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UNIQUENESS IN DETERMINING MULTIPLE POLYGONAL SCATTERERS OF MIXED TYPE

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Abstract. We prove that a polygonal scatterer in $\mathbb{R}^2$, possibly consisting of finitely many sound-soft and sound-hard polygons, is uniquely determined by a single far-field measurement.

1. Introduction.

We consider an acoustic scattering problem by an impenetrable obstacle $\mathbf{D}$, which is assumed to be a compact set in $\mathbb{R}^2$ with connected complement $\mathbf{G} = \mathbb{R}^N \backslash \mathbf{D}$. Let u^i, u^s and $u = u^i + u^s$ denote, respectively, the incident, scattered and total filed. Throughout, we take $u^i(x) = \exp\{ikx \cdot d\}$ to be a time-harmonic plane wave, with $\mathrm{i} = \sqrt{-1}$, incident direction $d \in \mathbb{S}^1 := \{x \in \mathbb{R}^2; |x| = 1\}$ and wave number $k > 0$. Then the direct scattering problem is described by the following Helmholtz equation

$$\Delta u + k^2 u = 0 \qquad \text{in} \quad \mathbf{G} = \mathbb{R}^2 \backslash \mathbf{D}. \tag{1}$$

The Helmholtz equation (1) is complemented by the following Sommerfeld radiation condition

$$\lim_{r \to \infty} r^{1/2} \left(\frac{\partial u^s}{\partial r} - \mathrm{i} k u^s \right) = 0, \tag{2}$$

with $r = |x|$ for any $x \in \mathbb{R}^2$, and either one of the following boundary conditions:

$$u = 0 \qquad \text{on} \quad \partial \mathbf{G} \text{ (the sound-soft obstacle);} \tag{3}$$

$$\frac{\partial u}{\partial \nu} = 0 \qquad \text{on} \quad \partial \mathbf{G} \text{ (the sound-hard obstacle);} \tag{4}$$

where ν is the unit normal to $\partial \mathbf{G}$ directed into the interior of $\mathbf{G}$.

It is known that for any Lipschitz domain $\mathbf{G}$, there exists a unique solution $u = u(\mathbf{D}; k, d) \in H^1_{loc}(\mathbf{G})$ to the above Helmholtz system, and u is analytic in $\mathbf{G}$

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(see [12]). Moreover, the asymptotic behavior at infinity of the scattered wave u^s is governed by

$$u^s(x) = \frac{e^{ik|x|}}{|x|^{1/2}} \left\{ u_\infty(\hat{x}) + \mathcal{O}(\frac{1}{|x|}) \right\} \quad \text{as } |x| \to \infty, \tag{5}$$

uniformly for all directions $\hat{x} = x/|x| \in \mathbb{S}^1$. The analytic function $u_\infty(\hat{x})$ is defined on the unit sphere $\mathbb{S}^1$, and often called the far-field pattern (see [3]). We shall write $u_\infty(\hat{x}; \mathbf{D}, d, k)$ to specify its dependence on the observation direction $\hat{x}$, the obstacle $\mathbf{D}$, the incident direction d and the wave number k. The inverse obstacle scattering problem is to determine $\partial\mathbf{G}$ from the measurement of u_∞, which widely occurs in my practical applications, e.g., radar, sonar and non-destructive testing. In the present paper, we are mainly concerned with the uniqueness issue in the inverse problem, i.e., *is the correspondence between* $u_\infty(\hat{x}; \mathbf{D}, k, d)$ *and* $\mathbf{D}$ *one to one?* This is a well-known problem and still largely remains open (see Problems 6.3 and 6.4, [9]). We also refer to [5] for a review of the existing uniqueness results. Recently, extensive study has been focused on establishing uniqueness for polyhedral (polygonal)-type scatterers by means of unique continuation along lines (planes) and reflection arguments for solutions of Hemlholtz equation (1) (see [1],[6],[8],[10]). It is now clear that a general sound-soft polyhedral scatterer in $\mathbb{R}^N (N \geq 2)$, possibly consisting of finitely many solid polyhedra and subsets of $(N-1)$-dimensional hyperplanes, is uniquely determined by the far-field pattern corresponding to a single incident plane wave at an arbitrarily fixed wave number and incident direction (see [1] and [10]). Whereas for the sound-hard case, such uniqueness is established in [10] by N far-field measurements corresponding to N incident plane waves given by a fixed wave number and N linearly independent incident directions. For a particular case with one solid two-dimensional sound-hard polygon, the uniqueness is verified in [8] by only one incoming wave. In the current work, we will establish the uniqueness by a single incident wave for a much more general sound-hard polygonal scatterer which consists of finitely many polygons. In fact, we have stepped further by proving the uniqueness in a much more challenging setting without knowing a prior the physical properties of the underlying scatterers. It is remarked that all the aforementioned existing uniqueness results are established under the a prior knowledge of the physical properties on the underlying obstacles, that is, the obstacle is known to be sound-soft or sound-hard. However, in more realistic applications, such a prior information may not be always available. An example of such a situation is the detection of buried objects from the far-field or near-field measurements (see, e.g., [2]), where it is naturally assumed that (i) the number of components of the scatterer under consideration is finite but unknown; (ii) the physical properties of each component are unknown and it may even happen that some components of the scatterer are sound-soft while the others are sound-hard. Hence, the uniqueness results established in the present paper would be of much more practical interest and importance.

