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An Augmented Lagrangian Method for Parameter Identifications in Parabolic Systems

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The identification of parameters in parabolic systems is formulated as a constrained minimization problem combining the output least squares and the equation error method. The minimization problem is then proved to be equivalent to the saddle-point problem of an augmented Lagrangian.

1. INTRODUCTION

In this paper, we investigate a new approach for the identification of the unknown coefficient $q(x)$ in the following parabolic equation

$$\frac{\partial u}{\partial t} - \nabla \cdot (q \nabla u) = f, \qquad \text{in } \Omega \times (0, T] \tag{1.1}$$

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with the initial condition

$$u(x,0) = u_0(x), \qquad \text{in } \overline{\Omega} \tag{1.2}$$

and the Dirichlet boundary condition

$$u(x,t) = 0, \qquad \text{on } \partial\Omega \times (0,T]. \tag{1.3}$$

Here Ω is a bounded domain in $\mathscr{R}^n (n \leq 3)$ with a piecewise smooth boundary $\partial\Omega$, and $f(x,t)$ and $u_0(x)$ are the given source function and the initial condition, respectively. The problem (1.1)–(1.3) may describe the flow of a fluid through some medium with the permeability q, or the heat transfer in a material with the conductivity q. In many practical applications, it is often easier to measure the solution u at various discrete points in the medium and at certain time, than to measure the physical parameter q itself; see Bank and Kunisch [2] and Engl *et al.* [7] and the references therein. The inverse problem to be considered here is to recover the conductivity $q(x)$ in Eq. (1.1), assuming that the observation data $z(x)$ of the terminal temperature $u(x,T)$ or a set of the observation data $\{z(x,t_i)\}$ of the temperature $u(x,t)$ at a set of discrete time points $\{t_i\}$ in a small time interval $[T-\sigma, T]$ are available. This problem is known to be highly ill-posed and has been widely investigated in the last several decades, although it remains still challenging due to the lack of efficiency and good stability of the existing methods. For the references in this aspect, we refer to Cannon [4], Bank and Kunisch [2], Chen and Zou [6], Engl *et al.* [7], Guenther *et al.* [9], Gutman [10], Ito and Kunisch [11], Kunisch and White [14], and Keung and Zou [13].

Recently, a very stable and efficient approach was proposed by Ito and Kunisch [11, 12] for the identification of the parameters $q(x)$ in the steady-state case of (1.1) (time-independent case), when the $q(x)$ are very smooth. The method was then generalized by Chen and Zou [6] to treat the non-smooth parameters in the steady-state system. The important novelty of this method is to combine the output least squares and the equation error method with the mathematical framework given by the augmented Lagrangian technique, which was widely used earlier in nonlinear constrained optimizations; see, e.g., Bertsekas [3], and Glowinski and Tallec [8]. This new approach has been proved to be very successful in the recovery of the parameters in the elliptic problems due to its fast convergence and nice stability. Unfortunately, people have still not found a reasonable way to apply this method, in particular to justify its mathematical formulation, for the identification of the parameters $q(x)$ in the parabolic system (1.1)–(1.3). In this paper we will make some efforts in this direction, and this seems to be the first time to study and rigorously justify the augmented Lagrangian formulation for a time-dependent inverse problem.

The paper is arranged as follows. In the next section, we establish some a priori estimates which will be used in the forthcoming discussions. In Section 3, we reformulate the parameter identifying problem as a nonlinear constrained minimization problem and show the existence of the global minimizers of the minimization system. In Section 4, we introduce an augmented Lagrangian functional and prove that the minimization problem is equivalent to the saddle-point problem associated with the Lagrangian functional. The augmented Lagrangian functional is quadratic and convex with respect to each of its variables, so it is much more convenient than the original highly nonlinear optimization problem from the viewpoint of the implementation of the identification process.

2. SOME A PRIORI ESTIMATES

In this section, we present some a priori estimates about the solution $u(x,t)$ of the parabolic system (1.1)–(1.3). For the ease of notation, we will use the following notations with any $r \geq 0$ and $p \geq 1$,

$$|v|_r = |v|_{H^r(\Omega)}, \qquad \|v\|_r = \|v\|_{H^r(\Omega)}, \qquad \|v\|_{r,p} = \|v\|_{W^{r,p}(\Omega)},$$

$$\|v\|_\infty = \|v\|_{L^\infty(\Omega)}, \quad |v|_{r,\infty} = |v|_{W^{r,\infty}(\Omega)}, \qquad \|v\|_{r,\infty} = \|v\|_{W^{r,\infty}(\Omega)},$$

and the notation $(v,w) = (v,w)_{L^2(\Omega)}$ and $\|v\| = \|v\|_{L^2(\Omega)}$.

With these notations, the weak formulation of the parabolic system (1.1)–(1.3) can be stated as follows.

Find $u \in L^2(0,T; H_0^1(\Omega)) \cap L^\infty(0,T; L^2(\Omega))$ such that the following holds for a.e. $t \in (0,T]$,

$$\begin{cases} \left(\dfrac{\partial u}{\partial t}, \phi\right) + (q\nabla u, \nabla\phi) = \langle f, \phi\rangle, & \forall \phi \in V, \\ u(x,0) = u_0(x), & \text{in } \overline{\Omega}, \end{cases} \tag{2.1}$$

where $V = H_0^1(\Omega)$ and $\langle \cdot, \cdot \rangle$ denotes the duality pairing between V and $V' = H^{-1}(\Omega)$.

Physically, it is very reasonable to search the parameters $q(x)$ among all positive functions which are bounded below and above by two roughly predicted fixed constants, say γ_1 and γ_2. Therefore we will only consider the parameters $q(x)$ which have the following bounds:

$$\gamma_1 \leq q(x) \leq \gamma_2 \qquad \text{for a.e. } x \in \Omega. \tag{2.2}$$

First of all, we know from the standard theory about the parabolic equation that if $u_0 \in L^2(\Omega)$ and $f \in L^2(0, T; H^{-1}(\Omega))$, then (2.1) has a unique solution u, which satisfies

$$\|u\|_{L^2(0,T;H^1(\Omega))} + \|u\|_{L^\infty(0,T;L^2(\Omega))} \le c\big(\|u_0\| + \|f\|_{L^2(0,T;H^{-1}(\Omega))}\big). \quad (2.3)$$

Hereafter, $c > 0$ denotes a generic constant depending only on γ_1, γ_2 and Ω.

