of non-convex obstacles. To that end, we first introduce the aforementioned class of non-convex obstacles.

In the sequel, we denote by $\boldsymbol{P}_S(\mathbf{x})$ the projection of a point $\mathbf{x} \in \mathbb{R}^3$ onto a set S. Let $\partial(\mathcal{CH}(\Omega)) = \{\Sigma_\ell | \ell = 1, \ldots, N\}$, where $\Sigma_\ell, \ell = 1, \ldots, N$ are the finitely many faces of $\mathcal{CH}(\Omega)$. Let $\mathcal{V}(\Omega)$ and $\mathcal{V}(\mathcal{CH}(\Omega))$ denote, respectively, the sets of vertices of Ω and $\mathcal{CH}(\Omega)$. It is known that $\mathcal{V}(\mathcal{CH}(\Omega)) \subset \mathcal{V}(\Omega)$. For any vertex $\mathbf{v} \in \mathcal{V}(\Omega) \backslash \mathcal{V}(\mathcal{CH}(\Omega))$, we consider the projection, $\boldsymbol{P}_{\Sigma_j}(\mathbf{v})$, where $\Sigma_j \subset \partial(\mathcal{CH}(\Omega))$ is a face. It is assumed that there exists at least one Σ_j such that $\mathbf{v} - \boldsymbol{P}_{\Sigma_j}(\mathbf{v}) \subset \mathbb{R}^3 \backslash \Omega$. Then for a face $\Sigma_l \subset \partial(\mathcal{CH}(\Omega))$ we say that $\mathbf{v} \vdash \Sigma_l$ if

$$\mathbf{v} - \boldsymbol{P}_{\Sigma_\ell}(\mathbf{v}) = \underset{\mathbf{w} - \boldsymbol{P}_{\Sigma_j}(\mathbf{w}) \in \mathbb{R}^3 \backslash \Omega, \forall \Sigma_j \subset \partial(\mathcal{CH}(\Omega))}{\arg\min} \left| \mathbf{w} - \boldsymbol{P}_{\Sigma_j}(\mathbf{w}) \right|. \tag{6.18}$$

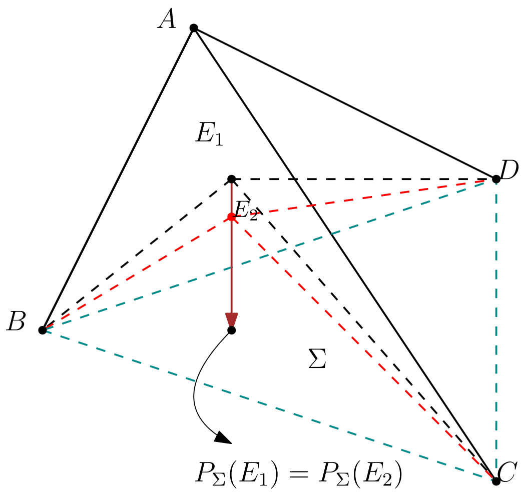

Figure 2. Schematic illustration of two different uniformly concave hexahedrons $ABCDE_1$ and $ABCDE_2$ with $\mathcal{CH}(ABCDE_1) = \mathcal{CH}(ABCDE_2) = ABCD$.

Definition 6.6. Let Ω be an admissible polyhedral obstacle, and Σ_l be a given face of $\partial(\mathcal{CH}(\Omega))$, and $\mathcal{V}_{\mathcal{C}}$ be a given set of finitely many, discrete and distinct points on Σ_l. Ω is said to be uniformly concave with respect to $\mathcal{V}_{\mathcal{C}}$ if for any $\mathbf{v} \in \mathcal{V}(\Omega)\backslash\mathcal{V}(\mathcal{CH}(\Omega))$, $\mathbf{v} \vdash \Sigma_l$ and

$$\{\boldsymbol{P}_{\Sigma_l}(\mathbf{v}) | \mathbf{v} \in \mathcal{V}(\Omega)\backslash\mathcal{V}(\mathcal{CH}(\Omega))\} = \mathcal{V}_{\mathcal{C}}.$$

As a simple illustrating example of definition 6.6, we consider two different uniformly concave hexahedrons $\Omega_1 := ABCDE_1$ and $\Omega_2 := ABCDE_2$; see figure 2. It is easy to see that Ω_1 and Ω_2 have the same convex hull, which is the tetrahedron $ABCD$. The vertexes E_1 and E_2 corresponding to Ω_1 and Ω_2 have the same projecting point on the face $\Sigma := BCD$ of the convex hull $ABCD$. It is pointed out that the vertex corner $\mathcal{V}(BE_2C, CE_2D, BE_2D, E_2) \in \partial\mathbf{G}$, where BE_2C, CE_2D, BE_2D are faces of Ω_2 and $\mathbf{G} = \mathbb{R}^3\backslash(\Omega_1 \cup \Omega_2)$.

