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Zeros of the Bessel and spherical Bessel functions and their applications for uniqueness in inverse acoustic obstacle scattering

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Some novel interlacing properties of the zeros for the Bessel and spherical Bessel functions are first presented and then applied to prove an interesting uniqueness result in inverse acoustic obstacle scattering. It is shown that in the resonance region, the shape of a sound-soft/sound-hard ball in $\mathbb{R}^3$ or a sound-soft/sound-hard disc in $\mathbb{R}^2$ is uniquely determined by a single far-field datum measured at some fixed spot corresponding to a single incident plane wave.

Keywords: inverse obstacle scattering; uniqueness; discs and balls.

1. Introduction

In this paper, we will be mainly concerned with an inverse acoustic obstacle scattering problem (IAOSP) of determining the shape of an unknown impenetrable obstacle D from its corresponding scattered far-field pattern. It is assumed that the obstacle $D \subset \mathbb{R}^N (N = 2, 3)$ is a bounded domain with connected complement $G := \mathbb{R}^N \backslash D$. Consider a given incident plane wave $u^{\mathrm{i}}(x) = \exp\{\mathrm{i}kx \cdot d\}$, where $\mathrm{i} = \sqrt{-1}$, $d \in \mathbb{S}^{N-1}$ is the incident direction and $k > 0$ is the wave number. The presence of the inhomogeneity D will scatter the incident wave and lead to a scattered wave u^{s}. Writing the total field as u, then $u = u^{\mathrm{i}} + u^{\mathrm{s}}$ and solves the following Helmholtz system:

$$\begin{cases} \Delta u + k^2 u = 0 \quad \text{in } G, \\ \lim_{r\to\infty} r^{(N-1)/2}\left(\frac{\partial u^{\mathrm{s}}}{\partial r} - \mathrm{i}ku^{\mathrm{s}}\right) = 0, \end{cases} \tag{1.1}$$

where $r = |x|$ for any $x \in \mathbb{R}^N$. System (1.1) will be associated with either of the following boundary conditions:

$$u = 0 \quad \text{on } \partial G \text{ (the sound-soft obstacle)}, \tag{1.2}$$

$$\frac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial G \text{ (the sound-hard obstacle)}, \tag{1.3}$$

where ν is the unit normal to ∂G directed into the interior of G.

We know that for any Lipschitz continuous boundary ∂G, there exists a unique solution $u = u(D; d, k) \in H^1_{\mathrm{loc}}(G)$ to the above Helmholtz system, and u is analytic on any compact set in G,

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see McLean (2000). Moreover, the asymptotic behaviour at infinity of the scattered wave u^s is given by

$$u^s(x) = \frac{e^{ik|x|}}{|x|^{(N-1)/2}} \left\{ u_\infty(\hat{x}) + O\left(\frac{1}{|x|}\right) \right\} \quad \text{as } |x| \to \infty \tag{1.4}$$

uniformly for all directions $\hat{x} = x/|x| \in \mathbb{S}^{N-1}$. The analytic function $u_\infty(\hat{x})$ is defined on the unit sphere $\mathbb{S}^{N-1}$, and often called the far-field pattern, see Colton & Kress (1998). We shall write $u_\infty(\hat{x}; D, d, k)$ to specify its dependence on the observation direction $\hat{x}$, the obstacle D, the incident direction d and the wave number k.

The IAOSP is to determine ∂G from the measurement of the far-field pattern $u_\infty(\hat{x}; D, d, k)$, i.e. we need to solve the following operator equation:

$$F(\partial G) = u_\infty(\hat{x}; D, d, k), \quad (\hat{x}, d, k) \in \mathbb{S}_0^{N-1} \times \widetilde{\mathbb{S}}_0^{N-1} \times \mathbb{K} \subset \mathbb{S}^{N-1} \times \mathbb{S}^{N-1} \times (0, +\infty),$$

where $\mathbb{S}_0^{N-1}$ and $\widetilde{\mathbb{S}}_0^{N-1}$ are subsets of $\mathbb{S}^{N-1}$, $\mathbb{K} \subset (0, \infty)$ and the non-linear operator F is defined by the Helmholtz equation system. This problem plays an indispensable role in many fields of science and technology such as radar and sonar, medical imaging, geophysical exploration and non-destructive testing, etc., see Colton & Kress (1998). An important theoretical issue in IAOSP is the 'uniqueness', i.e. how many measurement data may uniquely determine the object. It is observed that for a single incident wave at a fixed frequency and incident direction, the inverse problem is formally determined with measurement in every possible direction, since the measurement data depend on the same number of variables, $N-1$, as does the object. However, even up to now, this important theoretical and practical problem still remains largely open.

Since the first uniqueness result due to Schiffer for sound-soft general C^2-obstacle by countably many incident plane waves (see Colton & Kress, 1998; Lax & Phillips, 1967), there has been extensive study in this direction and a lot of results can be found in the literature; see Colton & Sleeman (1983), Gintides (2005), Kirsch & Kress (1993), Rondi (2003) and Sleeman (1982) for uniqueness with general smooth obstacles, see Alessandrini & Rondi (2005), Cheng & Yamamoto (2003), Elschner & Yamamoto (2006) and Liu & Zou (2006a,b; 2007) for uniqueness with polyhedral- type scatterers and see Liu (1997) and Yun (2001) for uniqueness with balls and Kress (1995a,b) and Mönch (1997) for uniqueness with obstacles of smooth planar curves.