About the solution of the parabolic system (2.1), we have the following further a priori estimates:

LEMMA 2.1. *If $u_0 \in H_0^1(\Omega)$ and $f \in L^2(0, T; L^2(\Omega))$, then the solution $u(x, t)$ of the system (2.1) has the following bounds*

$$\|u\|^2_{L^\infty(0,T;H^1(\Omega))} + \|u\|^2_{H^1(0,T;L^2(\Omega))} \le c\Big(|u_0|_1^2 + \|f\|^2_{L^2(0,T;L^2(\Omega))}\Big).$$

If, in addition, $q \in W^{1,\infty}(\Omega)$, then

$$\|\Delta u\|^2_{L^2(0,T;L^2(\Omega))} \le c\big(T|q|^2_{1,\infty} + 1\big)\Big(|u_0|_1^2 + \|f\|^2_{L^2(0,T;L^2(\Omega))}\Big).$$

Proof. We first take $\phi = \frac{\partial u}{\partial t}$ in (2.1) to obtain that

$$\left\|\frac{\partial u}{\partial t}\right\|^2 + \frac{d}{dt}(q\nabla u, \nabla u) \le \|f\|^2.$$

Integrating the above inequality with respect to t, yields that

$$\left\|\frac{\partial u}{\partial t}\right\|^2_{L^2(0,T;L^2(\Omega))} + \gamma_1 |u(t)|_1^2 \le \gamma_2 |u_0|_1^2 + \|f\|^2_{L^2(0,T;L^2(\Omega))},$$

$$\text{for a.e. } t \in (0, T].$$

Therefore

$$\|u\|^2_{L^\infty(0,T;H^1(\Omega))} + \|u\|^2_{H^1(0,T;L^2(\Omega))} \le c\Big(|u_0|_1^2 + \|f\|^2_{L^2(0,T;L^2(\Omega))}\Big).$$

Next, let $q \in W^{1,\infty}(\Omega)$. We have from (2.1) that in the sense of distributions,

$$\frac{\partial u}{\partial t} - \nabla q \cdot \nabla u - q\Delta u = f, \qquad \text{for a.e. } t \in (0, T]. \quad (2.4)$$

Multiplying both sides of (2.4) by $-\Delta u$ and integrating then over Ω, we have

$$\frac{d}{dt}|u|_1^2 + 2(q\Delta u, \Delta u) \le 2|(\nabla q \cdot \nabla u, \Delta u)| + \frac{\gamma_1}{2}\|\Delta u\|^2 + \frac{2}{\gamma_1}\|f\|^2.$$

Moreover, by the previous estimate for $|u(t)|_1$ we have

$$2|(\nabla q \cdot \nabla u, \Delta u)| \le \frac{\gamma_1}{2}\|\Delta u\|^2 + \frac{2}{\gamma_1}|q|_{1,\infty}^2 |u|_1^2$$
$$\le \frac{\gamma_1}{2}\|\Delta u\|^2 + c|q|_{1,\infty}^2\Big(|u_0|_1^2 + \|f\|_{L^2(0,T:L^2(\Omega))}^2\Big).$$

Thus

$$\frac{d}{dt}|u|_1^2 + (q\Delta u, \Delta u) \le c|q|_{1,\infty}^2\Big(|u_0|_1^2 + \|f\|_{L^2(0,T:L^2(\Omega))}^2\Big) + \frac{2}{\gamma_1}\|f\|^2.$$

Integrating the above inequality with respect to t, yields

$$|u(t)|_1^2 + \gamma_1 \int_0^t \|\Delta u(s)\|^2\, ds \le ct|q|_{1,\infty}^2\Big(|u_0|_1^2 + \|f\|_{L^2(0,T;L^2(\Omega))}^2\Big)$$
$$+ |u_0|_1^2 + \frac{2}{\gamma_1}\|f\|_{L^2(0,T,L^2(\Omega))}^2,$$

which implies the second result of Lemma 2.1. ∎

Remark 2.1. In many practical applications, the boundary $\partial\Omega$ often satisfies one of the three conditions: (i) $\partial\Omega$ is convex; (ii) $\partial\Omega$ is of $C^{1,1}$; (iii) $\partial\Omega$ is piecewisely of C^2, and the neighbourhoods of all corners are locally convex. Under either of these three conditions, we have the estimate

$$\|u\|_2 \le c(\|\Delta u\| + \|u\|), \qquad \forall u \in H^2(\Omega).$$

Then using the relation $-(u, \Delta u) = \|\nabla u\|^2$, and the Poincare inequality, we have

$$|u|_1^2 \le \|\Delta u\|\,\|u\| \le c\|\Delta u\|\,\|u\|_1 \le \alpha|u|_1^2 + \frac{c}{\alpha}\|\Delta u\|^2$$

for any $\alpha > 0$. Let α be sufficiently small. Then $|u|_1 \le c\|\Delta u\|$, and so $\|u\|_2 \le c\|\Delta u\|$. In this situation, we can replace the norm $\|\Delta u\|_{L^2(0,T;L^2(\Omega))}$ by the norm $\|u\|_{L^2(0,T;H^2(\Omega))}$ in the second result of Lemma 2.1.

Remark 2.2. We know that $L^2(\Omega)$ and $H^1(\Omega)$ are reflexive spaces, and we have the embedding $H^1(\Omega) \subset L^4(\Omega) \subset L^2(\Omega)$, and the injection $H^1(\Omega) \to L^4(\Omega)$ is compact. Thus by Lemma 5.1 of Lions [15], for any $\eta > 0$, there exists c_η depending on η such that

$$\|v\|_{L^4(\Omega)} \le \eta|v|_1 + c_\eta\|v\|. \tag{2.5}$$

Then in the case of Remark 2.1, we have that

$$\begin{aligned} 2|(\nabla q \cdot \nabla u, \Delta u)| &\leq \frac{\gamma_1}{4}\|\Delta u\|^2 + \frac{4}{\gamma_1}|q|^2_{W^{1,4}(\Omega)}\|\nabla u\|^2_{L^4(\Omega)} \\ &\leq \frac{\gamma_1}{2}\|\Delta u\|^2 + C\big(\gamma_1, |q|_{W^{1,4}(\Omega)}\big)|u|_1^2, \end{aligned}$$

where $C(\alpha, \beta)$ is a positive constant depending on α and β. By using this fact and following the same line as in the proof of Lemma 2.1, we can show that

$$\|u\|^2_{L^2(0,T;\,H^2(\Omega))} \leq C\big(\gamma_1, |q|_{W^{1,4}(\Omega)}\big)\Big(|u_0|_1^2 + \|f\|^2_{L^2(0,T;\,L^2(\Omega))}\Big).$$

