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Convergence rates of Tikhonov regularizations for elliptic and parabolic inverse radiativity problems

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Abstract

We shall study in this paper the convergence rates of the Tikhonov regularized solutions for the recovery of the radiativities in elliptic and parabolic systems in general dimensional spaces. The conditional stability estimates are first derived. Due to the difficulty of the verification of the existing source conditions or nonlinearity conditions of the inverse radiativity problems in high dimensional spaces, some new variational source conditions are proposed. The conditions are rigorously verified in general dimensional spaces under the conditional stability estimates. We will also derive the reasonable convergence rates under the new source conditions, and the results reveal the explicit relation between the regularity of the radiativities and the convergence rates.

Keywords: inverse radiativity problem, Tikhonov regularization, Lipschitz type stability, convergence rates, variational source condition

1. Introduction

In this work, we shall investigate the identification of the radiativities in the elliptic boundary value problem

$$\begin{cases} -\nabla\cdot(a(x)\nabla u)+q(x)u &= f(x) \quad \text{in } \Omega, \\ u &= g(x) \quad \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

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and the time-dependent parabolic system

$$\begin{cases} \partial_t u - \nabla\cdot(a(x)\nabla u) + q(x)u &= f(x,t) \quad \text{in } \Omega\times(0,T], \\ u(x,0) &= u_0(x) \quad \text{in } \Omega, \\ u(x,t) &= g(x,t) \quad \text{on } \partial\Omega\times(0,T], \end{cases} \tag{1.2}$$

where $\Omega \subset \mathbb{R}^d \quad (d=2,3)$ is the interested physical domain, an open bounded and connected domain with C^2 boundary $\partial\Omega$. The source density $f(x)$ or $f(x,t)$, the ambient temperature $g(x)$ or $g(x,t)$, the conductivity $a(x)$ and the initial temperature $u_0(x)$ in the model systems (1.1) and (1.2) are all given, while the radiativity $q(x)$ is the focus of our interest to be reconstructed in the following admissible constraint set

$$K = \left\{q \in L^2(\Omega);\ 0 < \underline{q} \leqslant q \leqslant \bar{q} \quad \text{a.e. in } \Omega\right\}. \tag{1.3}$$

Here $\underline{q}$ and $\bar{q}$ are two positive constants. For convenience, we often write the solutions of systems (1.1) and (1.2) as $u(q)$ to emphasize their dependence on the radiativities $q(x)$.

Throughout this work, we use δ to denote the noise level in the measurement data. Then the elliptic and parabolic inverse radiativity problems of our interest are stated as follows:

Elliptic Inverse Radiativity Problem. Given $a(x), f(x)$ and $g(x)$ in (1.1), recover the radiativity $q(x)$ in Ω from the available noisy data ∇z^δ (or z^δ) of ∇u (or u) in Ω.

Parabolic Inverse Radiativity Problem. Given $a(x), f(x,t)$, $g(x,t)$ and $u_0(x)$ in (1.2), recover the radiativity $q(x)$ in Ω from the available noisy data ∇z^δ (or z^δ) of ∇u (or u) in $\Omega\times I$, where I is a general open subinterval of the entire time range $(0,T]$.

Convergence rates have been well studied for Tikhonov regularizations for inverse conductivity and radiativity problems [11, 12, 20, 22, 26]. Most convergence results are established under the well recognised classical convergence theory for general inverse problems developed in [11]. This classical framework requires the forward map $u(q)$ to be Fréchet differentiable and the Fréchet differentive $u'(q)$ to be Lipschitz continuous. The essence of the classical regularization theory lies in its source condition which involves the adjoint operator $u'(q)^*$ and requires the existence of a small source function in certain sense. A new convergence theory was proposed in [12] for an inverse conductivity problem in a parabolic system to relax the restrictive requirements in the classical convergence theory [11]. A much simpler source condition was presented in [12], which involved only the forward map $u(q)$ itself, instead of its derivative and the adjoint, and does not require the smallness of the source function and the Fréchet differentiability of $u(q)$ and the Lipschitz continuity of the Fréchet differentive $u'(q)$. Same convergence rates as the ones from the classical theory were achieved under these much weaker and more realistic conditions. However, this new theory works only for the time-dependent inverse conductivity problems and does not apply to elliptic inverse problems, and more importantly, the proposed source conditions can be verified only in the one-dimensional spaces. Convergence rates of the Tikhonov regularizations were further studied in [20] for identifying conductivity and radiativity respectively in elliptic systems, where the identifying parameters were assumed to be known over all the boundaries and then the source conditions in [12] can be relaxed and the convergence rates can be established for elliptic systems. But the identifying parameters may not be accessible over the entire boundary in most applications. Moreover, there is a critical technical issue in the development in [20]; a linear mapping $\Theta : H_0^1(\Omega) \to H_0^1(\Omega)\cap H^2(\Omega)$ was defined there as $\Theta(\phi) = \Phi$ by $\langle\phi,\psi\rangle_{H^1(\Omega)} = \langle\Phi,\psi\rangle_{H^2(\Omega)}$ for any $\psi \in H_0^1(\Omega)\cap H^2(\Omega)$. The mapping Θ was claimed to be bijective, but its surjection is generally not easy to verify and may not hold. A novel convergence theory was developed in [22] for general nonlinear inverse operator equation, under a special source condition and a strong nonlinearity condition, which can also get rid of the smallness for the source function. Inverse conductivity problem was investigated [26] for a coupled elliptic and parabolic system, and the convergence rate was established for the H^1 regularization and mixed L^p–H^1 regularization, under a simple and easily interpretable source condition, again without smallness for the source function. As far as the stationary or instationary inverse conductivity and radiativity problems are concerned, the aforementioned convergence theories need some source conditions [11] or the required nonlinearity conditions [22], which may not be easy to verify in general dimensional spaces, unless adding some restrictive conditions on the identified parameters or forward solutions.

Variational source condition (VSC) and the resulting convergence rates results were initiated by Hofmann *et al* [21] and its extensions were proven independently in [5, 13, 14]. In comparison with the classical source condition, VSC does not involve the computation of Fréchet differentiability of the forward operator, and its resulting convergence rates for the regularized solutions follow immediately from VSC under an appropriate parameter choice rule (see e.g. [18]). But the verifications of VSC are a nontrivial and highly technical problem. For example, VSC was verified for some abstract linear inverse problems with ℓ^p penalties [2, 3, 6], in particular, for elastic-net regularizations in [6]. The main techniques used there are the operator theory and a delicate construction of index functions. For inverse problems of PDEs, Hohage and Weidling [15, 16] verified the validity of VSC for the Tikhonov regularization of inverse scattering problems, by using the conditional stability estimates via geometrical optics solutions and variational regularization theory. Recently, the validity of VSC was shown for the ill-posed backward nonlinear Maxwell's equations in [7], where the semigroup theory and extrapolation of Hilbert spaces were used. It is worth mentioning that the techniques in all these verifications are quite different in each case. For more details about the connections between VSC and classical source conditions, we refer the reader to [15–17]. In this work, we shall first derive some Lipschitz-type stability estimates for the proposed inverse problems and then propose some new variational source conditions to achieve reasonable convergence rates of the Tikhonov regularizations for the inverse problems. There are several important novelties in this work. The first one is its rigorous verification of the proposed VSC in general dimensional spaces under the newly established Lipschitz-type stabilities. The second one is that the convergence rates can be achieved without the knowledge of the identifying radiativities over the boundaries; this has significantly improved the results established in [20] for the inverse elliptic radiativity problem. Moreover, if we know the boundary information of the dentifying radiativities, higher convergence rates will be established (see remark 3.6). As the third novelty, our results reveal the relation between the regularity of the radiativities and the convergence rates. One more important novelty we like to emphasize in this work is that the main convergence results are established for both the pointwise and gradient-type measurement data, and when the measurement data is available over an arbitrary small range in time for the parabolic inverse problem.

The remainder of this work is arranged as follows. In section 2, some preliminaries are presented. In section 3, the conditional stability estimates are derived and some new VSCs are proposed for the elliptic inverse radiativity problem. We shall verify the VSCs rigorously and derive the results about the reasonable convergence rates. In section 4, we shall get some conditional stability estimates for the parabolic inverse radiativity problem and propose some new VSCs to achieve reasonable convergence rates. Some concluding remarks are given in section 5.

2. Preliminaries

In this section, we present some auxiliary notation and results for our subsequent use. We first recall some terminologies and notation. Given a linear operator $T : X \to X$ on a complex Hilbert space X, the notation $D(T)$, stand for the domain of T respectively. A linear operator $T : D(T) \subset X \to X$ is said to be closed, if its graph $\{(x, Tx), x \in D(T)\}$ is closed in $X \times X$. Furthermore, the adjoint of a densely defined operator $T : D(T) \subset X \to X$ is denoted by $T^* : D(T^*) \subset X \to X$. We call $T : D(T) \subset X \to X$ symmetric, if $Tx = T^*x$ holds true for all $x \in D(T)$, i.e. $(Tx, y)_X = (x, Ty)_X$ for all $x, y \in D(T)$. If a symmetric operator T satisfies that $D(T) = D(T^*)$, then T is said to be self-adjoint.