It is emphasized that for uniqueness studies, one can assume that the far-field data are given only on an open subset of $\mathbb{S}^{N-1}$, no matter how small the subset is, since we can always recover such data on the whole unit sphere by analytic continuation. In fact, all the existing results about uniqueness for the IAOSP have made use of the fact that far-field pattern is given on the whole unit sphere. However, in some cases, if the obstacle is of very simple geometric structures, e.g. it is a ball or a cube in $\mathbb{R}^N$, then its radius and centre can uniquely identify the object. Hence, for a ball or a disc, two measurement data are sufficient to formally determine the obstacle. This raises a natural question: 'could uniqueness be established in those cases mathematically'? In the present paper, we make a first step towards this important direction. We will prove that $u_\infty(d_0; D, d_0, k_0)$ with d_0 and k_0 fixed uniquely determines the radius of the obstacle D which can be a sound-soft/sound-hard ball in $\mathbb{R}^3$ or a sound-soft/sound-hard disc in $\mathbb{R}^2$ centred at origin, provided k_0 is in the resonance region. More accurately speaking, in the sound-soft case, if $k_0 R < \pi/2$, then the radius R of the underlying ball is uniquely determined by the single datum $u_\infty(d_0; D, d_0, k_0)$ (this condition becomes $k_0 R < 0.89357697$ in the 2D case); whereas in the sound-hard case, if $k_0 R < \sqrt{2}$, then R is uniquely determined by $u_\infty(d_0; D, d_0, k_0)$ (this condition becomes $k_0 R < 1$ in the 2D case). To our knowledge, this is the first result of such kind which uses only a single measurement datum. The proof is based on carefully studying the forward-scattering far-field data given by the series expansion of wave or spherical wave functions and some fine properties of the zeros of the Bessel and spherical Bessel functions. We wish to emphasize that there are uniqueness results by one single incident plane wave for a sound-hard or sound-soft ball in $\mathbb{R}^3$ (see Liu, 1997; Yun, 2001), but they rely heavily on the reflection of solutions to Helmholtz equation (1.1) corresponding to a ball, as well as require the measured far-field data on the whole unit sphere.

The plan of the paper is as follows: In Section 2, we collect and present some preliminary knowledge of the Bessel and spherical Bessel functions which is relevant to our investigation. Section 3 is devoted to the main uniqueness results.

2. Bessel and spherical Bessel functions

In this section, we shall present some novel properties of the Bessel and spherical Bessel functions, especially about the positive zeros of these functions and their derivatives for the subsequent use. We refer to Abramowitz & Stegun (1965), Tranter (1968) and Watson (1944) for other more elaborate and intensive studies on the Bessel functions and Colton & Kress (1998) and Lebedev (1965) for discussions about applications of the Bessel functions to wave scattering theory. In the following, let $n \in \mathbb{N} \cup \{0\}$ be an non-negative integer. The first- and second-kind Bessel functions of order n, namely, $J_n(t)$ and $Y_n(t)$ are, respectively, defined by the following series expansions:

$$J_n(t) = \sum_{p=0}^{\infty} \frac{(-1)^p \left(\frac{1}{2}t\right)^{n+2p}}{p!\Gamma(n+p+1)}, \tag{2.1}$$

$$\begin{aligned} Y_n(t) = \frac{2}{\pi}\left\{\ln\frac{t}{2} + C\right\} J_n(t) - \frac{1}{\pi}\sum_{p=0}^{n-1}\frac{(n-1-p)!}{p!}\left(\frac{2}{t}\right)^{n-2p} \\ - \frac{1}{\pi}\sum_{p=0}^{\infty}\frac{(-1)^p}{p!(n+p)!}\left(\frac{t}{2}\right)^{n+2p}\{\psi(p+n)+\psi(p)\}, \end{aligned} \tag{2.2}$$

where $\psi(p) := \sum_{m=1}^{p} 1/m$ for $p = 1, 2, \ldots,$ $\psi(0) := 0$ and $C \approx 0.5772$ is the Euler's constant. And in (2.2), the finite sum is set to be zero in the case $n = 0$. Usually, J_n is referred to as the Bessel function while Y_n as the Neumann function, and both are solutions to the Bessel differential equation

$$t^2 f''(t) + t f'(t) + [t^2 - n^2] f(t) = 0, \tag{2.3}$$

which arises when finding separable solutions to Laplace's equation and the Helmholtz equation in cylindrical and spherical coordinates. For our purpose, we are especially interested in the real positive zeros of the Bessel and the Neumann functions and their respective derivatives. Due to the Bessel–Lommel theorem (see Watson, 1944, Chapter XV), it is well-known that both the Bessel and the Neumann functions have infinitely many positive zeros, with no repetitions except for the possible zero at the origin. By Rolle's theorem, we know that both $J_n'(t)$ and $Y_n'(t)$ also have infinitely many positive zeros. From now on and throughout the rest of the paper, we shall use $j_{n,s}$ to denote the sth positive zero, arranged in ascending order of magnitude, of the function $J_n(x)$ and $j_{n,s}'$ the sth positive zero of $J_n'(x)$, except that in the case of $J_0'(t)$, we count $t = 0$ as its first 'positive' zero. Similarly, the sth positive zeros of $Y_n(t)$ and $Y_n'(t)$ are denoted by $y_{n,s}$ and $y_{n,s}'$, respectively. The following lemma summarizes some interlacing properties of these zero points, see Abramowitz & Stegun (1965, Section 9.5, pp. 370), Watson (1944, Chapter XV) and Liu & Zou (2006c, Theorems 3 and 4).

LEMMA 2.1 For any $n \in \mathbb{N} \cup \{0\}$, the positive zeros of $J_n(t)$ are interlaced with those of $J_{n+1}(t)$:

$$j_{n,1} < j_{n+1,1} < j_{n,2} < j_{n+1,2} < j_{n,3} < \cdots, \tag{2.4}$$

while the positive zeros of $Y_n(t)$ are interlaced with those of $Y_{n+1}(t)$:

$$y_{n,1} < y_{n+1,1} < y_{n,2} < y_{n+1,2} < y_{n,3} < \cdots. \tag{2.5}$$

Similarly, the positive zeros of $J_n'(t)$ are interlaced with those of $J_{n+1}'(t)$:

$$j_{n,1}' < j_{n+1,1}' < j_{n,2}' < j_{n+1,2}' < j_{n,3}' < \cdots, \tag{2.6}$$

while the positive zeros of $Y_n'(t)$ are interlaced with those of $Y_{n+1}'(t)$:

$$y_{n,1}' < y_{n+1,1}' < y_{n,2}' < y_{n+1,2}' < y_{n,3}' < \cdots. \tag{2.7}$$

By taking $n = 0, 1, 2, \ldots$ in Lemma 2.1, we immediately derive the following lemma with mathematical induction.