In particular,

$$\|u\|^2_{L^2(0,T;\,H^2(\Omega))} \leq C\big(\gamma_1, \|q\|_2\big)\Big(|u_0|_1^2 + \|f\|^2_{L^2(0,T;\,L^2(\Omega))}\Big).$$

3. THE MINIMIZATION FORMULATION

We now discuss how to formulate the identification problem of Section 1 into a minimization problem. In the following discussions we assume that

$$u_0 \in H_0^1(\Omega), \qquad f \in L^2\big(0,T; L^2(\Omega)\big) \tag{3.1}$$

and $\partial\Omega$ satisfies one of the three conditions mentioned in Remark 2.1. Let

$$\begin{aligned} W &= L^2\big(0,T; H^2(\Omega)\big) \cap L^\infty\big(0,T; H_0^1(\Omega)\big) \cap H^1\big(0,T; L^2(\Omega)\big), \\ K &= \big\{v \in H^2(\Omega) \text{ and } \gamma_1 \leq v \leq \gamma_2, \text{ a.e. in } \Omega\big\}. \end{aligned}$$

For any $(q, v) \in K \times W$, we define $e(q, v)$ to be a function in W which satisfies the initial condition $e(x, 0) = v(x, 0) - u_0(x)$ and solves the equation

$$\left(\frac{\partial e}{\partial t}, \phi\right) + (\nabla e, \nabla\phi) = \left(\frac{\partial v}{\partial t}, \phi\right) + (q\nabla v, \nabla\phi) - (f, \phi), \qquad \forall \phi \in V. \tag{3.2}$$

Note that for any $(q, v) \in K \times W$, we have by (2.5) that $\nabla \cdot (q\nabla v) \in L^2(0, T; L^2(\Omega))$. This fact with Lemma 2.1, and Remarks 2.1–2.2 implies that $e(q, v)$ is well-defined. Moreover, $e(q, v) = 0$ means that v is the unique solution of (2.1) corresponding to the heat conductivity $q(x)$.

Now let $z(x,t)$ be the observation data of $u(x,t)$ for t in a small interval $[T-\sigma,T]$ near the terminal time $t=T$. We assume (possibly after some interpolations of the discrete observation data) that

$$z \in L^2\big(T-\sigma,T;H^2(\Omega)\big) \cap L^\infty\big(T-\sigma,T;H_0^1(\Omega)\big) \cap H^1\big(T-\sigma,T;L^2(\Omega)\big).$$

Then we formulate the identification of the parameter q in the parabolic system (1.1)–(1.3) as the following minimization problem,

$$\text{minimize} J(q,v) = \frac{1}{2}\int_{T-\sigma}^{T} \left\| \frac{\partial}{\partial t}(v(t)-z(t)) - \nabla\cdot(q\nabla(v(t)-z(t))) \right\|^2 dt + \beta|q|_2^2 \tag{3.3}$$

subject to $(q,v)\in K\times W$ and $e(q,v)=0$ for a.e. $t\in[0,T]$. The constant $\beta>0$ in (3.2) is called a regularization parameter.

About the existence of the minimizers of the problem (3.3), we have the following result:

THEOREM 3.1. *There exists at least one minimizer to the problem* (3.3).

Proof. Let A be the admissible set

$$A = \{(q,v)\in K\times W;\ e(q,v)=0, \text{for a.e. } t\in[0,T]\}. \tag{3.4}$$

Clearly $A\neq\varnothing$ and $J(q,v)\geq 0$ on A. Thus there exists a sequence $(q_n,v_n)\in A$ such that

$$\lim_{n\to\infty} J(q_n,v_n) = \inf_{(q,v)\in A} J(q,v). \tag{3.5}$$

Since $J(q_n,v_n)\leq c$ for all $n\geq 0$ we have $|q_n|_2\leq c$. On the other hand, due to $q_n\in K$, we have $\|q_n\|\leq c$. By Lemma 5.1 of Lions [15], for any $\eta>0$ there exists c_η depending on η such that

$$\|v\|_1 \leq \eta|v|_2 + c_\eta\|v\|.$$

Hence $\|q_n\|_2\leq c$. Using the compactness of the injection from $H^2(\Omega)$ to $L^\infty(\Omega)\cap H^1(\Omega)$, we can extract a subsequence, denoted still by (q_n,v_n), such that for certain $q^*\in L^\infty(\Omega)\cap H^1(\Omega)$,

$$q_n \to q^*, \qquad \text{in } L^\infty(\Omega)\cap H^1(\Omega). \tag{3.6}$$

Due to $q_n \in K$, we also have $q^* \in K$. Next, by (3.2) and the fact that $e(q_n, v_n) = 0$, we have

$$\begin{cases} \left(\dfrac{\partial v_n}{\partial t}, \phi \right) + (q_n \nabla v_n, \nabla \phi) = (f, \phi), & \forall \phi \in V, \\ v_n(x,0) = u_0(x). \end{cases} \tag{3.7}$$

By virtue of Lemma 2.1 and Remarks 2.1 and 2.2,

$$\|v_n\|_{L^2(0,T;\,H^2(\Omega))} + \|v_n\|_{L^\infty(0,T;\,H^1(\Omega))} + \|v_n\|_{H^1(0,T;\,L^2(\Omega))} \le c^*, \tag{3.8}$$

where c^* is a positive constant depending only on γ_1, γ_2, Ω, u_0, and f. Thus there exists a subsequence, denoted still by $\{q_n, v_n\}$, such that for certain $(q^*, v^*) \in K \times W$,

$$\begin{aligned} & q_n \to q^* \text{ in } L^\infty(\Omega) \cap H^1(\Omega), \qquad q_n \rightharpoonup q^* \text{ in } H^2(\Omega), \\ & v_n \to v^* \text{ in } L^2\big(0,T; H_0^1(\Omega)\big) \cap L^\infty\big(0,T; L^2(\Omega)\big), \qquad v_n \rightharpoonup v^* \text{ in } W. \end{aligned} \tag{3.9}$$

Therefore it remains to prove that

$$e^* = e(q^*, v^*) = 0 \tag{3.10}$$

and

$$J(q^*, v^*) \le \inf_{(q,v) \in A} J(q,v). \tag{3.11}$$

We first prove (3.10). We have from (3.2) that

$$\begin{cases} \left(\dfrac{\partial e^*}{\partial t}, \phi \right) + (\nabla e^*, \nabla \phi) = \left(\dfrac{\partial v^*}{\partial t}, \phi \right) + (q^* \nabla v^*, \nabla \phi) - (f, \phi), \\ \qquad \forall \phi \in V, \text{ a.e. } t \in (0,T], \\ e^*(x,0) = v^*(x,0) - u_0(x). \end{cases} \tag{3.12}$$