Then for any $s \in (-\infty, \infty)$, we define the following fractional Sobolev space:

$$H^s(\mathbb{R}^d) := \left\{ u \in \mathcal{S}(\mathbb{R}^d)' | \|u\|^2_{H^s(\mathbb{R}^d)} := \int_{\mathbb{R}^d} (1 + |\xi|^2)^s |(\mathcal{F}u)(\xi)|^2 \mathrm{d}\xi < +\infty \right\},$$

where $\mathcal{F} : \mathcal{S}(\mathbb{R}^d)' \to \mathcal{S}(\mathbb{R}^d)'$ is the Fourier transform and $\mathcal{S}(\mathbb{R}^d)'$ denotes the tempted distribution space (see, e.g. [24, 31, 32]). For a bounded domain $U \subset \mathbb{R}^d$ with a Lipschitz boundary ∂U, the space $H^s(U)$ with a possibly non-integer exponent $s \geqslant 0$ is defined as the space of all complex-valued functions $v \in L^2(U)$ satisfying $V_{|U} = v$ for some $V \in H^s(\mathbb{R}^n)$, endowed with the norm

$$\|v\|_{s,U} := \inf_{\substack{V_{|U}=v \\ V \in H^s(\mathbb{R}^n)}} \|V\|_{H^s(\mathbb{R}^n)}.$$

When no confusion may be caused, we simply drop U in the subscription of $\| \cdot \|_{s,U}$. For every $s \in [0, \infty)$, we denote by $\lfloor s \rfloor \in [0, s]$ the largest integer less or equal to s. In the case of $s \in (0, \infty)$ with $s = \lfloor s \rfloor + \sigma$ and $0 < \sigma < 1$, the norm $\| \cdot \|_{s,U}$ is equivalent to (see [32])

$$\left(\sum_{|\alpha| \leqslant \lfloor s \rfloor} \|D^\alpha u\|^2_{L^2(U)} + \sum_{|\alpha| \leqslant \lfloor s \rfloor} \iint_{U \times U} \frac{|D^\alpha u(x) - D^\alpha u(y)|^2}{|x - y|^{n+2\sigma}} \mathrm{d}x\mathrm{d}y \right)^{\frac{1}{2}}.$$

If s is a non-negative integer, then $H^s(U)$ coincides with the classical Sobolev space. We set $H^s_0(U)$ to be the completion of $C^\infty_c(U)$ under the norm $\| \cdot \|_{s,U}$, and $H^{-s}(U)$ to be the dual space of $H^s_0(U)$ with respect to the inner product of $L^2(U)$. It is also well-known that the inner product $(\cdot, \cdot)_U = \int_U f\overline{g}\, \mathrm{d}x$ extends to a bounded sesquilinear form on $H^{-s}(U) \times H^s_0(U)$, where $\overline{g}$ denotes the complex conjugate of g, which satisfies

$$|\langle f, g \rangle_{H^{-s}(U), H^s_0(U)}| \leqslant \|f\|_{H^{-s}(U)} \|g\|_{H^s_0(U)} \quad \forall\, f \in H^{-s}(U),\ g \in H^s_0(U).$$

Throughout the paper, C is often used for a generic positive constant. We shall often use the symbol $\langle \cdot, \cdot \rangle$ for the general duality pairing.

Lemma 2.1. *The following estimates will be used frequently* [30, 32].

(*a*) *Let $r, s \in \mathbb{R}$ such that $r, s < \frac{d}{2}$ and $r + s > 0$. Then for $t = r + s - \frac{d}{2}$ and distributions $u \in H^r(\mathbb{R}^d)$, $v \in H^s(\mathbb{R}^d)$, we have $uv \in H^t(\mathbb{R}^d)$, with the following estimate*

$$\|uv\|_{t,\mathbb{R}^d} \leqslant C\|u\|_{s,\mathbb{R}^d} \|v\|_{r,\mathbb{R}^d}.$$

(*b*) *Let* $r > \frac{d}{2}$. *Then* $H^s(\Omega)$ *is an algebra under pointwise multiplication, i.e. for all functions* $u, v \in H^r(\Omega)$, *it holds*

$$\|uv\|_{H^r(\Omega)} \leqslant C\|u\|_{H^r(\Omega)}\|v\|_{H^r(\Omega)}.$$

We end this section by recalling the following two well-posedness results, which can be found, e.g. in [19] (*corollary 2.2.2.4*) *and* [25] (*chapter VI, section 9*) *for the elliptic system* (1.1) *and the parabolic system* (1.2) *respectively.*

Lemma 2.2. *Assume that* $a(x) \in W^{1,\infty}(\Omega)$ *with a positive lower bound,* $q(x) \in K$, $f(x) \in L^2(\Omega)$ *and* $g(x) \in H^{\frac{3}{2}}(\partial\Omega)$. *Then there exists a unique solution* $u \in H^2(\Omega)$ *to the system* (1.1) *with the estimate*

$$\|u\|_{2,\Omega} \leqslant C\left(\|f\|_{0,\Omega} + \|g\|_{\frac{3}{2},\partial\Omega}\right). \tag{2.1}$$

Lemma 2.3. *Assume that* $a(x) \in W^{1,\infty}(\Omega)$ *with a positive lower bound,* $q(x) \in K$, $f(x,t) \in L^2(0,T;L^2(\Omega))$, $g(x,t) \in L^2(0,T;H^{\frac{3}{2}}(\partial\Omega)) \cap H^{\frac{3}{4}}(0,T;L^2(\partial\Omega))$ *and* $u_0(x) \in H^1(\Omega)$. *Then there exists a unique solution* $u \in L^2(0,T;H^2(\Omega)) \cap H^1(0,T;L^2(\Omega))$ *to the system* (1.2) *with the estimate*

$$\begin{aligned}&\|u\|_{L^2(0,T;H^2(\Omega))} + \|u\|_{H^1(0,T;L^2(\Omega))}\\ &\qquad \leqslant C(\|f\|_{L^2(0,T;L^2(\Omega))} + \|g\|_{L^2(0,T;H^{\frac{3}{2}}(\partial\Omega))} + \|g\|_{H^{\frac{3}{4}}(0,T;L^2(\partial\Omega))} + \|u_0\|_{1,\Omega}).\end{aligned} \tag{2.2}$$

3. Convergence rates of Tikhonov regularization for elliptic inverse radiativity problem

We will study in this section the conditional stabilities and convergence rates of the Tikhonov regularization for the recovery of the radiativity in the elliptic system (1.1). Throughout this section, we always assume that $a(x) \in W^{1,\infty}(\Omega)$ has a positive lower bound, $f(x) \in L^2(\Omega)$ and $g(x) \in H^{\frac{3}{2}}(\partial\Omega)$ in (1.1).

3.1. Measurement data in gradient form

We assume that the measurement data ∇z^δ of $\nabla u(q)$ is available, with a noise level δ, namely

$$\|\nabla u(q^\dagger) - \nabla z^\delta\|_{0,\Omega} \leqslant \delta, \tag{3.1}$$

where $q^\dagger$ is the true physical radiativity. The elliptic inverse radiativity problem is highly ill-posed [10], and is usually transformed into an effective and stable minimisation system with Tikhonov regularization:

$$\min_{q\in K} J_{\delta,\beta}(q) = \min_{q\in K}\left(\frac{1}{2}\|\nabla u(q) - \nabla z^\delta\|^2_{0,\Omega} + \frac{\beta}{2}\|q - q^*\|^2_{0,\Omega}\right), \tag{3.2}$$

where $\beta > 0$ is the regularization parameter and q^* is an *a priori* estimate of the true parameter $q^\dagger$. Note that q^* may not need to be selected from the constraint set K, and q^* plays the role of a selection criterion, i.e. if the minimizer to (3.2) is not unique, the choice of q^* helps choose a more desired approximate parameter [12].

We refer to [8, 20] (theorem 3.1) for the following existence of the minimizers to the optimization problem (3.2).

Theorem 3.1. *There exists at least a minimizer q_β^δ to the optimization problem* (3.2).

In the following, we shall first derive a Lipschitz-type stability estimate for the elliptic inverse radiativity problem. This Lipschitz-type stability estimate does not look so strong, but it is of fundamental importance and sufficient enough for us to rigorously verify the subsequent VSC.

Theorem 3.2. *Assume $|u(q^\dagger)| \geqslant c_0$ a.e. in Ω for some constant $c_0 > 0$, then for a fixed $\epsilon \in \left(0, \frac{1}{2}\right)$, there exists a constant C such that*

$$\|q - q^\dagger\|_{H^{-1-\epsilon}(\Omega)} \leqslant C\|u(q) - u(q^\dagger)\|_{1,\Omega} \quad \forall q \in K. \tag{3.3}$$

Proof of Theorem 3.2. We know easily from system (1.1) that for all $q \in K$,

$$-\nabla \cdot (a(x)\nabla(u(q^\dagger) - u(q))) + q(u(q^\dagger) - u(q)) = u(q^\dagger)(q - q^\dagger). \tag{3.4}$$

Since $(u(q^\dagger) - u(q)) \in H_0^1(\Omega)$, we can multiply (3.4) by a function $\varphi \in H_0^1(\Omega)$ and obtain

$$\left|\int_\Omega u(q^\dagger)(q - q^\dagger)\varphi \,\mathrm{d}x\right| \leqslant \overline{q}\int_\Omega |(u(q^\dagger) - u(q))\varphi| \,\mathrm{d}x + \|a\|_{L^\infty(\Omega)}\|\nabla(u(q^\dagger) - u(q)) \cdot \nabla\varphi\|_{0,\Omega},$$

which implies that

$$c_1\|u(q^\dagger)(q - q^\dagger)\|_{H^{-1}(\Omega)} \leqslant \|u(q) - u(q^\dagger)\|_{H_0^1(\Omega)} \tag{3.5}$$

for some constant $c_1 > 0$. By the definition of the $H^{-1-\epsilon}(\Omega)$ norm, we have

$$\begin{aligned} \|q - q^\dagger\|_{H^{-1-\epsilon}(\Omega)} &= \sup_{\|\varphi\|_{H_0^{1+\epsilon}(\Omega)}=1} \left|\int_\Omega (q - q^\dagger)\varphi(x)\mathrm{d}x\right| \\ &= \sup_{\|\varphi\|_{H_0^{1+\epsilon}(\Omega)}=1} \left|\int_\Omega u(q^\dagger)(q - q^\dagger)\frac{\varphi(x)}{u(q^\dagger)}\mathrm{d}x\right|. \end{aligned} \tag{3.6}$$

Since $u(q^\dagger) \in H^2(\Omega)$ (by lemma 2.2) with $|u(q^\dagger)| \geqslant c_0$, it follows that $\frac{1}{u(q^\dagger)} \in H^2(\Omega)$ by Leibniz' rule.