Theorem 6.7. *Consider a fixed triplet of $k \in \mathbb{R}_+$, $\mathbf{d} \in \mathbb{S}^2$ and $\mathbf{p} \in \mathbb{R}^3\backslash\{\mathbf{0}\}$. Let (Ω, η) and $(\widetilde{\Omega}, \widetilde{\eta})$ be two uniformly concave irrational admissible polyhedral obstacles with respect to the set $\mathcal{V}_{\mathcal{C}}$, with $\mathbf{E}_\infty$ and $\widetilde{\mathbf{E}}_\infty$ being their corresponding far-field patterns. If $\mathbf{E}_\infty$ and $\widetilde{\mathbf{E}}_\infty$ satisfy (6.6), then*

$$\Omega = \widetilde{\Omega} \quad \text{and} \quad \boldsymbol{\eta} = \widetilde{\boldsymbol{\eta}}.$$

Proof. We prove this theorem by contradiction. Assume that $\Omega \neq \widetilde{\Omega}$ but (6.6) is still fulfiled. From corollary 6.5, we have $\mathcal{CH}(\Omega) = \mathcal{CH}(\widetilde{\Omega})$, which implies that the vertices of Ω contributing to $\mathcal{CH}(\Omega)$ are the same as the corresponding vertices of $\widetilde{\Omega}$ contributing to $\mathcal{CH}(\widetilde{\Omega})$. We shall prove that there must exist an edge-corner $\mathcal{E}(\Pi_1, \Pi_2, \mathbf{x}_c) \in \partial\mathbf{G}$, where $\mathbf{G}$ is the unbounded connected component of $\mathbb{R}^3\backslash\overline{(\Omega \cup \widetilde{\Omega})}$. Since $\Omega \neq \widetilde{\Omega}$, there exists an edge $\boldsymbol{l} \subset \partial\Omega\backslash\partial\widetilde{\Omega}$ or $\boldsymbol{l} \subset \partial\widetilde{\Omega}\backslash\partial\Omega$. Without loss of generality, we assume that $\boldsymbol{l} \subset \partial\widetilde{\Omega}\backslash\partial\Omega$. In the sequel, we let $\mathbf{a}_l$ and $\mathbf{b}_l$ denote the two vertices of the line segment $\boldsymbol{l}$. We divide our remaining proof into two separate cases.

Case 1. Suppose that $\mathbf{a}_{\boldsymbol{l}} \in \mathcal{V}(\mathcal{CH}(\widetilde{\Omega}))$ and $\mathbf{b}_{\boldsymbol{l}} \in \mathcal{V}(\mathcal{CH}(\widetilde{\Omega}))$. Therefore, $\boldsymbol{l} \subset \partial\mathbf{G} \cap \partial\widetilde{\Omega}$. There exists a point $\mathbf{x}_c \in \boldsymbol{l}$ and a sufficient small $h \in \mathbb{R}_+$ such that

$$B_h(\mathbf{x}_c) \cap \partial\widetilde{\Omega} = \widetilde{\Pi}_\ell, \quad \ell = 1, 2,$$

where $\widetilde{\Pi}_\ell$ are two flat subsets lying on the faces of $\widetilde{\Omega}$ that intersect at $\mathbf{x}_c$. Clearly, $\mathbf{x}_c \in \boldsymbol{l}$ is an edge-corner point.

Case 2. Suppose that there exists at least one of $\mathbf{a}_{\boldsymbol{l}}$ and $\mathbf{b}_{\boldsymbol{l}}$ belonging to $\mathcal{V}(\widetilde{\Omega})\backslash\mathcal{V}(\mathcal{CH}(\widetilde{\Omega}))$; namely, $\mathbf{x}_c \in \mathcal{V}(\widetilde{\Omega})\backslash\mathcal{V}(\mathcal{CH}(\widetilde{\Omega}))$, where $\mathbf{x}_c$ could be either $\mathbf{a}_{\boldsymbol{l}}$ or $\mathbf{b}_{\boldsymbol{l}}$. Since Ω and $\widetilde{\Omega}$ are uniformly concave admissible polyhedral obstacles with respect to the set $\mathcal{V}_{\mathcal{C}}$, there exists a face $\Sigma_\ell \Subset \partial(\mathcal{CH}(\Omega))$ such that $\mathbf{x}_c \vdash \Sigma_\ell$ and $\mathcal{V}_{\mathcal{C}} \Subset \Sigma_\ell$. Furthermore, we know that there exists a vertex $\mathbf{x}_{c,\Omega} \in \mathcal{V}(\Omega)\backslash\mathcal{V}\left(\mathcal{CH}(\Omega)\right)$ such that