Thanks to $v_n(x,0) = u_0(x)$, we have $e^*(x,0) = 0$. The combination of (3.7) and (3.12) leads to

$$\left(\frac{\partial e^*}{\partial t}, \phi \right) + (\nabla e^*, \nabla \phi) = A_1(\phi) + A_2(\phi), \qquad \forall \phi \in V, \text{ a.e. } t \in (0,T], \tag{3.13}$$

where

$$A_1(\phi) = \left(\frac{\partial v^*}{\partial t} - \frac{\partial v_n}{\partial t}, \phi \right), \qquad A_2(\phi) = (q^* \nabla v^* - q_n \nabla v_n, \nabla \phi).$$

Obviously, $A_1(\phi) \to 0$ as $n \to \infty$. On the other hand,

$$A_2(\phi) = \big((q^* - q_n)\nabla v_n, \nabla\phi\big) + \big(q^*\nabla(v^* - v_n), \nabla\phi\big).$$

Consequently,

$$|A_2(\phi)| \le \left(\int_\Omega |q^* - q_n|\,|\nabla v_n|^2 dx\right)^{\frac{1}{2}} \left(\int_\Omega |q^* - q_n|\,|\nabla\phi|^2 dx\right)^{\frac{1}{2}} + \big|\big(q^*\nabla(v^* - v_n), \nabla\phi\big)\big|.$$

By (3.8), (3.9), and the Lebesque dominant convergence theorem, we deduce that when $n \to \infty$,

$$\left(\int_\Omega |q^* - q_n|\,|\nabla v_n|^2 dx\right)^{\frac{1}{2}} \left(\int_\Omega |q^* - q_n|\,|\nabla\phi|^2 dx\right)^{\frac{1}{2}} \to 0, \qquad \text{for a.e. } t \in (0, T].$$

Since $v_n \rightharpoonup v^*$ in $L^\infty(0, T; H^1(\Omega))$ and $q^* \in L^\infty(\Omega)$,

$$\big(q^*\nabla(v^* - v_n), \nabla\phi\big) \to 0, \qquad \text{as } n \to \infty, \text{ for a.e. } t \in (0, T].$$

Thus $|A_2(\phi)| \to 0$ as $n \to \infty$. Therefore we obtain from (3.13) that

$$\left(\frac{\partial e^*}{\partial t}, \phi\right) + (\nabla e^*, \nabla\phi) = 0, \qquad \text{for a.e. } t \in (0, T]$$

which implies (3.10).

We next prove (3.11). Due to $e(q_n, v_n) = 0$, we have

$$\left(\frac{\partial v_n}{\partial t} - \nabla\cdot(q_n\nabla v_n) - f, \phi\right) = 0, \qquad \text{for a.e. } t \in (0, T],$$

and so

$$\left(\frac{\partial v_n}{\partial t} - \frac{\partial z}{\partial t} - \nabla\cdot\big(q_n\nabla(v_n - z)\big), \phi\right) = \left(-\frac{\partial z}{\partial t} + \nabla\cdot(q_n\nabla z) + f, \phi\right),$$
$$\text{for a.e. } t \in (0, T].$$

Let $J(q, v) = J_0(q, v) + \beta|q|_2^2$. We have from the above equation that in the sense of distributions,

$$\frac{\partial v_n}{\partial t} - \frac{\partial z}{\partial t} - \nabla\cdot\big(q_n\nabla(v_n - z)\big) = -\frac{\partial z}{\partial t} + \nabla\cdot(q_n\nabla z) + f. \quad (3.14)$$

Multiplying (3.14) by

$$\frac{\partial v_n}{\partial t} - \frac{\partial z}{\partial t} - \nabla\cdot\big(q_n(\nabla v_n - z)\big),$$

and integrating the resulting equation from $t = T - \sigma$ to $t = T$, we obtain that

$$J_0(q_n, v_n) = \int_{T-\sigma}^{T} \left(-\frac{\partial z}{\partial t} + \nabla \cdot (q_n \nabla z) + f, \frac{\partial v_n}{\partial t} - \frac{\partial z}{\partial t} - \nabla \cdot \left(q_n \nabla (v_n - z)\right)\right) dt. \tag{3.15}$$

Similarly,

$$J_0(q^*, v^*) = \int_{T-\sigma}^{T} \left(-\frac{\partial z}{\partial t} + \nabla \cdot (q^* \nabla z) + f, \frac{\partial v^*}{\partial t} - \frac{\partial z}{\partial t} - \nabla \cdot \left(q^* \nabla (v^* - z)\right)\right) dt. \tag{3.16}$$

By (3.9),

$$\int_{T-\sigma}^{T} \left(\frac{\partial v_n}{\partial t}, \nabla \cdot (q_n \nabla z) \right) dt \to \int_{T-\sigma}^{T} \left(\frac{\partial v^*}{\partial t}, \nabla \cdot (q^* \nabla z) \right) dt. \tag{3.17}$$

Next,

$$\begin{aligned}(\nabla \cdot (q_n \nabla z), \nabla \cdot (q_n \nabla v_n)) &= (\nabla q_n \cdot \nabla z, \nabla q_n \cdot \nabla v_n) + (q_n \Delta z, \nabla q_n \cdot \nabla v_n) \\ &\quad + (\nabla q_n \cdot \nabla z, q_n \Delta v_n) + (q_n \Delta z, q_n \Delta v_n).\end{aligned} \tag{3.18}$$

By (3.9) and the imbedding theory, we have $(\nabla q_n)^2 \rightharpoonup (\nabla q^*)^2$ in $L^2(\Omega)$ and $\nabla v_n \to \nabla v^*$ in $L^2(0, T; L^2(\Omega))$. Thus

$$\int_{T-\sigma}^{T} (\nabla q_n \cdot \nabla z, \nabla q_n \cdot \nabla v_n)\, dt \to \int_{T-\sigma}^{T} (\nabla q^* \cdot \nabla z, \nabla q^* \cdot \nabla v^*)\, dt. \tag{3.19}$$