LEMMA 2.2 For each $s = 1, 2, 3, \ldots$, the sequences $\{j_{n,s}\}_{n=0}^{\infty}$ and $\{y_{n,s}\}_{n=0}^{\infty}$ are strictly monotonically increasing, i.e. for each $s = 1, 2, 3, \ldots$,

$$j_{0,s} < j_{1,s} < \cdots < j_{n,s} < j_{n+1,s} < \cdots, \quad y_{0,s} < y_{1,s} < \cdots < y_{n,s} < y_{n+1,s} < \cdots. \tag{2.8}$$

Similarly, for each $s = 1, 2, 3, \ldots$, the sequences $\{j_{n,s}'\}_{n=0}^{\infty}$ and $\{y_{n,s}'\}_{n=0}^{\infty}$ are strictly monotonically increasing, i.e. for each $s = 1, 2, 3, \ldots$,

$$j_{0,s}' < j_{1,s}' < \cdots < j_{n,s}' < j_{n+1,s}' < \cdots, \quad y_{0,s}' < y_{1,s}' < \cdots < y_{n,s}' < y_{n+1,s}' < \cdots. \tag{2.9}$$

With the help of Lemma 2.2, we are able to show the following crucial results to our subsequent uniqueness investigation.

THEOREM 2.1 For the Bessel and the Neumann functions $J_n(t)$ and $Y_n(t)$, we have

$$J_n(t)Y_n(t) < 0 \quad \text{for } t \in (0, 0.89357697) \tag{2.10}$$

uniformly for all $n \in \mathbb{N} \cup \{0\}$, while for $J_n'(t)$ and $Y_n'(t)$, we have for $t \in (0, 1)$ that

$$J_0'(t)Y_0'(t) < 0, \quad J_n'(t)Y_n'(t) > 0, \quad n = 1, 2, 3, \ldots. \tag{2.11}$$

Proof. We first make the following observations that for each fixed n,

$$J_n(t) = \frac{t^n}{2^n \Gamma(n+1)}[1 + \mathrm{O}(t^2)] \quad \text{as } t \to +0, \quad n = 0, 1, 2, \ldots, \tag{2.12}$$

$$J_0'(t) = -\frac{t}{2}[1 + \mathrm{O}(t^2)] \quad \text{as } t \to +0, \tag{2.13}$$

$$J_n'(t) = \frac{nt^{n-1}}{2^n \Gamma(n+1)}[1 + \mathrm{O}(t^2)] \quad \text{as } t \to +0, \quad n = 1, 2, 3, \ldots, \tag{2.14}$$

whereas for n being a positive integer,

$$Y_n(t) = -\frac{2^n(n-1)!}{\pi t^n}[1 + \mathrm{O}(t)] \quad \text{as } t \to +0, \tag{2.15}$$

$$Y_n'(t) = \frac{2^n n!}{\pi t^{n+1}}[1 + \mathrm{O}(t)] \quad \text{as } t \to +0; \tag{2.16}$$

and finally for $n = 0$,

$$Y_0(t) = \frac{2}{\pi}\left[\ln\frac{t}{2} + C\right][1 + \mathrm{O}(t^2)] \quad \text{as } t \to +0, \tag{2.17}$$

$$Y_0'(t) = \frac{2}{\pi t}[1 + \mathrm{O}(t)] \quad \text{as } t \to +0. \tag{2.18}$$

Hence, it is seen from (2.12), (2.15) and (2.17) that for $n \in \mathbb{N} \cup \{0\}$ and sufficiently small $t > 0$, $J_n(t)$ is positive but $Y_n(t)$ is negative. This further implies that $J_n(t) > 0$ for $t \in (0, j_{n,1})$ and $Y_n(t) < 0$ for $t \in (0, y_{n,1})$. Next, by Lemma 2.2, we know that the sequences $\{j_{n,1}\}_{n=0}^{\infty}$ and $\{y_{n,1}\}_{n=0}^{\infty}$ are strictly monotonically increasing, i.e.

$$j_{0,1} < j_{1,1} < j_{2,1} < \cdots, \quad y_{0,1} < y_{1,1} < y_{2,1} < \cdots.$$

Therefore, the smallest positive zeros for $J_n(t)$ and $Y_n(t)$, $n = 0, 1, 2, \ldots$, are, respectively, given by (see Olver, 1960, Table I)

$$j_{0,1} = 2.40482556, \quad y_{0,1} = 0.89357697,$$

and then it is easily deduced that

$$J_n(t)Y_n(t) < 0 \quad \text{for } t \in (0, 0.89357697) \tag{2.19}$$

uniformly for $n = 0, 1, 2, \ldots$.

We proceed to prove the relation (2.11) for the derivatives $J_n'(t)$ and $Y_n'(t)$. By (2.13), (2.14), (2.16) and (2.18), it is seen that for $n \in \mathbb{N} \cup \{0\}$ and sufficiently small $t > 0$, both $J_n'(t)$ and $Y_n'(t)$ are positive with the only exceptional case $J_0'(t)$, which is negative near the origin. By Lemma 2.2, the sequence $\{j_{n,1}'\}_{n=0}^{\infty}$ is strictly monotonically increasing, i.e.

$$j_{0,1}' < j_{1,1}' < \cdots < j_{n,1}' < j_{n+1,1}' < \cdots.$$

Noting that $j_{0,1}' = 0$, which together with the fact that $j_{1,1}' < j_{0,2}'$ by Lemma 2.1, shows that the smallest positive zero for $J_n'(t)$, $n = 0, 1, 2, \ldots$, is given by $j_{1,1}' = 1.84118378$ (see Olver, 1960, Table II). Hence, we know that for $t \in (0, 1.84118378)$,

$$J_0'(t) < 0, \quad J_n'(t) > 0, \quad n = 1, 2, \ldots. \tag{2.20}$$

Similarly, by Lemma 2.2, the sequence $\{y_{n,1}'\}_{n=0}^{\infty}$ is strictly monotonically increasing and hence the smallest positive zero for $Y_n'(t)$ is given by $y_{0,1}' = 2.19714132$ (see also Olver, 1960, Table II). Hence, we have for $t \in (0, 2.19714132)$,

$$Y_n'(t) > 0, \quad n = 0, 1, 2, \ldots. \tag{2.21}$$

Combining (2.20) with (2.21) immediately gives (2.11). □

The above discussions for the Bessel and the Neumann functions will be mainly used for the later analysis of 2D inverse scattering problem, whereas for our study of the 3D inverse scattering problem, we need similar results for spherical Bessel functions. The first- and second-kind spherical Bessel functions of order n, namely, $j_n(t)$ and $y_n(t)$ are, respectively, given by the following series expansions:

$$j_n(t) := \sum_{p=0}^{\infty} \frac{(-1)^p t^{n+2p}}{2^p p! 1 \cdot 3 \cdots (2n+2p+1)}, \tag{2.22}$$

$$y_n(t) := -\frac{(2n)!}{2^n n!} \sum_{p=0}^{\infty} \frac{(-1)^p t^{2p-n-1}}{2^p p!(-2n+1)(-2n+3)\cdots(-2n+2p-1)}. \tag{2.23}$$