Now let $w \in H^2(\mathbb{R}^d)$ be an extension of $\frac{1}{u(q^\dagger)}$ such that $\|w\|_{H^2(\mathbb{R}^d)} \leqslant 2\|\frac{1}{u(q^\dagger)}\|_{2,\Omega}$, then $\varphi/u(q^\dagger) = w\varphi|_\Omega$. For the space dimension $d = 3$, one has $1 + \epsilon + (\frac{3}{2} - \frac{\epsilon}{2}) - \frac{d}{2} > 1$ and $1 + \epsilon \leqslant \frac{d}{2}$ for the fixed $\epsilon \in (0, 1/2)$, and we obtain by using lemma 2.1 (a) that

$$\|w\varphi\|_{1,\mathbb{R}^d} \leqslant \|w\varphi\|_{1+\frac{\epsilon}{2}+\frac{3}{2}-\frac{d}{2},\mathbb{R}^d} \leqslant C\|w\|_{\frac{3}{2},\mathbb{R}^d}\|\varphi\|_{H_0^{1+\epsilon}(\mathbb{R}^d)} \leqslant C\|\frac{1}{u(q^\dagger)}\|_{2,\Omega}\|\varphi\|_{H_0^{1+\epsilon}(\Omega)}, \tag{3.7}$$

where we have used the continuous embedding $H^{\frac{3}{2}}(\mathbb{R}^d) \Subset H^{\frac{3-\epsilon}{2}}(\mathbb{R}^d)$. For the space dimension $d = 2$, $3/2 > d/2$ and $1 + \epsilon > d/2$ for the fixed $\epsilon \in (0, 1/2)$. Then we obtain by lemma 2.1 (b) that for any $r \in (1, \min\{3/2, 1 + \epsilon\})$,

$$\|w\varphi\|_{1,\Omega} \leqslant \|w\varphi\|_{r,\Omega} \leqslant C\|w\|_{r,\Omega}\|\varphi\|_{r,\Omega} \leqslant C\|w\|_{\frac{3}{2},\Omega}\|\varphi\|_{H_0^{1+\epsilon}(\Omega)} \leqslant C\|\frac{1}{u(q^\dagger)}\|_{2,\Omega}\|\varphi\|_{H_0^{1+\epsilon}(\Omega)}.$$

From this estimate, (3.6) and (3.7) and noting that $\frac{1}{u(q^\dagger)} \in H^2(\Omega)$ and $\|\varphi\|_{H_0^{1+\epsilon}(\Omega)} = 1$, we get

$$\begin{aligned}
\|q - q^\dagger\|_{H^{-1-\epsilon}(\Omega)} &\leqslant \|u(q^\dagger)(q - q^\dagger)\|_{H^{-1}(\Omega)}\|\frac{\varphi(x)}{u(q^\dagger)}\|_{1,\Omega} \\
&\leqslant C\|u(q^\dagger)(q - q^\dagger)\|_{H^{-1}(\Omega)}\|\frac{1}{u(q^\dagger)}\|_{2,\Omega}\|\varphi\|_{H_0^{1+\epsilon}(\Omega)} \\
&\leqslant C\|u(q^\dagger)(q - q^\dagger)\|_{H^{-1}(\Omega)}.
\end{aligned} \tag{3.8}$$

This together with (3.5) implies that

$$\|q - q^\dagger\|_{H^{-1-\epsilon}(\Omega)} \leqslant C\|u(q^\dagger)(q - q^\dagger)\|_{H^{-1}(\Omega)} \leqslant C\|u(q) - u(q^\dagger)\|_{H_0^1(\Omega)}.$$

□

Remark 3.1. By the same arguments as in the proof of theorem 3.2, we can actually prove that if $w \in H^{3/2}(\Omega)$ and $\epsilon \in (0, 1/2)$, then

$$\|w\varphi\|_{1,\Omega} \leqslant C\|w\|_{3/2,\Omega}\|\varphi\|_{1+\epsilon,\Omega} \quad \forall \varphi \in C_0^\infty(\Omega).$$

Remark 3.2. Since the embedding $H^{-s}(\Omega) \Subset H^{-t}(\Omega)$ is continuous whenever $t > s \geqslant 0$, we obtain from theorem 3.2 that for all $s > 1$ and $q \in K$, it holds

$$\|q - q^\dagger\|_{H^{-s}(\Omega)} \leqslant C\|u(q) - u(q^\dagger)\|_{1,\Omega}.$$

Remark 3.3. As $u(q) - u(q^\dagger) \in H_0^1(\Omega)$, then by theorem 3.2 and the Poincáre's inequality, we have for all $s > 1$ and $q \in K$,

$$\|q - q^\dagger\|_{H^{-s}(\Omega)} \leqslant C\|u(q) - u(q^\dagger)\|_{H_0^1(\Omega)} \leqslant C\|\nabla u(q) - \nabla u(q^\dagger)\|_{0,\Omega}. \tag{3.9}$$

Remark 3.4. We present a sufficient condition to ensure that the true solution $u(q^\dagger)$ fulfills the required positiveness sassumptions in theorem 3.2. As $u(q^\dagger) \in H^2(\Omega)$ by lemma 2.2, we have by Sobolev's embedding theorem that $u(q^\dagger)$ is Hölder continuous. If f and g are both nonnegative and $u(q^\dagger) \neq 0$, then the result from [23, theorem 4] implies that $u(q^\dagger)$ is strictly positive. i.e. there exists a positive constant c_0 such that $u(q^\dagger) \geqslant c_0$ in $\overline{\Omega}$.

Now, following the general principle of the VSC [21] for inverse problems, we propose a variational source condition of the specific form for the least-squares formulation (3.2) with Tikhonov regularization:

$$\frac{1}{4}\|q - q^\dagger\|_{0,\Omega}^2 \leqslant \frac{1}{2}\|q - q^*\|_{0,\Omega}^2 - \frac{1}{2}\|q^\dagger - q^*\|_{0,\Omega}^2 + C\|u(q) - u(q^\dagger)\|_{1,\Omega}^\alpha \quad \forall q \in K, \tag{3.10}$$

where α is selected in theorem 3.3. Then using parallelogram law in Hilbert spaces, it is easy to see that (3.10) is equivalent to the following inner product form:

$$(q^\dagger - q^*, q^\dagger - q)_\Omega \leqslant \frac{1}{4}\|q - q^\dagger\|^2_{0,\Omega} + C\|u(q) - u(q^\dagger)\|^\alpha_{1,\Omega} \quad \forall\, q \in K. \tag{3.11}$$

The rest of this section is devoted to verifying the source condition (3.11) rigorously. Before this verification, we first introduce some auxiliary tools. Assume that $\mathcal{A} := -\Delta$ with domain $D(\mathcal{A}) = H^1_0(\Omega) \cap H^2(\Omega)$. It is well-known that the operator $\mathcal{A} : D(\mathcal{A}) \subset L^2(\Omega) \to L^2(\Omega)$ is densely defined, closed, self-adjoint and m-accretive. Then, in view of the compactness of the embedding $D(\mathcal{A}) \subset L^2(\Omega)$, we infer that there exists a complete orthonormal basis $\{e_n\}_{n=1}^\infty \subset L^2(\Omega)$ such that

$$(\mathcal{A}u, u)_{L^2(\Omega)} = \sum_{n=1}^\infty \lambda_n |(u, e_n)_{L^2(\Omega)}|^2 \quad \forall\, u \in D(\mathcal{A}), \tag{3.12}$$

where λ_n are the eigenvalues of $\mathcal{A}$ satisfying $0 < \lambda_1 \leqslant \lambda_2 \leqslant \cdots$, $\lim_{n\to\infty}\lambda_n = +\infty$, and for any n, e_n is the eigenfunction of $\mathcal{A}$ for the eigenvalue of λ_n, i.e. $\mathcal{A}e_n = \lambda_n e_n$. For every $\theta \in \mathbb{R}$, the fractional power $\mathcal{A}^\theta$ of $\mathcal{A}$ can be defined as

$$\mathcal{A}^\theta u := \sum_{n=1}^\infty \lambda_n^\theta (u, e_n)_{L^2(\Omega)} e_n \quad \forall\, u \in D(\mathcal{A}^\theta), \tag{3.13}$$

where the domain $D(\mathcal{A}^\theta)$ is given by

$$D(\mathcal{A}^\theta) = \left\{ u \in L^2(\Omega) | \sum_{n=1}^\infty \lambda_n^{2\theta} |(u, e_n)_{L^2(\Omega)}|^2 < \infty \right\}. \tag{3.14}$$

Moreover, $\mathcal{A}^\theta : D(\mathcal{A}^\theta) \subset L^2(\Omega) \to L^2(\Omega)$ is also self-adjoint, and $D(\mathcal{A}^\theta)$ is a Banach space equipped with the norm

$$\|u\|_{D(\mathcal{A}^\theta)} := \|\mathcal{A}^\theta u\|_{0,\Omega} = \left(\sum_{n=1}^\infty \lambda_n^{2\theta} |(u, e_n)_{L^2(\Omega)}|^2 \right)^{1/2} \quad \forall u \in D(\mathcal{A}^\theta), \tag{3.15}$$

which is also equivalent to the corresponding graph norm of $(\mathcal{A}^\theta, D(\mathcal{A}^\theta))$ (for more details, we refer to [31]). Let us mention that for all $\theta \in [0, 1/4) \bigcup (1/4, 3/4)$, it holds that [24]

$$D(\mathcal{A}^\theta) = H_0^{2\theta}(\Omega). \tag{3.16}$$

Remark 3.5. For completeness, we provide a proof of (3.16) and explain why it is not true for $\theta = 1/4$.