Also by (3.9), $q_n \nabla q_n \to q^* \nabla q^*$ in $L^2(\Omega)$ and $\Delta v_n \rightharpoonup \Delta v^*$ in $L^2(0, T; L^2(\Omega))$. So

$$\int_{T-\sigma}^{T} (\nabla q_n \cdot \nabla z, q_n \Delta v_n)\, dt \to \int_{T-\sigma}^{T} (q^* \cdot \nabla z, q^* \Delta v^*)\, dt.$$

We can pass the limits in the other terms of (3.18). Consequently

$$\int_{T-\sigma}^{T} (\nabla \cdot (q_n \nabla z), \nabla \cdot (q_n \nabla v_n))\, dt \to \int_{T-\sigma}^{T} (\nabla \cdot (q^* \nabla z), \nabla \cdot (q^* \nabla v^*))\, dt.$$

In the same manner, we can prove that the other terms in (3.15) tend to the corresponding terms in (3.16), respectively. Therefore

$$J_0(q^*, v^*) = \lim_{n\to\infty} J_0(q_n, v_n). \tag{3.20}$$

On the other hand, $|q_n|_2$ is a convex function of q, and so by the semi-continuity of convex functions,

$$|q^*|_2^2 \le \liminf_2 |q_n|_2^2. \tag{3.21}$$

Then the combination of (3.20) and (3.21) implies that

$$J(q^*, v^*) \le \lim_{n\to\infty} \inf J(q_n, v_n) = \inf_{(q,v)\in A} J(q, v).$$

This completes the proof. ∎

4. THE AUGMENTED LAGRANGIAN FORMULATION

In this section, we propose a new approach for solving the constrained minimization problem (3.3), namely transforming the problem (3.3), which is highly nonlinear and not convex, into an equivalent saddle-point problem for an augmented Lagrange functional. The saddle-point problem is much more convenient and easier to solve than the minimization problem (3.3) due to the fact that the augmented Lagrange functional is quadratic and convex with respect to each variable. This seems to be the first time to study and justify the mathematical formulation of the augmented Lagrangian method for recovering the parameters in a time-dependent system. The augmented Lagrangian method for the parameter identification in some elliptic systems was studied earlier by Ito and Kunisch [11] and Chen and Zou [6].

Now, let

$$\tilde{W} = \{v \in W;\ v(x,0) = u_0(x)\}, \qquad \tilde{H} = L^2(0,T; L^2(\Omega)),$$

and for any $r \ge 0$, we define the augmented functional $\mathscr{L}_r$: $K \times \tilde{W} \times \tilde{H} \to \mathscr{R}$ by

$$\mathscr{L}_r(q, v; \mu) = J(q, v) + \int_0^T \left(\frac{\partial}{\partial t} e(q, v) - \Delta e(q, v), \mu \right) dt$$
$$+ \frac{r}{2} \int_0^T \|\nabla e(q, v)\|^2 \, dt. \tag{4.1}$$

It is easy to see that for any $e(q, v) \in W$, $\mathscr{L}_r(q, v; \mu)$ is finite.

The rest of this section is to establish the relation between the minimization problem (3.3) and the saddle-point problem of the augmented Lagrange functional $\mathscr{L}_r$. The key point for this is to apply the Hahn–Banach theorem stated below; see, e.g., Balakrishnan [1].

LEMMA 4.1. *Let B_1 and B_2 be two convex subsets of Hilbert space $\mathscr{H}$ with the inner product $(\cdot,\cdot)_{\mathscr{H}}$. If $B_1 \cap B_2 = \emptyset$, and B_1 or B_2 contains at least one interior point, then there exists an element $z \in \mathscr{H}$, $z \neq 0$ such that*

$$\sup_{y\in B_1} (z,y)_{\mathscr{H}} \le \inf_{y\in B_2} (z,y)_{\mathscr{H}}.$$

In the following we will make some efforts so that we can apply Lemma 4.1 to our identification problem (3.3). For this purpose, we first define two convex subsets B_1 and B_2. Let (q^*,v^*) be a minimizer of (3.3). Then B_1 and B_2 are defined as

$$B_1 = \{(\gamma,0) \in \mathscr{R} \times \tilde{H},\ \gamma < 0\}, \tag{4.2}$$

$$B_2 = \left\{\left(J(q,v) - J(q^*,v^*) + s,\ \frac{\partial}{\partial t}e(q,v) - \Delta e(q,v)\right)\right.$$
$$\left. \in \mathscr{R} \times \tilde{H},\ (q,v) \in K \times \tilde{W},\ s \ge 0\right\}. \tag{4.3}$$

The following three lemmas are to verify the conditions of Lemma 4.1.

LEMMA 4.2. *B_1 and B_2 are two convex subsets in $\mathscr{R} \times \tilde{H}$.*

Proof. Clearly, B_1 is a convex subset in $\mathscr{R} \times \tilde{H}$. We now check the convexity of B_2. Let Q_1 and Q_2 be any two points in B_2, namely for $i = 1,2$ we have

$$Q_i = \left(J(q_i,v_i) - J(q^*,v^*) + s_i,\ \frac{\partial}{\partial t}e(q_i,v_i) - \Delta e(q_i,v_i)\right) \in \mathscr{R} \times \tilde{H},$$
$$s_i \ge 0.$$

Let $\alpha = (\alpha_1, \alpha_2)$ with $\alpha_i \in [0,1]$ and $\alpha_1 + \alpha_2 = 1$. We have to prove that

$$Q_\alpha = \alpha_1 Q_1 + \alpha_2 Q_2 \equiv (p_\alpha, w_\alpha) \in B_2, \tag{4.4}$$

where

$$p_\alpha = \alpha_1 J(q_1,v_1) + \alpha_2 J(q_2,v_2) - J(q^*,v^*) + \alpha_1 s_1 + \alpha_2 s_2, \tag{4.5}$$

$$w_\alpha = \alpha_1\left(\frac{\partial}{\partial t}e(q_1,v_1) - \Delta e(q_1,v_1)\right) + \alpha_2\left(\frac{\partial}{\partial t}e(q_2,v_2) - \Delta e(q_2,v_2)\right). \tag{4.6}$$

For simplicity, we will write $e(q_i, v_i)$ as e_i. We now take $q_\alpha = \alpha_1 q_1 + \alpha_2 q_2 \in K$ and $v_\alpha \in W$ as the solution of the variational problem

$$\begin{cases} \left(\dfrac{\partial v_\alpha}{\partial t}, \phi\right) + (q_\alpha \nabla v_\alpha, \nabla\phi) = \left(\alpha_1 \dfrac{\partial v_1}{\partial t} + \alpha_2 \dfrac{\partial v_2}{\partial t}, \phi\right) \\ \qquad + (\alpha_1 q_1 \nabla v_1 + \alpha_2 q_2 \nabla v_2, \nabla\phi), \qquad \forall \phi \in V, \text{ a.e. } t \in (0, T], \\ v_\alpha(x, 0) = \alpha_1 v_1(x, 0) + \alpha_2 v_2(x, 0). \end{cases} \tag{4.7}$$