The rest of the paper is organized as follows. In Section 2, we present the main uniqueness result. Section 3 is devoted to the proof of the main theorem. In Section 4, we give proofs of some of the key lemmata needed in Section 3. And the paper is concluded in Section 5.

2. **Main uniqueness result.** We start with a detailed description on the two-dimensional polygonal scatterers, which will be handled in this work.

Definition 2.1. $\mathbf{D} \subset \mathbb{R}^2$ is called a multiple polygonal scatterer of mixed type if

(i) $\mathbf{D}$ is a compact set with connected complement $\mathbf{G} = \mathbb{R}^2 \backslash \mathbf{D}$;

(ii) $\mathbf{D} = \bigcup_{j=1}^{m} D_j$, where each D_j, $j = 1, 2, \ldots, m$, is a compact polygon and $D_j \cap D_{j'} = \emptyset$ if $j \neq j'$;

(iii) The number m of components is finite but unknown. The physical property of each component obstacle D_j, $j = 1, 2, \ldots, m$, is unknown but must be either sound-soft or sound-hard.

Now, based on Definition 2.1, the direct problem associated with such a scatterer clearly consists of the Helmholtz system (1)-(2) and the following mixed boundary condition,

$$u = 0 \quad \text{on} \ \bigcup_{j=1}^{m'} \partial D_j \quad \text{and} \quad \partial_\nu u = 0 \quad \text{on} \ \bigcup_{j=m'+1}^{m} \partial D_j, \tag{6}$$

where $0 \leq m' \leq m$. This corresponds to the case that $D_1, \ldots, D_{m'}$ are sound-soft polygons, while $D_{m'+1}, \ldots, D_m$ are sound-hard type. Henceforth, we shall use $\mathfrak{B}u = 0$ to represent the above boundary conditions on $\partial\mathbf{D}$. In the sequel, without loss of generality, we may assume that the mixed-type polygonal scatterer $\mathbf{D}$ satisfies $1 \leq m' \leq m-1$, that is, there are both sound-soft and sound-hard components. In fact, if all the components of the scatterer $\mathbf{D}$ are of the same type, i.e., all sound-soft type, or all sound-hard type, then the uniqueness result given in Theorem 2.2 below can be easily seen from our subsequent proof with scatterers of really mixed type.

Finally, we would like to mention a fact on the forward scattering problem that will be implicitly used in the subsequent analysis: if $x_0 \in \partial\mathbf{G}$ is an interior point of one of the edges forming $\partial\mathbf{G}$, then it is a regular point for the problem and the total field u is infinitely smooth up to that point (see Chapter 4, [12]).

Next, we are ready to state the main result of this paper.