Usually, $j_n(t)$ is referred to as spherical Bessel function while $y_n(t)$ as spherical Neumann function, and both are solutions to the spherical Bessel differential equation

$$t^2 f''(t) + 2t f'(t) + [t^2 - n(n+1)] f(t) = 0. \tag{2.24}$$

By the relationship between spherical Bessel functions and Bessel functions of real orders, it is known that there are infinitely many positive zeros for both $j_n(t)$ and $y_n(t)$, hence also for $j_n'(t)$ and $y_n'(t)$ (see Watson, 1944; Liu & Zou, 2006c). Henceforth, we let the sth positive zeros of $j_n(t)$, $y_n(t)$, $j_n'(t)$ and $y_n'(t)$ for $n \in \mathbb{N} \cup \{0\}$ to be denoted by $a_{n,s}, b_{n,s}, a'_{n,s}$ and $b'_{n,s}$ respectively, except that for $n = 0$, we count $t = 0$ to be the first positive zero of $j_0'(t)$. Then, similar to Lemma 2.1, one can obtain the following interlacing relations, whose proof can be found in Liu & Zou (2006c).

LEMMA 2.3 For any $n \in \mathbb{N} \cup \{0\}$, we have

$$a_{n,1} < a_{n+1,1} < a_{n,2} < a_{n+1,2} < a_{n,3} < \cdots, \quad b_{n,1} < b_{n+1,1} < b_{n,2} < b_{n+1,2} < b_{n,3} < \cdots \tag{2.25}$$

and

$$a'_{n,1} < a'_{n+1,1} < a'_{n,2} < a'_{n+1,2} < a'_{n,3} < \cdots, \quad b'_{n,1} < b'_{n+1,1} < b'_{n,2} < b'_{n+1,2} < b'_{n,3} < \cdots. \tag{2.26}$$

By taking $n = 0, 1, 2, \ldots$ in Lemma 2.3, we easily derive the following lemma with mathematical induction.

LEMMA 2.4 For each $s = 1, 2, 3, \ldots$, the sequences $\{a_{n,s}\}_{n=0}^{\infty}$, $\{b_{n,s}\}_{n=0}^{\infty}$, $\{a'_{n,s}\}_{n=0}^{\infty}$ and $\{b'_{n,s}\}_{n=0}^{\infty}$ are all strictly monotonically increasing, i.e. for each $s = 1, 2, 3, \ldots$,

$$a_{0,s} < a_{1,s} < \cdots < a_{n,s} < a_{n+1,s} < \cdots, \quad b_{0,s} < b_{1,s} < \cdots < b_{n,s} < b_{n+1,s} < \cdots \tag{2.27}$$

and

$$a'_{0,s} < a'_{1,s} < \cdots < a'_{n,s} < a'_{n+1,s} < \cdots, \quad b'_{0,s} < b'_{1,s} < \cdots < b'_{n,s} < b'_{n+1,s} < \cdots. \tag{2.28}$$

Now, with Lemma 2.4, we can show the following theorem.

THEOREM 2.2 For spherical Bessel and spherical Neumann functions $j_n(t)$ and $y_n(t)$, we have

$$j_n(t) y_n(t) < 0 \quad \text{for } t \in \left(0, \frac{\pi}{2}\right) \tag{2.29}$$

uniformly for all $n \in \mathbb{N} \cup \{0\}$, while for $j_n'(t)$ and $y_n'(t)$, we have for $t \in (0, \sqrt{2})$ that

$$j_0'(t) y_0'(t) < 0, \quad j_n'(t) y_n'(t) > 0, \quad n = 1, 2, 3, \ldots. \tag{2.30}$$

Proof. The proof is similar to that for Theorem 2.1. Firstly, it is easily deduced from the series expansions (2.22) and (2.23) that for spherical Bessel functions and their derivatives, we have for each fixed n,

$$j_n(t) = \frac{t^n}{1 \cdot 3 \cdots (2n+1)}[1 + \mathrm{O}(t^2)] \quad \text{as } t \to +0, \quad n = 0, 1, 2, \ldots, \tag{2.31}$$

$$j_0'(t) = -\frac{t}{3}[1 + \mathrm{O}(t^2)] \quad \text{as } t \to +0, \tag{2.32}$$

$${j_n}'(t) = \frac{nt^{n-1}}{1 \cdot 3 \cdots (2n+1)}[1 + \mathrm{O}(t^2)] \quad \text{as } t \to +0, \quad n = 1, 2, \ldots; \tag{2.33}$$

whereas for spherical Neumann functions and their derivatives, we have for each fixed $n \in \mathbb{N} \cup \{0\}$,

$$y_n(t) = -\frac{(2n)!}{2^n n!}\frac{1}{t^{n+1}}[1 + \mathrm{O}(t^2)] \quad \text{as } t \to +0, \tag{2.34}$$

$$y_n'(t) = \frac{(2n)!}{2^n n!}\frac{n+1}{t^{n+2}}[1 + \mathrm{O}(t^2)] \quad \text{as } t \to +0. \tag{2.35}$$

That is, for sufficiently small $t > 0$, $j_n(t)$ is positive and $j_n'(t)$ is also positive with the only exception that $j_0'(t)$ is negative near the origin, whereas $y_n(t)$ is negative but $y_n'(t)$ is positive. Hence,

$$j_n(t) > 0 \ \text{ for } t \in (0, a_{n,1}) \quad \text{and} \quad y_n(t) < 0 \ \text{ for } t \in (0, b_{n,1}), \quad n = 0, 1, 2, \ldots. \tag{2.36}$$