Clearly, v_α is well-defined. Let $e_\alpha = e(q_\alpha, v_\alpha)$. We have by (3.2) and (4.7) that

$$\begin{aligned} &\left(\frac{\partial e_\alpha}{\partial t}, \phi\right) + (\nabla e_\alpha, \nabla\phi) \\ &\qquad = \left(\frac{\partial v_\alpha}{\partial t}, \phi\right) + (q_\alpha, \nabla v_\alpha, \nabla\phi) - (f, \phi) \\ &\qquad = \left(\alpha_1 \frac{\partial v_1}{\partial t} + \alpha_2 \frac{\partial v_2}{\partial t}, \phi\right) + (\alpha_1 q_1 \nabla v_1 + \alpha_2 q_2 \nabla v_2, \nabla\phi) - (f, \phi) \\ &\qquad = \left(\frac{\partial}{\partial t}(\alpha_1 e_1 + \alpha_2 e_2), \phi\right) + \big(\nabla(\alpha_1 e_1 + \alpha_2 e_2), \nabla\phi\big), \end{aligned} \tag{4.8}$$

and

$$e_\alpha(x, 0) = \alpha_1 v_1(x, 0) + \alpha_2 v_2(x, 0) - u_0(x) = \alpha_1 e_1(x, 0) + \alpha_2 e_2(x, 0). \tag{4.9}$$

This implies

$$e_\alpha = \alpha_1 e_1 + \alpha_2 e_2.$$

Moreover, by using (4.6), (4.9), and integrating the right hand side of (4.8) by parts, we assert that

$$w_\alpha = \frac{\partial}{\partial t} e(q_\alpha, v_\alpha) - \Delta e(q_\alpha, v_\alpha).$$

On the other hand, noting the fact $q_\alpha = \alpha_1 q_1 + \alpha_2 q_2 \in K$ and (4.7),

$$\begin{aligned} &\left(\frac{\partial}{\partial t}(v_\alpha - z) - \nabla \cdot (q_\alpha \nabla(v_\alpha - z)), \phi\right) \\ &\qquad = \alpha_1 \left(\frac{\partial}{\partial t}(v_1 - z) - \nabla \cdot (q_1 \nabla(v_1 - z)), \phi\right) \\ &\qquad\qquad + \alpha_2 \left(\frac{\partial}{\partial t}(v_2 - z) - \nabla \cdot (q_2 \nabla(v_2 - z)), \phi\right). \end{aligned}$$

Thus in the sense of distributions we have

$$\begin{aligned}
&\frac{\partial}{\partial t}(v_\alpha - z) - \nabla\cdot(q_\alpha \nabla(v_\alpha - z))\\
&\quad = \alpha_1\left(\frac{\partial}{\partial t}(v_1 - z) - \nabla\cdot(q_1\nabla(v_1 - z))\right)\\
&\qquad + \alpha_2\left(\frac{\partial}{\partial t}(v_2 - z) - \nabla\cdot(q_2\nabla(v_2 - z))\right).
\end{aligned}$$

Using the convexity of the norm $\|\cdot\|$, we obtain

$$\begin{aligned}
&\left\|\frac{\partial}{\partial t}(v_\alpha - z) - \nabla\cdot(q_\alpha \nabla(v_\alpha - z))\right\|^2\\
&\quad \le \alpha_1\left\|\frac{\partial}{\partial t}(v_1 - z) - \nabla\cdot(q_1\nabla(v_1 - z))\right\|^2\\
&\qquad + \alpha_2\left\|\frac{\partial}{\partial t}(v_2 - z) - \nabla\cdot(q_2\nabla(v_2 - z))\right\|^2.
\end{aligned}$$

This, combined with the convexity of the semi-norm $|q|_2$, leads to

$$J(q_\alpha, v_\alpha) \le \alpha_1 J(q_1, v_1) + \alpha_2 J(q_2, v_2).$$

Hence by (4.5),

$$p_\alpha = J(q_\alpha, v_\alpha) - J(q^*, v^*) + s_\alpha,$$

where

$$s_\alpha = \alpha_1 J(q_1, v_1) + \alpha_2 J(q_2, v_2) - J(q_\alpha, v_\alpha) + \alpha_1 s_1 + \alpha_2 s_2 \ge 0.$$

This proves (4.4) and completes the proof of Lemma 4.2. ∎

LEMMA 4.3. $B_1 \cap B_2 = \emptyset$.

Proof. Let $Q = (p, w) \in B_2$. Then

$$p = J(q,v) - J(q^*, v^*) + s,\ s \ge 0; \qquad w = \frac{\partial}{\partial t} e(q,v) - \Delta e(q,v).$$

If Q also belongs to B_1, then $w = 0$, and so $e(q,v) = 0$. Hence $(q,v) \in A$, which implies $p \ge s \ge 0$. But it follows from $Q \in B_1$ that $p < 0$. So we have a contradiction. ∎

LEMMA 4.4. *The subset B_2 has at least one interior point.*

Proof. Take $q = q^*$, $v = v^*$, and $s = s_0 > 0$ in (4.3). Then $Q_0 = (s_0, 0) \in B_2$. We shall show that Q_0 is an interior point of B_2. To do this, let $\varepsilon \in (0,1)$ and (s, w) be an arbitrary point of the ε-neighborhood of $(s_0, 0)$ in $\mathscr{R} \times \tilde{H}$. Then for all $t \in [0, T]$, we have

$$|s - s_0|^2 + \int_0^T \|w(t)\|^2 \, dt \le \varepsilon^2. \tag{4.10}$$

Let $\bar{v}$ be the solution of the problem

$$\begin{cases} \left(\dfrac{\partial \bar{v}}{\partial t}, \phi \right) + (q^* \nabla \bar{v}, \nabla \phi) = (w + f, \phi), & \forall \phi \in V, \text{a.e. } t \in (0, T], \\ \bar{v}(x, 0) = u_0(x). \end{cases} \tag{4.11}$$