Theorem 2.2. *Assume that $\mathbf{D}$ and $\widetilde{\mathbf{D}}$ are two mixed-type multiple polygonal scatterers as described in Definition 2.1, with respective boundary conditions $\mathfrak{B}$ and $\widetilde{\mathfrak{B}}$. If the far-field patterns for $\mathbf{D}$ and $\widetilde{\mathbf{D}}$ coincide for a single incident plane wave at one arbitrarily fixed incident direction and wave number, then $\mathbf{D} = \widetilde{\mathbf{D}}$ and $\mathfrak{B} = \widetilde{\mathfrak{B}}$.*

Remark 1. Compared to the uniqueness result in [8] concerning a single sound-hard polygon by a single far-field measurement, the novelty of our uniqueness result lies in two aspects: the uniqueness applies to scatterers composed of finitely many disjoint polygons; the uniqueness holds without a prior information on the physical property of each component of the underlying scatterer.

In fact, our uniqueness holds in a more general setting without the pairwise disjoint condition (see Theorem 5.1 in Subsection 5.1). But to ease our exposition, we will first concentrate on the relatively simpler disjoint case.

3. **Proof of Theorem 2.2.** We first fix some notations which shall be used throughout the rest of the paper. For two distinct points $P, Q \in \mathbb{R}^2$, PQ denotes the open line segment with the endpoints P and Q. Let $\mathbf{D}$ be a polygonal scatterer as defined in Definition 2.1, then for a line segment $PQ \subset \mathbf{G} = \mathbb{R}^2 \backslash \mathbf{D}$ with $Q \in \partial\mathbf{G}$, we denote by $\angle(PQ, \partial\mathbf{G})$ the least one among the two angles in $\mathbf{G}$ formed by PQ and $\partial\mathbf{G}$ at Q. $\triangle PQR$ represents the interior of the triangle with three vertices P, Q, R, which are non-collinear, and $\angle PQR$ stands for the interior angle at Q. We will denote an open disc in $\mathbb{R}^2$ with center x and radius r by $B_r(x)$, the closure of $B_r(x)$ by $\bar{B}_r(x)$

and the boundary of $B_r(x)$ by $S_r(x)$. Moreover, the notation $T_r(x)$ is defined to be an open square of edge length r, centered at x, while $\bar{T}_r(x)$ is its corresponding closure. Unless specified otherwise, ν shall always denote the outward normal to a concerned domain, or the normal to a line. Also we may often write $u(E) = 0$ for any subset $E \subset \mathbb{R}^2$ if $u(x) = 0$ for $x \in E$. The distance between two sets $\mathcal{A}$ and $\mathcal{B}$ is defined by $\mathbf{d}(\mathcal{A}, \mathcal{B}) = \inf_{x\in\mathcal{A}, y\in\mathcal{B}} |x - y|$. Finally, a curve $\gamma = \gamma(t)(t \geq 0)$ is said to be regular if it is C^1-smooth and $\frac{d}{dt}\gamma(t) \neq 0$.

Next, we recall some auxiliary results. The first one is about a fundamental property of a connected set, see e.g., Theorem 3.19.9 in [7].

Lemma 3.1. *Let $\mathbb{E}$ be a metric space, $\mathcal{A} \subset \mathbb{E}$ be a subset and $\mathcal{B} \subset \mathbb{E}$ be a connected set such that $\mathcal{A} \cap \mathcal{B} \neq \emptyset$ and $(\mathbb{E}\backslash\mathcal{A}) \cap \mathcal{B} \neq \emptyset$, then $\partial\mathcal{A} \cap \mathcal{B} \neq \emptyset$.*

Lemma 3.2. *Let E be a domain in $\mathbb{R}^2$ and $v \in H^1_{loc}(E)$ be a solution to $\Delta v + k^2 v = 0$ in E. If l_0 and l are two line segments in E such that $l_0 \subset l$ and $\partial_\nu v = 0$ (resp. $v = 0$) on l_0, then $\partial_\nu v = 0$ (resp. $v = 0$) on l.*

Proof. Noting v is analytic in E, the lemma is readily seen by analytic continuation. □

Lemma 3.3. *Let u be a solution to the system (1), (2) and (6). Then there exists no infinite straight half-line $l \subset \mathbf{G}$ such that $u = 0$ on l. And there cannot exist two infinite half-lines $l_1, l_2 \subset \mathbf{G}$ such that l_1 and l_2 are not parallel and $\partial_\nu u = 0$ on $l_1 \cup l_2$.*