By Lemma 2.4, we know that the sequences $\{a_{n,1}\}_{n=0}^{\infty}$ and $\{b_{n,1}\}_{n=0}^{\infty}$ are strictly monotonically increasing, i.e.

$$a_{0,1} < a_{1,1} < a_{2,1} < \cdots, \quad b_{0,1} < b_{1,1} < b_{2,1} < \cdots.$$

Noting that $j_0(t) = \sin t/t$ and $y_0(t) = -\cos t/t$, we have

$$a_{0,1} = \pi, \quad b_{0,1} = \frac{\pi}{2},$$

i.e. the smallest positive zeros for $j_n(t)$ and $y_n(t)$, $n = 0, 1, 2, \ldots$, are, respectively, given by π and $\pi/2$. Hence, for all $n \in \mathbb{N} \cup \{0\}$,

$$j_n(t) > 0 \ \text{ for } t \in (0, \pi) \quad \text{and} \quad y_n(t) < 0 \ \text{ for } t \in \left(0, \frac{\pi}{2}\right).$$

As a consequence, we have

$$j_n(t) y_n(t) < 0 \quad \text{for } t \in \left(0, \frac{\pi}{2}\right)$$

uniformly for $n \in \mathbb{N} \cup \{0\}$.

In the same way, we can show the relation (2.30). In fact, by Lemma 2.4, we know that the sequences $\{a'_{n,1}\}_{n=0}^{\infty}$ and $\{b'_{n,1}\}_{n=0}^{\infty}$ are both strictly monotonically increasing. Noting that $a'_{0,1} = 0$, which together with the fact that $a'_{1,1} < a'_{0,2}$ by Lemma 2.3, shows that the smallest positive zero for $j_n'(t)$, $n = 0, 1, 2, \ldots$, is given by $a'_{1,1} = 2.08157598$ (see Olver, 1960, Table III). Whereas the smallest positive zero for $y_n'(t)$ is given by $b'_{0,1} = 2.79838605$. Therefore, it is easily seen that for $t \in (0, 2.08157598)$,

$$j_0'(t) < 0, \quad j_n'(t) > 0, \quad n = 1, 2, 3, \ldots, \tag{2.37}$$

and for $t \in (0, 2.79838605)$,

$$y_n'(t) > 0, \quad n = 0, 1, 2, \ldots. \tag{2.38}$$

Combining (2.37) and (2.38), we immediately get (2.30). □

In the rest of this section, we briefly introduce the Legendre polynomials and Jacobi–Anger expansion which are needed for our subsequent investigation (for more details, see Colton & Kress, 1998).

The Legendre polynomials are generated by the function

$$\frac{1}{\sqrt{1-2tr+r^2}} \quad \text{for } -1 \leqslant t \leqslant 1,$$

which is analytic in r for $r \in [0, 1)$ and thus has the following series expansion:

$$\frac{1}{\sqrt{1-2tr+r^2}} = \sum_{n=0}^{\infty} P_n(t) r^n. \tag{2.39}$$

The coefficients functions $\{P_n(t)\}_{n=0}^{\infty}$ above are known as the 'Legendre polynomials' and they form a complete orthogonal system in $L^2[-1, 1]$.

The following is the 'Jacobi–Anger expansion' for plane waves:

$$\mathrm{e}^{\mathrm{i}kx\cdot d} = \sum_{n=0}^{\infty} \mathrm{i}^n (2n+1) j_n(k|x|) P_n(\cos\theta) \quad \text{for } x \in \mathbb{R}^3, \tag{2.40}$$

where d is a unit vector, $\theta = \angle(\hat{x}, d)$ denotes the angle between $\hat{x}$ and d $(0 \leqslant \theta \leqslant \pi)$ and the convergence of the series (2.40) is uniform on any compact subset of $\mathbb{R}^3$. In $\mathbb{R}^2$, the Jacobi–Anger expansion becomes

$$\mathrm{e}^{\mathrm{i}kx\cdot d} = J_0(k|x|) + 2\sum_{n=1}^{\infty} \mathrm{i}^n J_n(k|x|) \cos n\theta \quad \text{for } x \in \mathbb{R}^2, \tag{2.41}$$

where d is a unit vector and $\theta = \angle(\hat{x}, d)$.

3. Two uniqueness results

In this section, we are ready to present the main uniqueness results of this paper on determining a sound-soft/sound-hard ball in $\mathbb{R}^3$ or a sound-soft/sound-hard disc in $\mathbb{R}^2$ by one measurement datum. In the sequel, we assume the centre of the ball/ disc is known *a priori*, and this is natural since we cannot expect one measurement datum to determine more than one unknown. Hence, noting the Laplace operator is invariant with respect to rigid motions, without loss of generality, we may assume that the ball/disc is centred at origin, which is of radius R and denoted by B_R.

3.1 *Uniqueness for a sound-soft ball or disc*

In view of the Dirichlet boundary data (1.2) and the Jacobi–Anger expansion (2.40), the scattered wave for a sound-soft ball $B_R \subset \mathbb{R}^3$ corresponding to the incident plane wave $\exp\{\mathrm{i}kx \cdot d\}$ is given by (see Colton & Kress, 1998, Section 3.2)

$$u^{\mathrm{s}}(x) := -\sum_{n=0}^{\infty} \mathrm{i}^n (2n+1) \frac{j_n(kR)}{h_n^{(1)}(kR)} h_n^{(1)}(k|x|) P_n(\cos\theta) \quad \text{for } x \in \mathbb{R}^3 \backslash B_R, \tag{3.1}$$

where $\theta = \angle(\hat{x}, d)$ and $h_n^{(1)}(t) = j_n(t) + \mathrm{i}y_n(t)$ is the first-kind spherical Hankel function of order n, $n = 0, 1, 2, \ldots$. The corresponding far-field pattern is given by

$$u_\infty(\hat{x}; B_R, d, k) = \frac{\mathrm{i}}{k}\sum_{n=0}^{\infty}(2n+1)\frac{j_n(kR)}{h_n^{(1)}(kR)}P_n(\cos\theta). \tag{3.2}$$