Following the results [32, theorems 1.40 and 16.12], we have

$$D(\mathcal{A}^\theta) = [L^2(\Omega), D(\mathcal{A})]_\theta = \begin{cases} H_0^{2\theta}(\Omega) (= H^{2\theta}(\Omega)) & \theta \in (0, 1/4), \\ \{H^{2\theta}(\Omega) | \gamma u = 0\} & \theta \in (1/4, 3/4), \end{cases} \tag{3.17}$$

where $\gamma : H^{2\theta}(\Omega) \to H^{2\theta-1/2}(\partial\Omega)$ is the trace. On the other hand, the combination of theorems 1.4.5.2 and 1.5.1.2. in [19] ensures that

$$\{H^{2\theta}(\Omega)|\gamma u = 0\} = H_0^{2\theta}(\Omega) \tag{3.18}$$

for any $1/4 < \theta < 3/4$. Then, the relation (3.16) follows from (3.17) and (3.18).

Next, we show that if $\theta = 1/4$, the relation (3.16) is not true. From the first identity in (3.17), we have

$$D(\mathcal{A}^{1/4}) = [L^2(\Omega), D(\mathcal{A})]_{1/4}.$$

Then using the reiteration property of complex interpolation (see theorem 4.6.1. in [4]) and the identity (3.16) when $\theta = 1/2$, it follows that

$$D(\mathcal{A}^{1/4}) = [L^2(\Omega), D(\mathcal{A})]_{1/4} = [L^2(\Omega), D(\mathcal{A}^{1/2})]_{1/2} = [L^2(\Omega), H_0^1(\Omega)]_{1/2}.$$

However, it is known that if Ω is smooth, then one has

$$[L^2(\Omega), H_0^1(\Omega)]_{1/2} = H_{00}^{1/4}(\Omega),$$

and the Hilbert space $H_{00}^{1/4}(\Omega)$ is strictly contained in $H_0^{1/4}(\Omega)$ with strictly finer topology (see. e.g. [24, theorem 11.4, chapter 1] for more details).

We are now ready to verify the variational source condition (3.11).

Theorem 3.3. *Assume $|u(q^\dagger)| \geqslant c_0$ a.e. in Ω and $q^\dagger - q^* \in H_0^\kappa(\Omega)$ with $\kappa > 0$ and $\kappa \neq 1/2$, then VSC (3.11) holds with some positive parameter α such that*

$$\begin{cases} \alpha = 1 & \text{if } \kappa > 1, \\ \alpha < \dfrac{2\kappa}{1+\kappa} \ (\alpha \text{ can be chosen arbitrarily close to } \dfrac{2\kappa}{1+\kappa}) & \text{if } \kappa \in \left(0, \dfrac{1}{2}\right) \cup \left(\dfrac{1}{2}, 1\right]. \end{cases}$$

Proof. Firstly, it is immediate to see that (3.11) holds if $q^\dagger - q^* = 0$. In the sequel, we shall consider the case when $q^\dagger - q^* \neq 0$.

Now if $q^\dagger - q^* \neq 0$ and $\kappa > 1$, then by making use of theorem 3.2, we have

$$\begin{aligned} |(q^\dagger - q^*, q^\dagger - q)_\Omega| &\leqslant \|q^\dagger - q^*\|_{H_0^\kappa(\Omega)} \|q^\dagger - q\|_{H^{-\kappa}(\Omega)} \\ &\leqslant C\|q^\dagger - q^*\|_{H_0^\kappa(\Omega)} \|u(q^\dagger) - u(q)\|_{1,\Omega} \leqslant C\|u(q^\dagger) - u(q)\|_{1,\Omega}, \end{aligned}$$

which verifies (3.11) with $\alpha = 1$.

Next, we consider the case with $\kappa \in \left(0, \frac{1}{2}\right) \cup (\frac{1}{2}, 1]$. For each $\lambda > 0$, we define a family of orthogonal projections

$$P_\lambda u := \sum_{\lambda_n < \lambda} (u, e_n)_\Omega e_n.$$

And if $\lambda < \lambda_1$, we set $P_\lambda = 0$. Then the Young's inequality yields

$$|((I-P_\lambda)(q^\dagger-q^*),q^\dagger-q)_\Omega| \leqslant \|q^\dagger-q\|_{0,\Omega}\|(I-P_\lambda)(q^\dagger-q^*)\|_{0,\Omega} \leqslant \frac{\|q^\dagger-q\|_{0,\Omega}^2}{4}+\|(I-P_\lambda)(q^\dagger-q^*)\|_{0,\Omega}^2. \tag{3.19}$$

As $q^\dagger-q^*\in H_0^\kappa(\Omega)$, we have from (3.16) that $q^\dagger-q^*\in D(\mathcal{A}^{\kappa/2})$. Hence, by the definition of P_λ, it is ready to see that

$$\|(I-P_\lambda)(q^\dagger-q^*)\|_{0,\Omega}^2=\sum_{\lambda_n\geqslant\lambda}|(q^\dagger-q^*,e_n)_\Omega|^2 \leqslant \frac{\sum_{n\geqslant 1}\lambda_n^\kappa|(q^\dagger-q^*,e_n)|^2}{\lambda^\kappa}=\frac{\|q^\dagger-q^*)\|_{D(A^{\kappa/2})}^2}{\lambda^\kappa}. \tag{3.20}$$

On the other hand, let us fix $s\in\left(1,\frac{3}{2}\right)$, then it follows from theorem 3.2 that

$$|(P_\lambda(q^\dagger-q^*),q^\dagger-q)_\Omega| \leqslant \|\left(P_\lambda(q^\dagger-q^*)\right\|_{H_0^s(\Omega)}\|q^\dagger-q\|_{H^{-s}(\Omega)} \leqslant C\|\left(P_\lambda(q^\dagger-q^*)\right\|_{H_0^s(\Omega)}\|u(q^\dagger)-u(q)\|_{1,\Omega}. \tag{3.21}$$

We then estimate $\|\left(P_\lambda(q^\dagger-q^*)\right)\|_{H_0^s(\Omega)}$. Indeed, by (3.16) one has

$$\|\left(P_\lambda(q^\dagger-q^*)\right)\|_{H_0^s(\Omega)}^2 \leqslant C\|\left(P_\lambda(q^\dagger-q^*)\right)\|_{D(\mathcal{A}^{\frac{s}{2}})}^2=\sum_{\lambda_n<\lambda}\lambda_n^s|(q^\dagger-q^*,e_n)|^2 =\sum_{\lambda_n<\lambda}\lambda_n^{s-\kappa}\cdot\lambda_n^\kappa|(q^\dagger-q^*,e_n)|^2\leqslant\lambda^{s-\kappa}\|q^\dagger-q^*\|_{D(\mathcal{A}^{\kappa/2})}^2,$$

which, together with (3.21), implies

$$|(P_\lambda(q^\dagger-q^*),q^\dagger-q)_\Omega| \leqslant C\lambda^{\frac{s-\kappa}{2}}\|q^\dagger-q^*\|_{D(\mathcal{A}^{\kappa/2})}\|u(q^\dagger)-u(q)\|_{1,\Omega}. \tag{3.22}$$

Combining (3.19) and (3.20) with (3.22), we have

$$(q^\dagger-q^*,q^\dagger-q)_\Omega \leqslant \frac{\|q^\dagger-q\|^2}{4}+CA\inf_{\lambda>0}\left(\frac{A}{\lambda^\kappa}+\lambda^{\frac{s-\kappa}{2}}\|u(q^\dagger)-u(q)\|_{1,\Omega}\right)\quad\forall q\in K, \tag{3.23}$$

where $A=\|(q^\dagger-q^*)\|_{D(\mathcal{A}^{\kappa/2})}$. Since $\|u(q^\dagger)-u(q)\|_{1,\Omega}\neq 0$ when $q\neq q^\dagger$ by theorem 3.2, we can choose $\lambda>0$ such that

$$\frac{A}{\lambda^\kappa}=\lambda^{\frac{s-\kappa}{2}}\|u(q^\dagger)-u(q)\|_{1,\Omega},$$

i.e. $\lambda^{\frac{s+\kappa}{2}} = \frac{A}{\|u(q^\dagger)-u(q)\|_{1,\Omega}}$ in (3.23), and deduce that

$$(q^\dagger - q^*, q^\dagger - q)_\Omega \leqslant \frac{\|q^\dagger - q\|^2}{4} + 2CAA^{\frac{s-\kappa}{s+\kappa}} \|u(q^\dagger) - u(q)\|^{\frac{2\kappa}{s+\kappa}} \quad \forall\, q \in K. \tag{3.24}$$

On the other hand, if $q = q^\dagger$, (3.24) still holds. Since $s > 1 \geqslant k > 0$, then we have $AA^{\frac{s-\kappa}{s+\kappa}} \leqslant C$ and $\frac{2\kappa}{s+\kappa} < \frac{2\kappa}{1+\kappa}$, which completes the proof.

□

We end this section by establishing the following main results about the convergence rates.

Theorem 3.4. *Assume $|u(q^\dagger)| \geqslant c_0$ a.e. in Ω and $q^\dagger - q^* \in H_0^\kappa(\Omega)$ with $\kappa > 0$ and $\kappa \neq 1/2$, and α is the parameter chosen as in theorem 3.3, then we have the following convergence rates:*

$$\|\nabla u(q_\beta^\delta) - \nabla u(q^\dagger)\|_{0,\Omega} = O(\delta), \tag{3.25}$$

$$\|q_\beta^\delta - q^\dagger\|_{0,\Omega} = O(\delta^{\frac{\alpha}{2}}), \tag{3.26}$$

under the parameter choice $\beta = O(\delta^{2-\alpha})$.