Proof. From the asymptotic expression (5), one can easily see that $u^s(x) \to 0$ as $|x| \to \infty$. Furthermore, by applying the Green's formula on the sphere of a sufficiently large ball containing $\mathbf{D}$, one directly derives that $\lim_{|x|\to\infty} |\nabla u^s(x)| = 0$ (see the proof of Lemma 2 in [6]). The lemma then follows readily from Lemma 3.1 in [1] and Lemma 2 in [6]. □

Lemma 3.4. *Let Ω be a connected polygonal domain in $\mathbb{R}^2$, OA be one of its sides such that Ω is located at one side of OA, and R be the reflection in $\mathbb{R}^2$ with respect to the extended straight line of OA. Let E be a domain in $\mathbb{R}^2$ such that $\Omega \cup R(\Omega) \subset E$. If $v \in H^1(E)$ satisfies $\Delta v + k^2 v = 0$ in E, then we have*

$$\begin{aligned} &v(x) = -Rv(x) \ \text{in } \Omega \cup R(\Omega) \quad \text{if } v = 0 \text{ on } OA; \\ &v(x) = Rv(x) \ \text{in } \Omega \cup R(\Omega) \quad \text{if } \partial_\nu v = 0 \text{ on } OA, \end{aligned} \tag{7}$$

where $Rv(x) = v(R(x))$. Consequently, if $v = 0 (\partial_\nu v = 0)$ on any other side BC of $\partial\Omega$ except OA, then $v = 0$ $(\partial_\nu v = 0)$ on $R(BC)$.

Proof. This lemma is known as the reflection principle for the Helmholtz equation and can be verified by a combination of the proofs of Theorem 1 and Theorem 2 in [10]. See also Lemma 1 in [8] and Lemma 3 in [6]. □

Our proof of Theorem 2.2 will be based on a careful study of the behaviors of some line segments in $\mathbf{G} := \mathbb{R}^2\backslash\mathbf{D}$, on which u assumes homogeneous Dirichlet or Neumann data. For this purpose, we now introduce those special line segments.

Define

$$\mathcal{S}_1 := \{l; l \text{ is a finite open line segment extended to maximum length in } \mathbf{G} \text{ such that } u = 0 \text{ or } \partial_\nu u = 0 \text{ on } l\}, \quad (8)$$

$$\mathcal{S}_2 := \{l; l \text{ is an infinite open line segment extended to maximum length in } \mathbf{G} \text{ such that } u = 0 \text{ or } \partial_\nu u = 0 \text{ on } l\}. \quad (9)$$

It is easily seen that any line segment in $\mathcal{S}_1$ must have its two endpoints on $\partial\mathbf{D}$, and they may lie on a single component polygon of $\mathbf{D}$, or two different component polygons of $\mathbf{D}$. Whereas for any $l \in \mathcal{S}_2$, it is either a straight line in $\mathbf{G}$ or an infinite half-line in $\mathbf{G}$ having one endpoint on $\partial\mathbf{D}$. Furthermore, from Lemma 3.3, we see that it must have that $\partial u_\nu(l) = 0$ if $l \in \mathcal{S}_2$, and $\mathcal{S}_2$ cannot have two different segments which are not parallel to each other.

Starting from now on, we shall use the notation L_j, with j being an integer, to represent the straight line in $\mathbb{R}^2$ containing some line segment $l_j \subset \mathcal{S}_1$ and denote by R_j the reflection in $\mathbb{R}^2$ with respect to L_j.

For our convenience, we shall reclassify the finite line segments in $\mathcal{S}_1$ as below. For any line segment $l_0 \in \mathcal{S}_1$, we fix a point $x_0 \in l_0$. Since $x_0 \in \mathbf{G}$, we can take a sufficiently small ball $B_{r_0}(x_0)$ with $r_0 > 0$ such that $\bar{B}_{r_0}(x_0) \subset \mathbf{G}$. Then, we take a point $A \in S_{r_0}(x_0)\backslash l_0$ and let B be the symmetric point to A with respect to L_0. Now, let $\mathbf{G}_0^+$ be the connected component of $\mathbf{G}\backslash l_0$ containing A and $\mathbf{G}_0^-$ be the connected component of $\mathbf{G}\backslash l_0$ containing B. It is remarked that it may happen that $\mathbf{G}_0^+ = \mathbf{G}_0^-$. Next, let E_0^+ be the connected component of $\mathbf{G}_0^+ \cap R_0(\mathbf{G}_0^-)$ containing A and E_0^- be the connected component of $\mathbf{G}_0^- \cap R_0(\mathbf{G}_0^+)$ containing B. We then introduce a symmetric set with respect to L_0:

$$E_0 = E_0^+ \cup l_0 \cup E_0^-. \quad (10)$$

We know that E_0 must be a connected set with its boundary composed of some line segments on $\partial\mathbf{D}$ and $R_0(\partial\mathbf{D})$ and $\bar{B}_{r_0}(x_0) \subset E_0$. Here, it is noted that the set E_0 is independent of the choice of r_0. Then we introduce two subsets of $\mathcal{S}_1$:

$$\mathcal{G}_1 := \{l_0 \in \mathcal{S}_1; \text{the connected set defined in (10) associated with } l_0 \text{ is bounded}\},$$

$$\mathcal{G}_2 := \{l_0 \in \mathcal{S}_1; \text{the connected set defined in (10) associated with } l_0 \text{ is unbounded}\}.$$

Since ∂E_0, $\partial\mathbf{G}_0^\pm$ and $R_0(\mathbf{G}_0^\pm)$ are all bounded by our construction, we see that if $l_0 \in \mathcal{G}_2$, then E_0 would contain $\mathbb{R}^2\backslash B_r(x_0)$ with $x_0 \in l_0$ and $r > 0$ being sufficiently large.

Next we shall present some crucial properties of the special line segments introduced above, which form the key lemmata in proving Theorem 2.2. But since the proofs for most of those lemmata involves rather lengthy and technical arguments, we would leave them for the subsequent Section 4 and focus ourselves on the proof of Theorem 2.2 in the current section.

In the following, two line segments $l_j, l_{j'} \subset \mathcal{G}_2$ are called the "same" if $L_j = L_{j'}$, otherwise, they are "different". Now, we have

Lemma 3.5. *There exists no line segment $l_0 \in \mathcal{G}_2$ such that $u = 0$ on l_0. Furthermore, the set $\mathcal{G}_2$ do not contain two "different" line segments.*

Lemma 3.6. *For each line segment $l \in \mathcal{G}_2$ such that $u(l) = 0$ (resp. $\partial_\nu u(l) = 0$), there corresponds an infinite half-line $\widetilde{L}$ which is collinear to l such that $u(\widetilde{L}) = 0$ (resp. $\partial_\nu u(\widetilde{L}) = 0$).*

Lemma 3.7. *There cannot exist two infinite half-lines $l_1, l_2 \subset \mathbf{G}$ which are not collinear such that $\partial_\nu u = 0$ on $l_1 \cup l_2$, disregarding whether or not they are parallel to each other. Moreover, there cannot exist two non-collinear line segments l_1, l_2 such that $l_1 \in \mathcal{G}_2$ while $l_2 \in \mathcal{S}_2$.*

Here we would like to make some important observations. Let $l_0, l_0' \subset \mathbf{G}$ such that $\partial_\nu u = 0$ on $l_0 \cup l_0'$. Then by Lemma 3.7, we know that either l_0 and l_0' are collinear; or $l_0, l_0' \in \mathcal{G}_1$; or $l_0 \in \mathcal{G}_1$ while $l_0' \in \mathcal{S}_2$. For the third case, we have

Lemma 3.8. *There exist no line segments $l_0 \in \mathcal{S}_1$ and $l_0' \in \mathcal{S}_2$ such that l_0 intersects l_0' at a unique point $P \in \mathbf{G}$.*

We proceed to present a lemma concerning the relationship among line segments in the set $\mathcal{S}_1$.

Lemma 3.9. *For each line segment $l_0 \in \mathcal{G}_1$, there corresponds a line segment $l_0' \in \mathcal{G}_2$.*

The following is a lemma on the finiteness of the set $\mathcal{S}_1$.