$$\mathbf{x}_{c,\Omega} \vdash \Sigma_\ell, \qquad \boldsymbol{P}_{\Sigma_\ell}\left(\mathbf{x}_{c,\Omega}\right) = \boldsymbol{P}_{\Sigma_\ell}\left(\mathbf{x}_c\right) \in \mathcal{V}_{\mathcal{C}}.$$

Since $\mathbf{x}_{c,\Omega}$ and $\mathbf{x}_c$ are distinct, it holds that

$$\mathrm{d}\left(\mathbf{x}_c, \Sigma_\ell\right) \neq \mathrm{d}\left(\mathbf{x}_{c,\Omega}, \Sigma_\ell\right),$$

where $\mathrm{d}\left(\mathbf{x}_c, \Sigma_\ell\right)$ is the distance between $\mathbf{x}_c$ and Σ_ℓ. Without loss of generality, we may assume that $\mathrm{d}\left(\mathbf{x}_c, \Sigma_\ell\right) < \mathrm{d}\left(\mathbf{x}_{c,\Omega}, \Sigma_\ell\right)$. Hence, one can conclude that $\mathbf{x}_c \in \partial\mathbf{G}$, which also indicates that $\mathbf{x}_c$ lies outside Ω. Let $h \in \mathbb{R}_+$ be sufficiently small such that $B_h(\mathbf{x}_c) \Subset \mathbb{R}^2\backslash\overline{\Omega}$, then due to the fact that $\mathcal{V}_{\mathcal{C}}$ is discrete and distinct we can conclude that

$$B_h(\mathbf{x}_c) \cap \partial\widetilde{\Omega} = \widetilde{\Pi}_\ell, \quad \ell = 1, 2,$$

where $\widetilde{\Pi}_\ell$ are two plane cells lying on the faces of $\widetilde{\Omega}$ that intersect at $\mathbf{x}_c$.

The remaining proof is similar to the that of theorem 6.4, which is omitted. $\square$

It is remarked that in this section, we only consider the case that the underlying obstacle is irrational in order to make use of the strong unique continuation principle in theorem 5.2. That is, in the contradiction argument in proving theorems 6.4 and 6.7, one can find an edge-corner that can lead to the vanishing of the total wave field outside the obstacle by the strong unique continuation principle in theorem 5.2. However, we would like to emphasize that the same argument would work for the case that the underlying obstacle is of a general polyhedral shape, subject to a some slight modification. In fact, in such a case, it may happen that the edge-corner in the contradiction argument is rational, and hence instead of theorem 5.2, one would need to make use of the finite vanishing order results in theorems 3.6, 4.2, 4.4 and 4.5 to obtain that the total wave field is ‘small’ around the edge-corner (compared to the totally vanishing in the irrational case). Hence, a contradiction can be obtained if one requires that the total wave field outside the obstacle is everywhere ‘big’, which can be fulfiled in certain scenarios of practical interest, see e.g. [7]. Nevertheless, we shall not explore this direction any further in this paper. Finally, we would like to mention that in a recent paper [15], the uniqueness was established in determining a strictly convex polyhedral impedance obstacle by a single far-field pattern. The argument therein relies on a reflection principle which requires that the obstacle should lie completely on one side of a plane extended by any one of its faces as well as that the impedance parameter is constant. In our study, we localize our analysis around a corner which can cover more general (even non-convex) polyhedral geometries as well as more general impedance parameters.