In $\mathbb{R}^2$, noting the Jacobi–Anger expansion (2.41), the scattered wave for B_R is now given by

$$u^{\mathrm{s}}(x) = -\frac{J_0(kR)}{H_0^{(1)}(kR)}H_0^{(1)}(k|x|) - 2\sum_{n=1}^{\infty}\mathrm{i}^n\frac{J_n(kR)}{H_n^{(1)}(kR)}H_n^{(1)}(k|x|)\cos n\theta, \tag{3.3}$$

where $H_n^{(1)}(t) = J_n(t) + \mathrm{i}Y_n(t)$ is the first-kind Hankel function of order n, $n = 0, 1, 2, \ldots$, and its far-field pattern is easily deduced (see Colton & Kress, 1998, Section 3.4) to be

$$u_\infty(\hat{x}; B_R, d, k) = -\mathrm{e}^{-\mathrm{i}\frac{\pi}{4}}\sqrt{\frac{2}{\pi k}}\left[\frac{J_0(kR)}{H_0^{(1)}(kR)} + 2\sum_{n=1}^{\infty}\frac{J_n(kR)}{H_n^{(1)}(kR)}\cos n\theta\right]. \tag{3.4}$$

From (3.2) and (3.4), it is observed that if $u_\infty(\hat{x}; B_R, d_0, k_0)$ is known for the fixed d_0 and k_0 and $\hat{x} \in \widetilde{\mathbb{S}}_0^{N-1} \subset \mathbb{S}^{N-1}$ $(N = 2, 3)$ such that

$$\{\angle(\hat{x}, d_0); \hat{x} \in \widetilde{\mathbb{S}}_0^{N-1}\} \supset (a, b), \quad 0 \leqslant a < b \leqslant \pi, \tag{3.5}$$

then the far-field pattern can be recovered by the unique continuation on the whole unit sphere $\mathbb{S}^{N-1}$, and therefore by the uniqueness result in Liu (1997), both the location and the shape of the ball/disc can be uniquely identified. In practical applications, the subset $\widetilde{\mathbb{S}}_0^{N-1}$ in (3.5) could be chosen to be an open portion of $\Pi_{d_0} \cap \mathbb{S}^{N-1}$ with Π_{d_0} being any hyperplane in $\mathbb{R}^N$ that contains d_0. It is noted that Condition (3.5) contains an infinite set of observation data. Then, a natural question arises: if the observation data are available only at a finite discrete set of the observation directions on $\mathbb{S}^{N-1}$, can the ball be uniquely determined? This question is surely important and meaningful to real applications. Next, we will give a definite answer to the question. In fact, surprisingly we are able to show that only one single observation datum is sufficient to uniquely determine the shape of the ball/disc. This is the first result of such kind, using a single observation datum.

THEOREM 3.1 The far-field datum $u_\infty(d_0; B_R, d_0, k_0)$ uniquely determines the radius of a sound-soft B_R in $\mathbb{R}^N$ provided

$$0 < k_0 R < \begin{cases} \pi/2, & \text{for } N = 3, \\ 0.89357697, & \text{for } N = 2. \end{cases} \tag{3.6}$$

Therefore, one measurement datum at one observation direction corresponding to a single incident plane wave uniquely identifies the radius of a sound-soft ball or disc.

Proof. By contradiction, suppose there exist two sound-soft balls B_{R_1} and B_{R_2} with $R_1 > R_2 > 0$ such that

$$u_\infty\left(d_0; B_{R_1}, d_0, k_0\right) = u_\infty\left(d_0; B_{R_2}, d_0, k_0\right). \tag{3.7}$$

In the $\mathbb{R}^3$ case, we have from (3.2) with $\theta = \angle(d_0, d_0) = 0$ that

$$u_\infty(d_0; B_R, d_0, k_0) = \frac{\mathrm{i}}{k_0}\sum_{n=0}^{\infty}(2n+1)\frac{j_n(k_0R)}{h_n^{(1)}(k_0R)}P_n(1). \tag{3.8}$$

From series expansion (2.39), we see that $P_n(1) = 1$ for all $n \in \mathbb{N} \cup \{0\}$. Then, it follows from (3.7) and (3.8) that

$$\sum_{n=0}^{\infty}(2n+1)\frac{j_n(t_1)}{h_n^{(1)}(t_1)} = \sum_{n=0}^{\infty}(2n+1)\frac{j_n(t_2)}{h_n^{(1)}(t_2)}, \tag{3.9}$$

where $t_1 = k_0R_1$, $t_2 = k_0R_2$ and $0 < t_2 < t_1 < \pi/2$ by (3.6).

Set

$$\Theta_n(t) := \frac{j_n(t)}{h_n^{(1)}(t)} = \frac{j_n(t)}{j_n(t) + \mathrm{i}y_n(t)} = \alpha_n(t) - \mathrm{i}\widetilde{\alpha}_n(t),$$

for $t \in (0, +\infty)$ and $n = 0, 1, 2, \ldots$. It is easy to find out that

$$\alpha_n(t) = \frac{j_n^2(t)}{j_n^2(t) + y_n^2(t)}, \quad \widetilde{\alpha}_n(t) = \frac{j_n(t)y_n(t)}{j_n^2(t) + y_n^2(t)}, \tag{3.10}$$

and we know by (3.9) that

$$\sum_{n=0}^{\infty}(2n+1)\alpha_n(t_1) = \sum_{n=0}^{\infty}(2n+1)\alpha_n(t_2). \tag{3.11}$$

Next, using the definition of $\alpha_n(t)$ in (3.10) and the Wronskian relation for spherical Bessel functions (see (2.36) in Colton & Kress, 1998), we can derive

$$\alpha_n'(t) = \frac{2j_n(t)y_n(t)[j'_n(t)y_n(t) - j_n(t)y'_n(t)]}{[j_n^2(t) + y_n^2(t)]^2} = -\frac{2}{t^2}\frac{j_n(t)y_n(t)}{[j_n^2(t) + y_n^2(t)]^2}. \tag{3.12}$$

Then by the first statement in Theorem 2.2, we conclude that for $n = 0, 1, \ldots$,

$$\alpha_n'(t) > 0 \quad \text{for } t \in \left(0, \frac{\pi}{2}\right),$$

which implies that $\alpha_n(t)$ is strictly monotonically increasing for $t \in (0, \pi/2)$ uniformly in $n \in \mathbb{N} \cup \{0\}$. Therefore, seeing that $0 < t_2 < t_1 < \pi/2$, we have

$$\alpha_n(t_1) > \alpha_n(t_2) > 0, \quad n = 0, 1, \ldots.$$

But this obviously contradicts (3.11).