Proof. By the definition of q_β^δ in (3.2) and using (3.1), we have

$$\begin{aligned}\frac{1}{2}\|\nabla u(q_\beta^\delta) - \nabla z^\delta\|_{0,\Omega}^2 + \frac{\beta}{2}\|q_\beta^\delta - q^*\|_{0,\Omega}^2 \leqslant& \frac{1}{2}\|\nabla u(q^\dagger) - \nabla z^\delta\|_{0,\Omega}^2 + \frac{\beta}{2}\|q^\dagger - q^*\|_{0,\Omega}^2 \\ \leqslant& \frac{1}{2}\delta^2 + \frac{\beta}{2}\|q^\dagger - q^*\|_{0,\Omega}^2,\end{aligned} \tag{3.27}$$

which implies

$$\frac{1}{2}\|q_\beta^\delta - q^*\|_{0,\Omega}^2 - \frac{1}{2}\|q^\dagger - q^*\|_{0,\Omega}^2 \leqslant \frac{\delta^2}{2\beta} - \frac{1}{2\beta}\|\nabla u(q_\beta^\delta) - \nabla z^\delta\|_{0,\Omega}^2 \leqslant \frac{\delta^2}{2\beta}. \tag{3.28}$$

Using (3.10), (3.28) and triangle inequality, we have

$$\begin{aligned}0 \leqslant& \frac{1}{2}\|q_\beta^\delta - q^*\|_{0,\Omega}^2 - \frac{1}{2}\|q^\dagger - q^*\|_{0,\Omega}^2 + C\|u(q_\beta^\delta) - u(q^\dagger)\|_{1,\Omega}^\alpha \\ \leqslant& \frac{1}{2\beta}\left(\delta^2 - \|\nabla u(q_\beta^\delta) - \nabla z^\delta\|_{0,\Omega}^2\right) + C\|u(q_\beta^\delta) - u(q^\dagger)\|_{1,\Omega}^\alpha \\ \leqslant& \frac{1}{2\beta}\left(2\delta^2 - \frac{1}{2}\|\nabla u(q_\beta^\delta) - \nabla u(q^\dagger)\|_{0,\Omega}^2\right) + C\|u(q_\beta^\delta) - u(q^\dagger)\|_{1,\Omega}^\alpha.\end{aligned} \tag{3.29}$$

As $u(q_\beta^\delta) - u(q^\dagger) \in H_0^1(\Omega)$, then by the Poincáre's inequality, we have

$$\|u(q_\beta^\delta) - u(q^\dagger)\|_{1,\Omega} \leqslant C\|\nabla u(q_\beta^\delta) - \nabla u(q^\dagger)\|_{0,\Omega},$$

which together with (3.29) implies that

$$\|\nabla u(q_\beta^\delta) - \nabla u(q^\dagger)\|_{0,\Omega}^2 \leqslant 4\delta^2 + C\beta\|\nabla u(q_\beta^\delta) - \nabla u(q^\dagger)\|_{0,\Omega}^\alpha. \tag{3.30}$$

Now if $\|\nabla u(q_\beta^\delta) - \nabla u(q^\dagger)\|_{0,\Omega} < \delta$, then we readily have the convergence rate (3.25). Otherwise, if $\|\nabla u(q_\beta^\delta) - \nabla u(q^\dagger)\|_{0,\Omega} \geqslant \delta$, one has (noting that $\alpha \leqslant 1$)

$$\|\nabla u(q_\beta^\delta) - \nabla u(q^\dagger)\|_{0,\Omega}^\alpha \leqslant \|\nabla u(q_\beta^\delta) - \nabla u(q^\dagger)\|_{0,\Omega}\delta^{\alpha-1}.$$

Taking the above inequality into (3.30) and choosing $\beta = O(\delta^{2-\alpha})$, we get

$$\begin{aligned}\|\nabla u(q_\beta^\delta) - \nabla u(q^\dagger)\|_{0,\Omega}^2 &\leqslant 4\delta^2 + C\delta^{2-\alpha}\|\nabla u(q_\beta^\delta) - \nabla u(q^\dagger)\|_{0,\Omega}\delta^{\alpha-1} \\ &= 4\delta^2 + C\delta\|\nabla u(q_\beta^\delta) - \nabla u(q^\dagger)\|_{0,\Omega} \\ &\leqslant (4+C)\delta\|\nabla u(q_\beta^\delta) - \nabla u(q^\dagger)\|_{0,\Omega},\end{aligned}$$

which implies (3.25).

Finally, using (3.10), (3.28) and Poincáre's inequality, we obtain

$$\begin{aligned}\frac{1}{4}\|q_\beta^\delta - q^\dagger\|_{0,\Omega}^2 &\leqslant \frac{1}{2}\|q_\beta^\delta - q^*\|_{0,\Omega}^2 - \frac{1}{2}\|q^\dagger - q^*\|_{0,\Omega}^2 + C\|u(q_\beta^\delta) - u(q^\dagger)\|_{1,\Omega}^\alpha \\ &\leqslant \frac{\delta^2}{2\beta} + C\|\nabla u(q_\beta^\delta) - \nabla u(q^\dagger)\|_{0,\Omega}^\alpha.\end{aligned}$$

Then choosing $\beta = O(\delta^{2-\alpha})$ and using (3.25), we obtain

$$\frac{1}{4}\|q_\beta^\delta - q^\dagger\|_{0,\Omega}^2 \leqslant \frac{\delta^2}{2C\delta^{2-\alpha}} + C\delta^\alpha \leqslant C\delta^\alpha,$$

which verifies (3.26). □

Remark 3.6. Recalling the results about the convergence rate in theorem 3.4 of [20], it was assumed that $\frac{q^\dagger - q^*}{u(q^\dagger)} \in H^1(\Omega)$ for the convergence rate $\|q_\beta^\delta - q^\dagger\|_{0,\Omega} = O(\sqrt{\delta})$. In order to verify the source condition in [20], the rediativity $q^\dagger$ needs to be known on the whole boundary. It is well-known that $H_0^\kappa(\Omega) = H^\kappa(\Omega)$ for $\kappa \in (0, 1/2)$ (see, e.g. [32, theorem 1.40]). Therefore, we do not need to assume any priori knowledge of $q^\dagger$ on the boundary, but only that $q^\dagger - q^* \in H^\kappa(\Omega)$ for $\kappa \in (0, 1/2)$.

For the case with $\kappa > 1/2$, if the boundary knowledge of $q^\dagger$ is unknown, i.e. $q^\dagger - q^* \in H^\kappa(\Omega)$, we have $q^\dagger - q^* \in H^s(\Omega) = H_0^s(\Omega)$ for any $s \in (0, 1/2)$. Hence, from theorem 3.4, we can still obtain the convergence rate $\|q_\beta^\delta - q^\dagger\|_{0,\Omega} = O(\delta^{\frac{\gamma}{2}})$ for any $\gamma < \frac{2s}{1+s} < \frac{2}{3}$. Moreover, if the boundary knowledge of $q^\dagger$ is given, i.e. $q^\dagger - q^* \in H_0^\kappa(\Omega)$, then the higher convergence rate (3.26) can be achieved.

Remark 3.7. We like to mention two existing related results in the literature.

(a) Convergence rates were estimated in [9] under some conditional stability. But in order to apply the convergence results there, we need to prove the conditional stability estimates between any two element $q_1, q_2 \in K$, which we do not know if it is true. Instead, our conditional stability estimate is required only between the true parameter $q^\dagger$ and any other feasible $q \in K$.

Now let us assume that we can establish the stability estimate

$$\|q_1 - q_2\|_{-1-\epsilon,\Omega} \leqslant C\|\nabla u(q_1) - \nabla u(q_2)\|_{0,\Omega} \quad \forall\, q_1, q_2 \in K,$$

and let $q_\beta^\delta \in K$ be chosen such that

$$F(q_\beta^\delta) \leqslant \inf_{q\in K} F(q) + C_0\delta^2$$

where $C_0 > 0$ is a constant and $F(q) := \|\nabla u(q_1) - \nabla u(q_2)\|_{0,\Omega}^2 + \beta\|q\|_{0,\Omega}^2$. By choosing $\beta \sim \delta^2$ as $\delta \to 0$, we can then get the following convergence rate from [9]:

$$\|q_\beta^\delta - q^\dagger\|_{-1-\epsilon,\Omega} = O(\delta) \quad \text{as } \delta \to 0.$$

But we can not derive the convergence in L^2-norm as in our results.
(b) We may also consider the following *a posteriori* parameter choice strategy. For the prescribed τ_1, τ_2 with $1 \leqslant \tau_1 \leqslant \tau_2$, the Morozov discrepancy principle suggests to choose the regularization parameter $\beta_* = \beta_{\mathrm{SDP}}$ such that $q^\delta_{\beta_{\mathrm{SDP}}}$ satisfies

$$\tau_1\delta \leqslant \|\nabla u(q^\delta_{\beta_{\mathrm{SDP}}}) - \nabla u^\delta\|_{0,\Omega} \leqslant \tau_2\delta. \tag{3.31}$$

Then, we can infer from [29, theorem 4.13] the following convergence rate

$$\|q^\delta_{\beta_{\mathrm{SDP}}} - q^\dagger\|_{0,\Omega} = O(\delta^{\frac{\alpha}{2}}) \quad \text{as } \delta \to 0^+.$$

Corollary 3.1. *Under the hypothesises and settings of theorem 3.4, we have the convergence rate for* $2 \leqslant p < +\infty$,

$$\|q_\beta^\delta - q^\dagger\|_{L^p(\Omega)} = O(\delta^{\frac{\alpha}{p}}), \tag{3.32}$$

under the parameter choice $\beta = O(\delta^{2-\alpha})$.