By Lemma 2.1 and Remarks 2.1 and 2.2,

$$\|\bar{v}\|_{L^2(0,T; H^2(\Omega)) \cap L^\infty(0,T; H^1(\Omega)) \cap H^1(0,T; L^2(\Omega))} \le c^*. \tag{4.12}$$

Next we rewrite s as

$$s = J(q^*, \bar{v}) - J(q^*, v^*) + \bar{s},$$

where

$$\bar{s} = J(q^*, v^*) - J(q^*, \bar{v}) + s.$$

Due to $e(q^*, v^*) = 0$, we have that

$$\left(\frac{\partial}{\partial t} v^*, \phi \right) + (q^* \nabla v^*, \nabla \phi) = (f, \phi), \qquad \forall \phi \in V.$$

Subtracting this from (4.11) yields

$$\begin{cases} \left(\dfrac{\partial}{\partial t} (\bar{v} - v^*), \phi \right) + (q^* \nabla (\bar{v} - v^*), \nabla \phi) = (w, \phi), & \forall \phi \in V, \\ \bar{v}(x, 0) - v^*(x, 0) = 0. \end{cases}$$

By virtue of (4.10), Lemma 2.1, and Remarks 2.1 and 2.2,

$$\|\bar{v} - v^*\|^2_{L^2(0,T; H^2(\Omega)) \cap L^\infty(0,T; H^1(\Omega)) \cap H^1(0,T; L^2(\Omega))} \le c \varepsilon^2. \tag{4.13}$$

By some careful calculations, we can write

$$J(q^*, v^*) - J(q^*, \bar{v}) = \frac{1}{2} \sum_{i=1}^{18} \int_{T-\sigma}^{T} G_i(s) \, ds, \tag{4.14}$$

where

$$G_1 = \left(\frac{\partial}{\partial t}(v^* - \bar{v}), \frac{\partial}{\partial t}(v^* + \bar{v})\right), \qquad G_2 = -2\left(\frac{\partial}{\partial t}(v^* - \bar{v}), \frac{\partial z}{\partial t}\right),$$

$$G_3 = (\nabla q^* \cdot \nabla(v^* - \bar{v}), \nabla q^* \cdot \nabla(v^* + \bar{v})),$$

$$G_4 = -2(\nabla q^* \cdot \nabla(v^* - v), \nabla q^* \cdot \nabla z),$$

$$G_5 = \left((q^*)^2 \Delta(v^* - \bar{v}), \Delta(v^* + \bar{v})\right), \qquad G_6 = -2\left((q^*)^2 \Delta(v^* - \bar{v}), \Delta z\right),$$

$$G_7 = 2(\nabla q^* \cdot \nabla(v^* - \bar{v}), q^* \Delta v^*), \qquad G_8 = 2(\nabla q^* \cdot \nabla \bar{v}, q^* \Delta(v^* - \bar{v})),$$

$$G_9 = -2(\nabla q^* \cdot \nabla z, q^* \Delta(v^* - \bar{v})), \qquad G_{10} = -2(\nabla q^* \cdot \nabla(v^* - \bar{v}), q^* \Delta z),$$

$$G_{11} = -2\left(\frac{\partial}{\partial t}(v^* - \bar{v}), q^* \Delta v^*\right), \qquad G_{12} = -2\left(\frac{\partial \bar{v}}{\partial t}, q^* \Delta(v^* - \bar{v})\right),$$

$$G_{13} = 2\left(\frac{\partial z}{\partial t}, q^* \Delta(v^* - \bar{v})\right), \qquad G_{14} = 2\left(\frac{\partial}{\partial t}(v^* - \bar{v}), q^* z\right),$$

$$G_{15} = -2\left(\frac{\partial}{\partial t}(v^* - \bar{v}), \nabla q^* \cdot \nabla v^*\right), \qquad G_{16} = -2\left(\frac{\partial \bar{v}}{\partial t}, \nabla q^* \cdot (v^* - \bar{v})\right),$$

$$G_{17} = 2\left(\frac{\partial z}{\partial t}, \nabla q^* \cdot \nabla(v^* - \bar{v})\right), \qquad G_{18} = 2\left(\frac{\partial}{\partial t}(v^* - \bar{v}), \nabla q^* \cdot \nabla z\right).$$

All the terms $G_i (1 \le i \le 18)$ above can be bounded by a factor of ε. For instance, by (4.12), (4.13), and the imbedding theory we can derive

$$\begin{aligned}
\int_{T-\sigma}^{T} |G_3(s)| ds &\le c \|q^*\|_{1,4}^2 \left\{ \int_{T-\sigma}^{T} |v^* - \bar{v}|_{1,4}^2 ds \right\}^{\frac{1}{2}} \left\{ \int_{T-\sigma}^{T} |v^* + \bar{v}|_{1,4}^2 ds \right\}^{\frac{1}{2}} \\
&\le c \|q^*\|_2^2 \left\{ \int_{T-\sigma}^{T} \|v^* - \bar{v}\|_2^2 ds \right\}^{\frac{1}{2}} \left\{ \int_{T-\sigma}^{T} \|v^* + \bar{v}\|_2^2 ds \right\}^{\frac{1}{2}} \le b^* \varepsilon,
\end{aligned}$$

$$\begin{aligned}
\int_{T-\sigma}^{T} |G_5(s)| ds &\le c \gamma_2^2 \left\{ \int_{T-\sigma}^{T} \|\Delta(u^* - \bar{v})\|^2 ds \right\}^{\frac{1}{2}} \left\{ \int_{T-\sigma}^{T} \|\Delta(u^* + \bar{v})\|^2 ds \right\}^{\frac{1}{2}} \\
&\le b^* \varepsilon,
\end{aligned}$$

$$\int_{T-\sigma}^{T} |G_{11}(s)| ds \le c \gamma_2^2 \left\{ \int_{T-\sigma}^{T} \left\| \frac{\partial}{\partial t}(v^* - \bar{v}) \right\|^2 ds \right\}^{\frac{1}{2}} \left\{ \int_{T-\sigma}^{T} \|\Delta u^*\|^2 ds \right\}^{\frac{1}{2}} \le b^* \varepsilon,$$

where b^* is a positive constant depending on the norms of v^* and $\bar{v}$ in the space W. Thus $\bar{s} > 0$, if ε is sufficiently small. That implies $(s, w) \in B_2$. ∎

We are now in a position to state the main result of this section.