For the $\mathbb{R}^2$ case, we derive by using the relation (3.4) and the assumption (3.7) that

$$\frac{J_0(t_1)}{H_0^{(1)}(t_1)} + 2\sum_{n=1}^{\infty}\frac{J_n(t_1)}{H_n^{(1)}(t_1)} = \frac{J_0(t_2)}{H_0^{(1)}(t_2)} + 2\sum_{n=1}^{\infty}\frac{J_n(t_2)}{H_n^{(1)}(t_2)}, \tag{3.13}$$

where $t_1 = k_0 R_1$, $t_2 = k_0 R_2$ and $0 < t_2 < t_1 < 0.89357697$ by (3.6). Now, we set

$$\Omega_n(t) := \frac{J_n(t)}{H_n^{(1)}(t)} = \frac{J_n(t)}{J_n(t) + \mathrm{i}Y_n(t)} = \lambda_n(t) - \mathrm{i}\widetilde{\lambda}_n(t),$$

for $t \in (0, +\infty)$ and $n = 0, 1, 2, \ldots$, and it is easy to check that

$$\lambda_n(t) = \frac{J_n^2(t)}{J_n^2(t) + Y_n^2(t)}, \quad \widetilde{\lambda}_n(t) = \frac{J_n(t)Y_n(t)}{J_n^2(t) + Y_n^2(t)}. \tag{3.14}$$

Then we obtain from (3.13),

$$\lambda_0(t_1) + 2\sum_{n=1}^{\infty} \lambda_n(t_1) = \lambda_0(t_2) + 2\sum_{n=1}^{\infty} \lambda_n(t_2). \tag{3.15}$$

But by definition of $\lambda_n(t)$ in (3.14) and the Wronskian relation for the Bessel functions (see (3.56) in Colton & Kress, 1998), we find that

$$\lambda'_n(t) = \frac{2J_n(t)Y_n(t)[J'_n(t)Y_n(t) - J_n(t)Y'_n(t)]}{[J_n^2(t) + Y_n^2(t)]^2} = -\frac{4}{\pi t}\frac{J_n(t)Y_n(t)}{[J_n^2(t) + Y_n^2(t)]^2}. \tag{3.16}$$

Then by the first statement in Theorem 2.1, we conclude that for $n = 0, 1, \ldots$,

$$\lambda'_n(t) > 0 \quad \text{for } t \in (0, 0.89357697),$$

and this implies that $\lambda_n(t)$ is strictly monotonically increasing for $t \in (0, 0.89357697)$ uniformly in $n \in \mathbb{N} \cup \{0\}$. Hence, noting that $0 < t_2 < t_1 < 0.89357697$, we come to

$$\lambda_n(t_1) > \lambda_n(t_2) > 0, \quad n = 0, 1, \ldots,$$

which clearly contradicts with the equality (3.15). □

3.2 *Uniqueness for a sound-hard ball or disc*

In view of the Neumann boundary data (1.3) and the Jacobi–Anger expansion (2.40), the scattered wave for a sound-hard ball $B_R \subset \mathbb{R}^3$ corresponding to the incident plane wave $\exp\{\mathrm{i}kx \cdot d\}$ is easily derived to be (see Yun, 2001)

$$u^{\mathrm{s}}(x) = -\sum_{n=0}^{\infty} \mathrm{i}^n(2n+1)\frac{j_n{}'(kR)}{h_n^{(1)\prime}(kR)} h_n^{(1)}(k|x|)P_n(\cos\theta) \quad \text{for } x \in \mathbb{R}^3 \backslash B_R, \tag{3.17}$$

where $\theta = \angle(d, \hat{x})$. Its corresponding far-field pattern is given by (see Colton & Kress, 1998, Theorem 2.15)

$$u_\infty(\hat{x}; B_R, d, k) = \frac{\mathrm{i}}{k}\sum_{n=0}^{\infty}(2n+1)\frac{j_n{}'(kR)}{h_n^{(1)\prime}(kR)} P_n(\cos\theta). \tag{3.18}$$

In $\mathbb{R}^2$, the scattered field and its far-field pattern are given by

$$u^{\mathrm{s}}(x) = -\frac{J_0'(kR)}{H_0^{(1)\prime}(kR)} H_0^{(1)}(k|x|) - 2\sum_{n=1}^{\infty} \mathrm{i}^n \frac{J_n'(kR)}{H_n^{(1)\prime}(kR)} H_n^{(1)}(k|x|)\cos n\theta, \tag{3.19}$$

$$u_\infty(\hat{x}; B_R, d, k) = -\mathrm{e}^{-\mathrm{i}\frac{\pi}{4}}\sqrt{\frac{2}{\pi k}}\left[\frac{J_0'(kR)}{H_0^{(1)\prime}(kR)} + 2\sum_{n=1}^{\infty}\frac{J_n'(kR)}{H_n^{(1)\prime}(kR)}\cos n\theta\right]. \tag{3.20}$$

Similarly to the sound-soft case, using the uniqueness result in Yun (2001), it is observed from (3.18) and (3.20) that the far-field data corresponding to a single incident plane wave on $\widetilde{\mathbb{S}}_0^{N-1}$ satisfying (3.5) uniquely identify the ball/disc B_R. It is also similar to the sound-soft case that the determination of a sound-hard ball by a single observation datum is not only mathematically interesting but also practically important and meaningful. This is answered by the following theorem.