Proof. Since $\|q_\beta^\delta - q^\dagger\|_{L^\infty(\Omega)} \leqslant 2\bar{q}$ for all $q_\beta^\delta \in K$, we can obtain by Hölder's inequality that for any $2 \leqslant p < +\infty$,

$$\|q_\beta^\delta - q^\dagger\|_{L^p(\Omega)} \leqslant (2\bar{q})^{\frac{p-2}{p}} \|q_\beta^\delta - q^\dagger\|_{L^2(\Omega)}^{\frac{2}{p}},$$

which, together with theorem 3.4, completes the proof. □

3.2. Measurement data in L^2-norm

In this subsection, we aim at recovering $q(x)$ from the L^2-noisy data of $u(q^\dagger)$. We assume that the measurable data z^δ of $u(q)$ is available with a noise level δ, namely

$$\|u(q^\dagger) - z^\delta\|_{0,\Omega} \leqslant \delta, \tag{3.33}$$

where $q^\dagger$ is the true physical radiativity. The elliptic inverse radiativity problem is transformed into an effective and stable minimisation system with Tikhonov regularization:

$$\min_{q\in K} J_{\delta,\beta}(q) = \min_{q\in K} \left(\frac{1}{2}\|u(q) - z^\delta\|_{0,\Omega}^2 + \frac{\beta}{2}\|q - q^*\|_{0,\Omega}^2\right), \tag{3.34}$$

where $\beta > 0$ is the regularization parameter and q^* is an *a priori* estimate of the true solution. We shall denote by q_β^δ the minimizer of (3.34). To study the convergence rate of q_β^δ, we first present some Hölder-type stability estimate for the elliptic inverse radiativity problem.

Theorem 3.5. *Assume* $|u(q^\dagger)| \geqslant c_0$ *a.e. in* Ω*, then for fixed* $\epsilon \in \left(0, \frac{1}{2}\right)$*, we have*

$$\|q - q^\dagger\|_{H^{-1-\epsilon}(\Omega)} \leqslant C\|u(q) - u(q^\dagger)\|_{0,\Omega}^{\frac{1}{2}} \quad \forall q \in K. \tag{3.35}$$

Proof. From theorem 3.2 and the well-known interpolation inequality (see [1, theorem 5.2])

$$\|u\|_{1,\Omega} \leqslant C\|u\|_{2,\Omega}^{\frac{1}{2}}\|u\|_{0,\Omega}^{\frac{1}{2}} \quad \forall\, u \in H^2(\Omega), \tag{3.36}$$

it suffices to show that

$$\|u(q) - u(q^\dagger)\|_{2,\Omega} \leqslant C \quad \forall q \in K. \tag{3.37}$$

In view of (3.4), we know that

$$-\nabla \cdot (a(x)\nabla(u(q^\dagger) - u(q))) + q(u(q^\dagger) - u(q)) = u(q^\dagger)(q - q^\dagger).$$

Then by lemma 2.2, we have

$$\|u(q^\dagger) - u(q)\|_{2,\Omega} \leqslant C\|u(q^\dagger)(q - q^\dagger)\|_{0,\Omega} \leqslant 2C\overline{q}\|u(q^\dagger)\|_{0,\Omega}, \tag{3.38}$$

which yields (3.37). □

Then we can prove the following analogue of theorem 3.3, whose proof is basically the same except that we use the results in theorem 3.5 here instead of theorem 3.2.

Theorem 3.6. *Assume $|u(q^\dagger)| \geqslant c_0$ a.e. in Ω and $q^\dagger - q^* \in H_0^\kappa(\Omega)$ with $\kappa > 0$ and $\kappa \neq 1/2$, then the following VSC holds for some constant $C > 0$,*

$$\frac{1}{4}\|q - q^\dagger\|_{0,\Omega}^2 \leqslant \frac{1}{2}\|q - q^*\|_{0,\Omega}^2 - \frac{1}{2}\|q^\dagger - q^*\|_{0,\Omega}^2 + C\|u(q) - u(q^\dagger)\|_{0,\Omega}^\alpha \quad \forall\, q \in K, \tag{3.39}$$

where the parameter α behaves as follows:

$$\begin{cases} \alpha = 1/2 & \text{if } \kappa > 1, \\ \alpha < \dfrac{\kappa}{1+\kappa} \ (\alpha \text{ can be chosen arbitrarily close to } \dfrac{\kappa}{1+\kappa}) & \text{if } \kappa \in \left(0, \dfrac{1}{2}\right) \cup \left(\dfrac{1}{2}, 1\right]. \end{cases}$$

With the aid of theorem 3.6, we can establish the convergence of $u(q_\beta^\delta)$, whose proof is basically the same as in theorem 3.4 and we only need to replace $\|u(q_\beta^\delta) - u(q^\dagger)\|_{1,\Omega}$ (or $\|\nabla u(q_\beta^\delta) - \nabla u(q^\dagger)\|_{0,\Omega}$) with $\|u(q_\beta^\delta) - u(q^\dagger)\|_{0,\Omega}$, which is valid due to (3.39).

Theorem 3.7. *Assume $|u(q^\dagger)| \geqslant c_0$ in Ω and $q^\dagger - q^* \in H_0^\kappa(\Omega)$ with $\kappa > 0$ and $\kappa \neq 1/2$, and α is the parameter chosen as in theorem 3.6, then we have the following convergence rates:*

$$\|u(q_\beta^\delta) - u(q^\dagger)\|_{0,\Omega} = O(\delta)$$

$$\|q_\beta^\delta - q^\dagger\|_{0,\Omega} = O(\delta^{\frac{\alpha}{2}}),$$

under the parameter choice $\beta = O(\delta^{2-\alpha})$.

Following the same arguments used in the proof of theorems 3.4 and 3.7, we can establish a more general result. For a general Hilbert space Y that is continuously embedded into $H^2(\Omega)$, we consider the following Tikhonov regularization:

$$\min_{q \in K} J_{\delta,\beta}(q) = \min_{q \in K} \left(\frac{1}{2}\|u(q) - z^\delta\|_Y^2 + \frac{\beta}{2}\|q - q^*\|_{0,\Omega}^2\right), \tag{3.40}$$

where the noisy data $z^\delta \in Y$ satisfies

$$\|z^\delta - u(q^\dagger)\|_Y \leqslant \delta.$$

Then we have the following results.

Theorem 3.8. *If there exist some $s_0 \in (0,1]$, $C > 0$ and $\alpha_0 \geqslant 0$ such that*

$$\|q^\dagger - q\|_{H^{-s_0}(\Omega)} \leqslant C\|u(q^\dagger) - u(q)\|_Y^{\alpha_0} \quad \forall\, q \in K \tag{3.41}$$

holds, and $q^\dagger - q^* \in H^{s_0\theta}(\Omega)$ for some $\theta \in (0,1]$ with $\theta s_0 \neq \frac{1}{2}$, then the minimizer q_β^δ of (3.40) enjoys the convergence rate $\|q_\beta^\delta - q^\dagger\|_{0,\Omega} = O(\delta^{\alpha/2})$ with $\alpha = \alpha_0\theta$ under a priori parameter choice $\beta = O(\delta^{2-\alpha})$.

In conclusion, the space Y in (3.41) quantifies the regularity condition on the noisy measurable data z^δ and α_0 is the 'maximal' convergence rate when $q^\dagger - q^* \in H_0^{s_0}(\Omega)$. If the regularity of $q^\dagger - q^*$ is weaker, then the convergence rate is smaller.

4. Convergence rates of Tikhonov regularization for parabolic inverse radiativity problem

In this section, we shall study the parabolic inverse radiativity problem proposed in section 1. Throughout this section, we always assume that $a(x) \in W^{1,\infty}(\Omega)$, $f(x,t) \in L^2(0,T;L^2(\Omega))$, $g(x,t) \in L^2(0,T;H^{\frac{3}{2}}(\partial\Omega)) \cap H^{\frac{3}{4}}(0,T;L^2(\partial\Omega))$ and $u_0 \in H^1(\Omega)$.

4.1. Measurement data in gradient form

We assume that the measurement data ∇z^δ of $\nabla u(q)$ is available in $\Omega \times I$, with a noise level δ, namely

$$\int_I \|\nabla u(q^\dagger) - \nabla z^\delta\|_{0,\Omega}^2 \mathrm{d}t \leqslant \delta^2, \tag{4.42}$$

where $q^\dagger$ is the true physical radiativity, and I is an arbitrary open subinterval of the entire time range $(0,T]$. We transform the inverse problem into the following output least-squares formulation with Tikhonov regularization:

$$\min_{q\in K} J_{\delta,\beta}(q) = \min_{q\in K}\left(\frac{1}{2}\int_I\int_\Omega |\nabla u(q) - \nabla z^\delta|^2 \mathrm{d}x\,\mathrm{d}t + \frac{\beta}{2}\|q - q^*\|_{0,\Omega}^2\right), \tag{4.43}$$

where $\beta > 0$ is the regularization parameter and q^* is an *a priori* estimate of the true parameter $q^\dagger$.

The following theorem states the existence of the minimizers to the optimization problem (4.43), whose proof is omitted and is very similar to the one of theorem 2.1 in [27, 28].

Theorem 4.1. *There exists at least a minimizer q_β^δ to the optimization problem* (4.43).

We are now going to derive some Lipschitz-type stability estimate for the parabolic inverse radiativity problem. This Lipschitz-type stability estimate is very critical in the subsequent rigorous verification of the VSC.

Lemma 4.1. *Assume $|u(q^\dagger)| \geqslant \bar{c}_0$ a.e. in $I \times \Omega$ for some constant $\bar{c}_0 > 0$.*

(*a*) *If the space dimension* $d = 2$*, then it holds for fixed* $\epsilon \in (0, 1/2)$ *that*

$$\|q - q^\dagger\|_{H^{-1-\epsilon}(\Omega)} \leqslant C\|u(q) - u(q^\dagger)\|_{L^2(I;H_0^1(\Omega))} \quad \forall\, q \in K. \tag{4.44}$$

(*b*) *If the space dimension* $d = 3$ *and*

$$\partial_t u(q^\dagger) \in L^2(I; L^3(\Omega)), \tag{4.45}$$

then (4.44) *also holds for fixed* $\epsilon \in (0, 1/2)$.