THEOREM 4.1. *$(q^*, v^*) \in K \times W$ is a minimizer of the problem (3.3) if and only if there exists a $\mu^* \in \tilde{H}$ such that $(q^*, v^*, \mu^*) \in K \times \tilde{W} \times \tilde{H}$ is a saddle-point of the augmented Lagrangian functional $\mathscr{L}_r(q, v; \mu)$, namely,*

$$\mathscr{L}_r(q^*, v^*; \mu) \le \mathscr{L}_r(q^*, v^*; \mu^*) \le \mathscr{L}_r(q, v; \mu^*),$$
$$\forall (q, v, \mu) \in K \times \tilde{W} \times \tilde{H}. \quad (4.15)$$

Proof. We first show that if (q^*, v^*, μ^*) is a saddle-point, then (q^*, v^*) is a minimizer to the problem (3.3). Indeed, by the first inequality of (4.15) we have for any $\mu \in \tilde{H}$,

$$\int_0^T \left(\frac{\partial}{\partial t} e(q^*, v^*) - \Delta e(q^*, v^*), \mu \right) dt$$
$$\le \int_0^T \left(\frac{\partial}{\partial t} e(q^*, v^*) - \Delta e(q^*, v^*), \mu^* \right) dt. \quad (4.16)$$

Let

$$\mu = \frac{\partial}{\partial t} e(q^*, v^*) - \Delta e(q^*, v^*) \in \tilde{H}.$$

Then

$$\int_0^T \left(\frac{\partial}{\partial t} e(q^*, v^*) - \Delta e(q^*, v^*), \mu^* \right) dt \ge 0.$$

On the other hand, if we take $\mu = 2\mu^*$ in (4.16), then

$$\int_0^T \left(\frac{\partial}{\partial t} e(q^*, v^*) - \Delta e(q^*, v^*), \mu^* \right) dt \le 0.$$

Therefore

$$\int_0^T \left(\frac{\partial}{\partial t} e(q^*, v^*) - \Delta e(q^*, v^*), \mu^* \right) dt = 0.$$

This yields

$$\mathscr{L}_r(q^*, v^*; \mu^*) = J(q^*, v^*) + \frac{r}{2} \int_0^T \|\nabla(q^*, v^*)\|^2 dt. \quad (4.17)$$

Furthermore, by the second inequality of (4.15) and (4.17), we have for any $(q, v) \in A$,

$$J(q^*, v^*) + \frac{r}{2} \int_0^T \|\nabla(q^*, v^*)\|^2 dt \le \mathscr{L}_r(q, v; \mu^*) = J(q, v). \quad (4.18)$$

Thus $J(q^*, v^*) \le J(q, v)$. Finally, taking $(q, v) = (q^*, v^*)$ in (4.18), we obtain $e(q^*, v^*) = 0$ and so $(q^*, v^*) \in A$. This completes the proof of the first part.

Now, assume that (q^*, v^*) is a minimizer to the problem (3.3); that is, for all $(q, v) \in A$, we have

$$J(q^*, v^*) \le J(q, v). \tag{4.19}$$

We need to prove that there is a Lagrange multiplier $\mu^* \in \tilde{H}$ such that (4.15) is fulfilled. According to Lemmas 4.2–4.4, we can use Lemma 4.1 with $\mathscr{H} = \tilde{H}$. So there exists a pair $(\alpha_0, \mu_0) \in \mathscr{R} \times \tilde{H}$ with $(\alpha_0, \mu_0) \neq (0, 0)$ such that

$$\begin{aligned}&\alpha_0(J(q,v) - J(q^*,v^*) + s)\\ &\quad + \int_0^T \left(\frac{\partial}{\partial t} e(q,v) - \Delta e(q,v), \mu_0 \right) dt \ge \alpha_0 \gamma, \qquad \gamma < 0.\end{aligned} \tag{4.20}$$

Taking $(q, v) = (q^*, v^*)$, $s = 1$, and $\gamma = -1$ in (4.20), we get $\alpha_0 \ge 0$. While taking $s = 0$ and letting $\gamma \to 0^-$, we obtain that

$$\begin{aligned}&\alpha_0(J(q,v) - J(q^*,v^*)) + \int_0^T \left(\frac{\partial}{\partial t} e(q,v) - \Delta e(q,v), \mu_0 \right) dt \ge 0,\\ &\qquad\qquad \forall (q,v) \in K \times \tilde{W}.\end{aligned} \tag{4.21}$$

We now claim that $\alpha_0 > 0$. Otherwise if $\alpha_0 = 0$, then it follows from (4.21) that

$$\int_0^T \left(\frac{\partial}{\partial t} e(q,v) - \Delta e(q,v), \mu_0 \right) dt \ge 0, \qquad \forall (q,v) \in K \times \tilde{W}. \tag{4.22}$$

But by (3.2) we know in the sense of distributions that

$$\frac{\partial}{\partial t} e(q,v) - \Delta e(q,v) = \frac{\partial}{\partial t} v - \nabla \cdot (q \nabla v) - f.$$

This with (4.22) leads to

$$\int_0^T \left(\frac{\partial}{\partial t} v - \nabla \cdot (q \nabla v) - f, \mu_0 \right) dt \ge 0, \qquad \forall (q,v) \in K \times \tilde{W}. \tag{4.23}$$

Now take $q = q^* \in K$ and let $v \in W$ be the solution of the problem

$$\left(\frac{\partial v}{\partial t}, \phi \right) + (q^* \nabla v, \nabla \phi) = (f - \mu_0, \phi), \qquad \forall \phi \in V \text{ and for a.e. } t \in (0, T).$$

Then in the sense of distributions,

$$\frac{\partial}{\partial t} v - \nabla \cdot (q^* \nabla v) - f = -\mu_0.$$

Accordingly we obtain from (4.23) with $q = q^*$ that

$$-\int_0^T \| \mu_0 \|^2 ds \geq 0.$$

This leads to $(\alpha_0, \mu_0) = (0, 0)$, which is a contradiction. Therefore $\alpha_0 > 0$. Taking $\mu^* = \mu_0 / \alpha_0$ in (4.21), we obtain

$$J(q^*, v^*) \leq J(q, v) + \int_0^T \left(\frac{\partial}{\partial t} e(q, v) - \Delta e(q, v), \mu^* \right) dt \leq \mathscr{L}_r(q, v; \mu^*). \tag{4.24}$$

Moreover, since (q^*, v^*) is a solution of (3.3), we have

$$\mathscr{L}_r(q^*, v^*; \mu) = J(q^*, v^*) = \mathscr{L}_r(q^*, v^*; \mu^*).$$

This fact with (4.24) completes the proof of the second part of Theorem 4.1. ∎

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