THEOREM 3.2 The far-field datum $u_\infty(d_0; B_R, d_0, k_0)$ determines the shape of a sound-hard ball/disc B_R in $\mathbb{R}^N$ uniquely provided

$$0 < k_0 R < \begin{cases} \sqrt{2} & \text{for } N = 3, \\ 1 & \text{for } N = 2. \end{cases} \tag{3.21}$$

Proof. We suppose by contradiction that there exist two sound-hard balls B_{R_1} and B_{R_2} with $R_1 > R_2 > 0$ such that

$$u_\infty\left(d_0; B_{R_1}, d_0, k_0\right) = u_\infty\left(d_0; B_{R_2}, d_0, k_0\right). \tag{3.22}$$

In the case of $\mathbb{R}^3$, we see from (3.18) and (3.22) that

$$\sum_{n=0}^{\infty}(2n+1)\frac{j_n'(t_1)}{h_n^{(1)\prime}(t_1)} = \sum_{n=0}^{\infty}(2n+1)\frac{j_n'(t_2)}{h_n^{(1)\prime}(t_2)}, \tag{3.23}$$

where $t_1 = k_0R_1$, $t_1 = k_0R_1$ and $0 < t_2 < t_1 < \sqrt{2}$ by (3.21).

Set

$$\Upsilon_n(t) := \frac{j_n{}'(t)}{h_n^{(1)\prime}(t)} = \frac{j_n{}'(t)}{j_n{}'(t) + \mathrm{i}y_n{}'(t)} = \beta_n(t) - \mathrm{i}\widetilde{\beta}_n(t).$$

Clearly,

$$\beta_n(t) = \frac{j_n'^2(t)}{j_n'^2(t) + Y_n'^2(t)}, \quad \widetilde{\beta}_n(t) = \frac{j_n'(t)y_n'(t)}{j_n'^2(t) + y_n'^2(t)}. \tag{3.24}$$

Then, we have from (3.23) that

$$\sum_{n=0}^{\infty}(2n+1)\beta_n(t_1) = \sum_{n=0}^{\infty}(2n+1)\beta_n(t_2). \tag{3.25}$$

By straightforward calculations, we derive

$$\beta'_n(t) = \frac{2j_n'(t)y_n'(t)[j_n''(t)y_n'(t) - j_n'(t)y_n''(t)]}{\left[j_n'^2(t) + y_n'^2(t)\right]^2}. \tag{3.26}$$

Noting that both j_n and y_n satisfy the spherical Bessel differential equation (2.24),

$$j_n'' = -\frac{2}{t} j_n' - \frac{t^2 - n(n+1)}{t^2} j_n,$$

$$y_n'' = -\frac{2}{t} y_n' - \frac{t^2 - n(n+1)}{t^2} y_n,$$

thus by the Wronskian relation for spherical Bessel functions (see (2.36) in Colton & Kress, 1998), we obtain

$$j_n'' y_n' - j_n' y_n'' = \frac{t^2 - n(n+1)}{t^2}(j_n' y_n - j_n y_n') = \frac{n(n+1) - t^2}{t^4}. \tag{3.27}$$

Using this, we get from (3.26) that

$$\beta_n'(t) = \frac{n(n+1) - t^2}{t^4} \frac{2 j_n'(t) y_n'(t)}{\left[{j_n'}^2(t) + {y_n'}^2(t) \right]^2}.$$

By the second statement in Theorem 2.2, one can easily check that for $n = 0, 1, \ldots$,

$$\beta'_n(t) > 0 \quad \text{for } 0 < t < \sqrt{2},$$

which implies that $\beta_n(t)$ is strictly monotonically increasing for $0 < t < \sqrt{2}$ uniformly in $n \in \mathbb{N} \cup \{0\}$. Therefore, we have

$$\beta_n(t_1) > \beta_n(t_2) > 0 \quad \text{for } n = 0, 1, \ldots,$$

but this clearly contradicts with the equality (3.25).

In the case of $\mathbb{R}^2$, we readily see the following from the expression (3.20) and the assumption (3.22):

$$\frac{J_0'(t_1)}{{H_0^{(1)}}'(t_1)} + 2\sum_{n=1}^{\infty} \frac{J_n'(t_1)}{{H_n^{(1)}}'(t_1)} = \frac{J_0'(t_2)}{{H_0^{(1)}}'(t_2)} + 2\sum_{n=1}^{\infty} \frac{J_n'(t_2)}{{H_n^{(1)}}'(t_2)}, \tag{3.28}$$

where $t_1 = k_0 R_1$, $t_2 = k_0 R_2$ and $0 < t_2 < t_1 < 1$ by (3.21). Now, we set

$$\Lambda_n(t) := \frac{{J_n}'(t)}{{H_n^{(1)}}'(t)} = \frac{{J_n}'(t)}{{J_n}'(t) + \mathrm{i}{Y_n}'(t)} = \gamma_n(t) - \mathrm{i}\widetilde{\gamma}_n(t).$$

It is easy to see

$$\gamma_n(t) = \frac{{J_n'}^2(t)}{{J_n'}^2(t) + {Y_n'}^2(t)}, \quad \widetilde{\gamma}_n(t) = \frac{J_n'(t) Y_n'(t)}{{J_n'}^2(t) + {Y_n'}^2(t)}. \tag{3.29}$$

So we have from (3.28) that

$$\gamma_0(t_1) + 2\sum_{n=1}^{\infty} \gamma_n(t_1) = \gamma_0(t_2) + 2\sum_{n=1}^{\infty} \gamma_n(t_2). \tag{3.30}$$

Now using the Wronskian relation for the Bessel functions (see (3.56) in Colton & Kress, 1998) and the Bessel differential equation (2.3) for $J_n(t)$ and $Y_n(t)$, we can derive

$$\begin{aligned}\gamma'_n(t) &= \frac{2J_n'(t)Y_n'(t)[J_n''(t)Y_n'(t) - J_n'(t)Y_n''(t)]}{\left[J_n'^2(t) + Y_n'^2(t)\right]^2} \\ &= \frac{n^2 - t^2}{\pi t^3}\frac{2J_n'(t)Y_n'(t)}{\left[J_n'^2(t) + Y_n'^2(t)\right]^2}.\end{aligned}$$

Then by the second statement in Theorem 2.1, we obtain

$$\gamma'_n(t) > 0 \quad \text{for } 0 < t < 1.$$

Hence, $\gamma_n(t)$ is strictly monotonically increasing for $0 < t < 1$ uniformly in $n \in \mathbb{N} \cup \{0\}$. So we have

$$\gamma_n(t_1) > \gamma_n(t_2) > 0, \quad n = 0, 1, \ldots,$$

but this clearly contradicts with the relation (3.30). □

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