Lemma 3.10. *The set $\mathcal{S}_1 = \mathcal{G}_1 \cup \mathcal{G}_2$ contains at most finitely many segments.*

Finally, for our uniqueness argument, we need to make another classification of the line segments in $\mathcal{S}_1$ and introduce the following two subsets:

$$\mathcal{S}_{11} := \{l; l \in \mathcal{S}_1 \text{ and the two endpoints of } l \text{ lie on a single component polygon of } \mathbf{D}\}, \tag{11}$$

$$\mathcal{S}_{12} := \{l; l \in \mathcal{S}_1 \text{ and the two endpoints of } l \text{ lie on two different component polygons of } \mathbf{D}\}. \tag{12}$$

Apparently,

$$\mathcal{S}_1 = \mathcal{S}_{11} \cup \mathcal{S}_{12}.$$

For $\mathcal{S}_{11}$ and $\mathcal{S}_{12}$, we have

Lemma 3.11. *If $l \in \mathcal{S}_{11}$, then $\mathbf{G}$ is divided by l into two (open) connected components, where one is bounded and the other is unbounded. Whereas if $l \in \mathcal{S}_{12}$, then $\mathbf{G}\backslash l$ is connected; that is, $\mathbf{G}\backslash l$ has only one (open) connected component which is $\mathbf{G}\backslash l$ itself.*

We are in a position to the state the proof of Theorem 2.2.

Proof of Theorem 2.2. We first show that $\mathfrak{B} = \widetilde{\mathfrak{B}}$ if $\mathbf{D} = \widetilde{\mathbf{D}}$. Set $\mathbf{D}_0 = \mathbf{D} = \widetilde{\mathbf{D}}$ and assume contrarily that $\mathfrak{B} \neq \widetilde{\mathfrak{B}}$. Let $u_\infty(\hat{x}; \mathbf{D}, d_0, k_0) = u_\infty(\hat{x}; \widetilde{\mathbf{D}}, d_0, k_0)$ for $\hat{x} \in \mathbb{S}^1$ with d_0 and k_0 fixed, and we have by Rellich's theorem (see, Lemma 2.11 in [3]) that $u(\mathbf{D}) = u(\widetilde{\mathbf{D}}) := u(\mathbf{D}_0)$. Noting $\mathfrak{B} \neq \widetilde{\mathfrak{B}}$, there must exist an edge of $\mathbf{D}_0$ such that $u(\mathbf{D})$ and $u(\widetilde{\mathbf{D}})$ assume different boundary conditions; that is $u(\mathbf{D}_0)$ satisfies both homogeneous Dirichlet and Neumann boundary conditions on this edge. In view of the Holmgren's theorem (see Theorem 6.12 in [4]), we immediately get that $u(\mathbf{D}_0) = 0$ in $\mathbb{R}^2\backslash\mathbf{D}_0$, which is certainly not true.

To demonstrate Theorem 2.2, it remains to show the major part, i.e., $\mathbf{D} = \widetilde{\mathbf{D}}$. We will do this by contradiction and divide the proof into the following two steps:

Step 1: Non-empty of the set $\mathcal{G}_1$

Assume that $u_\infty(\hat{x}; \mathbf{D}, d_0, k_0) = u_\infty(\hat{x}; \widetilde{\mathbf{D}}, d_0, k_0)$ for $\hat{x} \in \mathbb{S}^1$ with d_0 and k_0 fixed but $\mathbf{D} \neq \widetilde{\mathbf{D}}$. We shall write respectively $u = u(\mathbf{D})$ and $\tilde{u} = u(\widetilde{\mathbf{D}})$ to represent the total fields corresponding to $\mathbf{D}$ and $\widetilde{\mathbf{D}}$. Furthermore, let Ω be the (unique) unbounded component of $\mathbb{R}^2\backslash(\mathbf{D}\cup\widetilde{\mathbf{D}})$. We have by Rellich's theorem (see, Lemma 2.11 in [3]) that $u=\tilde{u}$ in Ω. Moreover, using a standard argument, e.g., see the first part of the proof of Theorem 1 in [10], we can assume without loss of generality that there is an open segment PQ such that

$$PQ \subset \partial\Omega \cap (\mathbb{R}^2\backslash\mathbf{D}) = \partial\Omega \cap \mathbf{G}. \tag{13}$$

Since $\partial\Omega$ consists of segments of $\partial\mathbf{D}$ and $\partial\widetilde{\mathbf{D}}$ and $u=\tilde{u}$ in Ω, we have $u(PQ)=\tilde{u}(PQ)=0$ if PQ lies on a sound-soft component of $\widetilde{\mathbf{D}}$, and $\partial_\nu u(PQ)=\partial_\nu\tilde{u}(PQ)=0$ if PQ lies on a sound-hard component of $\widetilde{\mathbf{D}}$.