6.2. Information-encoding for inverse problems and generalised Holmgren's uniqueness principle

We recall the classical Holmgren's theorem for an elliptic partial differential operator $\mathcal{P}$ with real-analytic coefficients (see [24]). If $\mathcal{P}\mathbf{u}$ is real analytic in a connected open neighbourhood of Ω, then $\mathbf{u}$ is also real-analytic. The Holmgren's theorem applied to $\mathbf{u} = (\mathbf{E}, \mathbf{H})$ in (1.1), we immediately see that $(\mathbf{E}, \mathbf{H})$ is real-analytic in Ω. Let Γ be an analytic surface in Ω. Suppose that

$$\nu \wedge \mathbf{E} = \mathbf{0} \quad \text{and} \quad \nu \wedge \mathbf{H} = 0 \quad \text{on } \Gamma, \tag{6.19}$$

then by the Cauchy–Kowalevski theorem, one readily has that $\mathbf{E} = \mathbf{H} \equiv 0$ in Ω. This is known as the Holmgren's uniqueness principle. In fact, in the proofs of theorems 6.4 and 6.7, we have made use of the Holmgren's principle in the case that Γ is an open subset of a plane. In the sequel, to ease the exposition and with a bit abuse of notation, we simply refer to Γ as a plane in such a case, though it may actually be an open subset of a plane. Our results established in theorems 3.6, 4.2, 4.4, 4.5 and 5.2 can be regarded as generalizing the Holmgren's uniqueness principle as discussed in what follows.

Suppose that there are two planes $\widetilde{\Pi}_1$ and $\widetilde{\Pi}_2$ which intersect at a line segment $\boldsymbol{l}$ within Ω (see figure 1), and

$$\nu \wedge \mathbf{E} = \mathbf{0} \quad \text{on } \widetilde{\Pi}_1 \quad \text{and} \quad \nu \wedge \mathbf{H} = \mathbf{0} \quad \text{on } \widetilde{\Pi}_2. \tag{6.20}$$

Let $\angle(\widetilde{\Pi}_1, \widetilde{\Pi}_2) = \alpha\pi$. Suppose that $\alpha = 1/N$ with $N \in \mathbb{N}$. Then according to theorem 4.2, we know that the vanishing order of $\mathbf{E}$ around $\boldsymbol{l}$ is at least N. Letting $N \to \infty$, we see that in the limiting case, one has (6.19) with $\widetilde{\Pi}_1 = \widetilde{\Pi}_2 = \Gamma$ as well as that the vanishing order becomes infinity. That is, the classical Holmgren's uniqueness principle associated with a plane Γ for the Maxwell system (1.1) is the limiting case of our result in theorem 4.2. It is surprisingly interesting that we have generalised such an observation in three aspects. First, the angle between the two intersecting planes is not infinitesimal and hence the vanishing order may be finite. Second, if the angle is irrational, not necessarily infinitesimal, the vanishing order is still infinity. Third, the homogeneous condition on the plan can be the much more general impedance condition.

The application to inverse problem of the above observation can be described as follows. In inverse problems with electromagnetic probing, one usually sends a pair of incident fields and then collects the corresponding scattered wave data away from the inhomogeneous object; see (6.5) associated with (6.2). In the following, we first take (6.5) as a specific example to elucidate the basic idea. Usually, the collection of the data is made on an analytic surface, say Γ, in the form $(\nu \wedge \mathbf{E}|_\Gamma, \nu \wedge \mathbf{H}|_\Gamma)$. Then by the Holmgren's principle, we know that the information encoded into $(\nu \wedge \mathbf{E}|_\Gamma, \nu \wedge \mathbf{H}|_\Gamma)$ is equivalent to knowing the electromagnetic fields outside the scattering obstacle, namely $\mathbb{R}^3\backslash\overline{\Omega}$, and hence is equivalent to the far-field pattern $\mathbf{E}_\infty/\mathbf{H}_\infty$. According to theorem 5.2, the measurement data can also be collected as $(\nu \wedge \mathbf{H} + \boldsymbol{\eta}_1 \nu \wedge \mathbf{E}|_{\widetilde{\Pi}_1}, \nu \wedge \mathbf{H} + \boldsymbol{\eta}_2 \nu \wedge \mathbf{H}|_{\widetilde{\Pi}_2})$ as long as $\widetilde{\Pi}_1$ and $\widetilde{\Pi}_2$ can intersect within $\mathbb{R}^3\backslash\overline{\Omega}$ with an irrational angle. Clearly, due to the analytic extension, it is not necessary for $\widetilde{\Pi}_1$ and $\widetilde{\Pi}_2$ to really intersect each other. The irrational intersection seems to be too restrictive and one can relax it to be a rational intersection with a large degree. Clearly, this conceptual information encoding technique also work for the other inverse electromagnetic scattering problem where the underlying object is not necessarily an impenetrable obstacle as that considered in (6.5). It is very interesting to find some practical applications of these results.

Data availability statement

All data that support the findings of this study are included within the article (and any supplementary files).

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