Proof. It is easy to see from (1.2) that

$$\partial_t u(q) - \partial_t u(q^\dagger) - \nabla \cdot (a\nabla \cdot (u(q) - u(q^\dagger))) + q(u(q) - u(q^\dagger)) = u(q^\dagger)(q^\dagger - q) \text{ in } I \times \Omega.$$

Multiplying both sides of the equation by an arbitrary function $\phi \in H_0^1(I, L^2(\Omega)) \cap L^2(I; H_0^1(\Omega))$ and integrating over $I \times \Omega$, we have

$$\left|\int_I\int_\Omega (q^\dagger - q)u(q^\dagger)\phi \mathrm{d}x\,\mathrm{d}t\right| \leqslant \left|\int_I\int_\Omega (u(q) - u(q^\dagger))\partial_t\phi \mathrm{d}x\,\mathrm{d}t\right| + \left|\int_I\int_\Omega a\nabla(u(q) - u(q^\dagger)) \cdot \nabla\phi \mathrm{d}x\,\mathrm{d}t\right| + \left|\int_I\int_\Omega q(u(q) - u(q^\dagger))\phi \mathrm{d}x\,\mathrm{d}t\right|,$$

from which we obtain by the Cauchy–Schwarz inequality that

$$\begin{aligned}
&\left|\int_I\int_\Omega (q^\dagger - q)u(q^\dagger)\phi\,\mathrm{d}x\,\mathrm{d}t\right| \\
&\leqslant \|u(q) - u(q^\dagger)\|_{L^2(I;L^2(\Omega))}\|\partial_t\phi\|_{L^2(I;L^2(\Omega))} + \|a\|_{L^\infty(\Omega)}\|\nabla(u(q) - u(q^\dagger))\|_{L^2(I;L^2(\Omega))} \\
&\quad \times \|\nabla\phi\|_{L^2(I;L^2(\Omega))} + \|q\|_{L^\infty(\Omega)}\|u(q) - u(q^\dagger)\|_{L^2(I;L^2(\Omega))}\|\phi\|_{L^2(I;L^2(\Omega))} \qquad (4.46)\\
&\leqslant (1 + \|a\|_{L^\infty(\Omega)} + \|q\|_{L^\infty(\Omega)})\|u(q) - u(q^\dagger)\|_{L^2(I;H_0^1(\Omega))}(\|\partial_t\phi\|_{L^2(I;L^2(\Omega))} + \|\phi\|_{L^2(I;H_0^1(\Omega))}).
\end{aligned}$$

Further, let us fix some $\varphi \in C_c^\infty(I)$ such that $\int_I \varphi(t)\mathrm{d}t = 1$ and set

$$\phi_h := hG, \quad \text{with } G := \frac{\varphi}{u(q^\dagger)} \tag{4.47}$$

for any $h \in H_0^{1+\epsilon}(\Omega)$. From the hypothesis $|u(q^\dagger)| \geqslant \bar{c}_0$ and the fact that $u(q^\dagger) \in L^2(I; H^2(\Omega))$ (by lemma 2.3), we can infer by Leibniz's rule that

$$G \in L^2(I; H^2(\Omega)) \cap H_0^1(I; L^2(\Omega)). \tag{4.48}$$

Then using remark 3.1 and (4.48), we have

$$\begin{aligned}
\|\phi_h\|^2_{L^2(I;H_0^1(\Omega))} &= \int_I \|hG\|^2_{H_0^1(\Omega)}\,\mathrm{d}t \leqslant C\int_I \|h\|^2_{1+\epsilon,\Omega}\|G\|^2_{2,\Omega}\,\mathrm{d}t \\
&= C\|h\|^2_{1+\epsilon,\Omega}\|G\|^2_{L^2(I;H^2(\Omega))} \leqslant C\|h\|^2_{1+\epsilon,\Omega}.
\end{aligned} \tag{4.49}$$

For the space dimension $d = 2$, we know that $h \in L^\infty(\Omega)$ and $\|h\|_{L^\infty(\Omega)} \leqslant C\|h\|_{1+\epsilon,\Omega}$ for all $h \in H_0^{1+\epsilon}(\Omega)$ by the Sobolev embedding theorem. Therefore, we obtain

$$\|\partial_t\phi_h\|^2_{L^2(I,L^2(\Omega))} = \int_I \|h\partial_t G\|^2_{L^2(\Omega)}\,\mathrm{d}t \leqslant \|h\|^2_{L^\infty(\Omega)}\int_I \|\partial_t G\|^2_{L^2(\Omega)}\,\mathrm{d}t \leqslant C\|h\|^2_{1+\epsilon,\Omega}. \tag{4.50}$$

For the space dimension $d = 3$, from the hypothesis $|u(q^\dagger)| \geqslant \bar{c}_0$ and (4.45), we get $\partial_t G \in L^2(I; L^3(\Omega))$. Therefore, it holds by the Hölder inequality and the Sobolev embedding theorem $H^1(\Omega) \Subset L^6(\Omega)$ that

$$\|\partial_t\phi_h\|^2_{L^2(I,L^2(\Omega))} = \int_I \|h\partial_t G\|^2_{L^2(\Omega)}\,\mathrm{d}t \leqslant \|h\|^2_{L^6(\Omega)}\int_I \|\partial_t G\|^2_{L^3(\Omega)}\,\mathrm{d}t \leqslant C\|h\|^2_{1+\epsilon,\Omega}. \tag{4.51}$$

Taking $\phi = \phi_h$ in (4.47) and using the estimates (4.49)–(4.51), we can conclude that for all $q \in K$ and $h \in H_0^{1+\epsilon}(\Omega)$,

$$\begin{aligned}\left|\int_I\int_\Omega (q^\dagger - q)u(q^\dagger)\phi_h\,\mathrm{d}x\,\mathrm{d}t\right| &= \left|\int_I\int_\Omega (q^\dagger - q)h\varphi\,\mathrm{d}x\,\mathrm{d}t\right| \\ &= \left|\int_\Omega (q^\dagger - q)h\,\mathrm{d}x\right| \leqslant C\|u(q) - u(q^\dagger)\|_{L^2(I;H_0^1(\Omega))}\|h\|_{1+\epsilon,\Omega},\end{aligned} \tag{4.52}$$

which implies (4.44).

□

Remark 4.1. The assumption (4.45) holds provided that the source term $f(x,t)$, the ambient temperature $g(x,t)$ and the initial value $u_0(x)$ are smooth enough. For example, if $f(x,t) \in L^3(I\times\Omega)$, $g(x,t) \in W_3^{\frac{5}{3},\frac{5}{6}}(I\times\partial\Omega)$ and $u_0(x) \in W_3^{4/3}(\Omega)$, then $\partial_t u(q^\dagger) \in L^3(I\times\Omega)$ (see [25, chapter VI]) and hence (4.45) is true. For the definitions of the Sobolev–Slobodeckij type spaces $W_3^{\frac{5}{3},\frac{5}{6}}(I\times\partial\Omega)$ and $W_3^{4/3}(\Omega)$, we refer to [25].

Remark 4.2. We present a sufficient condition under which the true solution $u(q^\dagger)$ fulfills the required positiveness assumption. If f, g and u_0 are nonnegative a.e. in Ω, then $u(q^\dagger)$ is nonnegative by theorem 1 in [23]. Hence, it follows that

$$\begin{aligned}&\partial_t u(q^\dagger) - \nabla\cdot(a(x)\nabla u(q^\dagger)) + \bar{q}u(q^\dagger) \\ &\quad\geqslant \partial_t u(q^\dagger) - \nabla\cdot(a(x)\nabla u(q^\dagger)) + q^\dagger u(q^\dagger) = f \geqslant 0.\end{aligned} \tag{4.53}$$

Assuming that there exists some $c^* > 0$ such that $u_0 \geqslant c^*$ a.e. in Ω and $g(t,x) \geqslant c^*$ a.e. on $\partial\Omega\times(0,T)$ and letting $w := u(q^\dagger) - c^*\mathrm{e}^{-\bar{q}t}$, we get from (4.53) that

$$\begin{aligned}&\partial_t w - \nabla\cdot(a(x)\nabla w) + \bar{q}w \\ &\quad= \partial_t(u(q^\dagger) - c^*\mathrm{e}^{-\bar{q}t}) - \nabla\cdot(a(x)\nabla(u(q^\dagger) - c^*\mathrm{e}^{-\bar{q}t})) + \bar{q}(u(q^\dagger) - c^*\mathrm{e}^{-\bar{q}t}) \\ &\quad= \partial_t u(q^\dagger) + \bar{q}c^*\mathrm{e}^{-\bar{q}t} - \nabla\cdot(a(x)\nabla u(q^\dagger)) + \bar{q}u(q^\dagger) - \bar{q}c^*\mathrm{e}^{-\bar{q}t} \\ &\quad= \partial_t u(q^\dagger) - \nabla\cdot(a(x)\nabla u(q^\dagger)) + \bar{q}u(q^\dagger) \geqslant 0.\end{aligned}$$

Since $w(t,x) = g(t,x) - c^*\mathrm{e}^{-\bar{q}t} \geqslant c^* - c^* = 0$ a.e. on $\partial\Omega\times(0,T)$ and $w(0,x) = u_0 - c^* \geqslant 0$ a.e in Ω, we can infer that $w = u(q^\dagger) - c^*\mathrm{e}^{-\bar{q}t} \geqslant 0$ a.e. in $\Omega\times(0,T)$ by theorem 1 in [23]. This implies $u(q^\dagger) \geqslant c^*\mathrm{e}^{-\bar{q}t} \geqslant c^*\mathrm{e}^{-\bar{q}T}$ a.e. in $\Omega\times(0,T)$.

Now we shall propose the following variational source condition: for any $q \in K$,

$$\frac{1}{4}\|q-q^{\dagger}\|_{0,\Omega}^{2} \leqslant \frac{1}{2}\|q-q^{*}\|_{0,\Omega}^{2} - \frac{1}{2}\|q^{\dagger}-q^{*}\|_{0,\Omega}^{2} + C\|u(q)-u(q^{\dagger})\|_{L^{2}(I;H_{0}^{1}(\Omega))}^{\alpha} \tag{4.54}$$

and its equivalent form

$$(q^{\dagger}-q^{*}, q^{\dagger}-q)_{\Omega} \leqslant \frac{1}{4}\|q-q^{\dagger}\|_{0,\Omega}^{2} + C\|u(q)-u(q^{\dagger})\|_{L^{2}(I;H_{0}^{1}(\Omega))}^{\alpha}, \tag{4.55}$$

where the parameter α behaves as follows:

$$\begin{cases} \alpha = 1 & \text{if } \kappa > 1 \\ \alpha < \dfrac{2\kappa}{1+\kappa} \ (\alpha \text{ can be chosen arbitrarily close to } \dfrac{2\kappa}{1+\kappa}) & \text{if } \kappa \in \left(0, \dfrac{1}{2}\right) \cup \left(\dfrac{1}{2}, 1\right). \end{cases} \tag{4.56}$$

Following the same arguments used as in theorem 3.3 and using lemma 4.1, we can obtain the following results.

Theorem 4.2. *Assume $|u(q^{\dagger})| \geqslant \bar{c}_0$ a.e. in $I \times \Omega$, $q^{\dagger} - q^{*} \in H_0^{\kappa}(\Omega)$ with $\kappa > 0$ and $\kappa \neq 1/2$, and in addition, (4.45) holds for the space dimension $d = 3$, then the VSC (4.55) holds.*

With the aid of the proposed VSC (4.54), we are ready to establish the following results about the convergence rate, whose proof follows the same techniques used in theorem 3.4.

Theorem 4.3. *Assume $|u(q^{\dagger})| \geqslant \bar{c}_0$ in $I \times \Omega$, $q^{\dagger} - q^{*} \in H_0^{\kappa}(\Omega)$ with $\kappa > 0$ and $\kappa \neq 1/2$, and in addition, (4.45) holds for the space dimension $d = 3$. Let α be the parameter defined in (4.56), then the following convergences hold under the parameter choice $\beta = O(\delta^{2-\alpha})$:*

$$\|\nabla u(q_{\beta}^{\delta}) - \nabla u(q^{\dagger})\|_{L^{2}(I;L^{2}(\Omega))} = O(\delta), \tag{4.57}$$

$$\|q_{\beta}^{\delta} - q^{\dagger}\|_{0,\Omega} = O(\delta^{\frac{\alpha}{2}}). \tag{4.58}$$

Similar to corollary 3.1, we can obtain the following result.

Corollary 4.1. *Under the hypothesises and settings of theorem 4.3, we have the convergence rate for $2 \leqslant p < +\infty$,*

$$\|q_{\beta}^{\delta} - q^{\dagger}\|_{L^{p}(\Omega)} = O(\delta^{\frac{\alpha}{p}}) \tag{4.59}$$

under the parameter choice $\beta = O(\delta^{2-\alpha})$.

4.2. Measurement data in L^2-norm

In this subsection, we aim at recovering $q(x)$ from the L^2-noisy data of $u(q^{\dagger})$ in $I \times \Omega$. We assume that the measurable data z^{δ} of $u(q)$ is available with a noise level δ, namely

$$\int_I \int_{\Omega} |u(q^{\dagger}) - z^{\delta}|^2 \mathrm{d}x\, \mathrm{d}t \leqslant \delta^2. \tag{4.60}$$

Let q_β^δ is the minimizer of the output least-squares formulation with Tikhonov regularization:

$$\min_{q\in K} J_{\delta,\beta}(q) = \min_{q\in K}\left(\frac{1}{2}\int_I\int_\Omega |u(q)-z^\delta|^2 \mathrm{d}x\,\mathrm{d}t + \frac{\beta}{2}\|q-q^*\|_{0,\Omega}^2\right), \tag{4.61}$$

where $\beta > 0$ is the regularization parameter and q^* is an *a priori* estimate of the true parameter $q^\dagger$. Our goal is to study the convergence rate of the regularized solution q_β^δ. To this end, we follow the same procedure used in subsection 4.1 and first establish the following Hölder type estimate for the parabolic inverse radiativity problem.

Lemma 4.2. *Assume* $|u(q^\dagger)| \geqslant \bar{c}_0$ *a.e. in* $I\times\Omega$ *for some constant* $\bar{c}_0 > 0$.

(*a*) *If the space dimension* $d = 2$, *then it holds for fixed* $\epsilon \in (0, 1/2)$ *that*

$$\|q-q^\dagger\|_{H^{-1-\epsilon}(\Omega)} \leqslant C\|u(q)-u(q^\dagger)\|_{L^2(I;L^2(\Omega))}^{\frac{1}{2}} \quad \forall\, q\in K. \tag{4.62}$$

(*b*) *If the space dimension* $d = 3$ *and* (4.45) *is fulfilled, then* (4.62) *also holds for fixed* $\epsilon \in (0, 1/2)$.

Proof. From (3.36) and the Cauchy–Schwarz inequality, it follows that

$$\|u\|_{L^2(I;H^1(\Omega))} \leqslant C\|u\|_{L^2(I;H^2(\Omega))}^{\frac{1}{2}}\|u\|_{L^2(I;L^2(\Omega))}^{\frac{1}{2}} \quad \forall\, u\in L^2(I;H^2(\Omega)). \tag{4.63}$$

Hence, from (4.44), it suffices to show that

$$\|u(q)-u(q^\dagger)\|_{L^2(I;H^2(\Omega))} \leqslant C. \tag{4.64}$$

In view of (1.2), we know that $w = u(q) - u(q^\dagger)$ satisfies

$$\begin{cases} \partial_t w - \nabla\cdot(a(x)\nabla w) + q(x)w & = u(q^\dagger)(q^\dagger - q) \quad \text{in} \quad \Omega\times(0,T], \\ w(x,0) & = 0 \quad \text{in} \quad \Omega, \\ w(x,t) & = 0 \quad \text{on} \quad \partial\Omega\times(0,T], \end{cases} \tag{4.65}$$

Then by making use of lemma 2.3, we get

$$\|u(q)-u(q^\dagger)\|_{L^2(I;H^2(\Omega))} \leqslant C\|u(q^\dagger)(q^\dagger-q)\|_{L^2(0,T;L^2(\Omega))} \leqslant 2C\overline{q}\|u(q^\dagger)\|_{L^2(0,T;L^2(\Omega))},$$

which infers (4.64). □

We are now ready to establish the analogues of theorems 4.2 and 4.3, whose proofs are basically the same except that we use the results in lemma 4.2 here instead of lemma 4.1.

Theorem 4.4. *Assume* $|u(q^\dagger)| \geqslant \bar{c}_0$ *a.e. in* $\Omega\times I$, $q^\dagger - q^* \in H_0^\kappa(\Omega)$ *with* $\kappa > 0$ *and* $\kappa \neq 1/2$, *and in addition,* (4.45) *holds for the space dimension* $d = 3$, *then the following VSC holds:*

$$\frac{1}{4}\|q-q^\dagger\|_{0,\Omega}^2 \leqslant \frac{1}{2}\|q-q^*\|_{0,\Omega}^2 - \frac{1}{2}\|q^\dagger-q^*\|_{0,\Omega}^2 + C\|u(q)-u(q^\dagger)\|_{L^2(I;L^2(\Omega))}^\alpha, \tag{4.66}$$

where the parameter α *is specified by*

$$\begin{cases} \alpha = 1/2 & \textit{if } \kappa > 1 \\ \alpha < \dfrac{\kappa}{1+\kappa} \ (\alpha \textit{ can be chosen arbitrarily close to } \dfrac{\kappa}{1+\kappa}) & \textit{if } \kappa \in \left(0, \dfrac{1}{2}\right) \cup \left(\dfrac{1}{2}, 1\right). \end{cases} \tag{4.67}$$

Theorem 4.5. *Assume* $|u(q^\dagger)| \geqslant \bar{c}_0$, $q^\dagger - q^* \in H_0^\kappa(\Omega)$ *with* $\kappa > 0$ *and* $\kappa \neq 1/2$*, and in addition,* (4.45) *holds for the space dimension* $d = 3$*. Let* α *be the parameter defined in* (4.67)*, then the following convergences hold under the parameter choice* $\beta = O(\delta^{2-\alpha})$:

$$\|u(q_\beta^\delta) - u(q^\dagger)\|_{L^2(I;L^2(\Omega))} = O(\delta),$$

$$\|q_\beta^\delta - q^\dagger\|_{0,\Omega} = O(\delta^{\frac{\alpha}{2}}).$$

5. Concluding remarks

We have established some important Lipschitz-type stability estimates for both the elliptic and parabolic inverse radiativity problems, which are then applied to help us rigorously verify the variational source conditions in general dimensional spaces. With these variational source conditions, reasonable convergence rates are achieved for both the elliptic and parabolic inverse radiativity problems. The convergence results have explicitly revealed the relation between the regularity of the radiativities and the convergence rates we may achieve.

The analytical strategies developed in this work, especially for the Lipschitz-type stability estimates and convergence rates, appear to be very promising for the future development of the variational source conditions and convergence rates for other nonlinear inverse problems, including the elliptic and parabolic inverse conductivity problems, inverse acoustic and Maxwell medium scattering problems.

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