By Lemma 3.2, we know that $u=0$ (respectively $\partial_\nu u=0$) on the maximum extension of PQ in $\mathbf{G}$. Henceforth, the notations $\mathcal{S}_1,\mathcal{S}_2,\mathcal{G}_1,\mathcal{G}_2$, etc. are naturally introduced in the earlier part of this section. Now, we turn to the key point of this step to show

Lemma 3.12. *The set $\mathcal{G}_1$ is not empty.*

Proof. We start with the line segment PQ from (13). If $u=0$ on PQ, we know by Lemmata 3.3 and 3.5 that PQ can not be extended to an infinite half-line in $\mathbf{G}$, nor can be extended to a line segment in $\mathcal{G}_2$, therefore the maximum extension of PQ belongs to $\mathcal{G}_1$, and the lemma is proved. Without loss of generality, we may now assume that $\partial_\nu u=0$ on PQ, i.e., PQ lies on a sound-hard component polygon of $\widetilde{\mathbf{D}}$. Denoting by l_{PQ} the maximum extension of PQ in $\mathbf{G}$, we know $\partial_\nu u=0$ on l_{PQ} by Lemma 3.2. If l_{PQ} is a finite line segment having its two endpoints on a single component of $\mathbf{D}$, i.e., $l_{PQ}\in\mathcal{S}_{11}$, thus $l_{PQ}\in\mathcal{G}_1$ by Lemma 3.11, and so $\mathcal{G}_1$ is not empty. Hence we assume below that $l_{PQ}\notin\mathcal{S}_{11}$, that is, either $l_{PQ}\in\mathcal{S}_{12}$ or $l_{PQ}\in\mathcal{S}_2$. We next consider these two cases.

Case I: $l_{PQ}\in\mathcal{S}_2$. By definition, l_{PQ} is an infinite half-line segment in $\mathbf{G}$. Noting $PQ\subset\partial\Omega\cap\mathbf{G}\subset\partial\widetilde{\mathbf{D}}\backslash\mathbf{D}$, the edge of $\partial\Omega$ containing PQ has to be separated from l_{PQ} at a vertex V of $\partial\Omega$, lying in $\mathbf{G}$. Hence there exists a point $V'\in\mathbf{G}$ such that $V'V\subset\partial\Omega\cap\mathbf{G}$ and VV' is not parallel to PQ. Furthermore, by recalling the fact that $u=\tilde{u}$ in Ω, we may assume that $\partial_\nu u=\partial_\nu\tilde{u}=0$ on $V'V$ (otherwise we must have $u(V'V)=\tilde{u}(V'V)=0$, this implies that the extension of $V'V$ is in $\mathcal{G}_1$ by Lemma 3.5). Then by Lemma 3.3, we see that $V'V$ cannot be extended to infinity in $\mathbf{G}$ by noting that l_{PQ} is another infinite half-line in $\mathbf{G}$ on which $\partial_\nu u=0$. On the other hand, the maximum extension of $V'V$ cannot belong to $\mathcal{G}_2$; otherwise by Lemma 3.6, we would get an infinite half-line $\widetilde{L}_{VV'}$ such that $\partial_\nu u=0$ on $\widetilde{L}_{VV'}$. Obviously, this infinite half-line is not collinear to PQ, contradicting Lemma 3.7. Therefore, if we denote the maximum extension of $V'V$ in $\mathbf{G}$ by $l_{VV'}$, then we must have $l_{VV'}\in\mathcal{G}_1$, thus $\mathcal{G}_1$ is not empty. But we remark that in this case, one has $l_{PQ}\in\mathcal{S}_2$,$l_{VV'}\in\mathcal{G}_1$ and $l_{PQ}\cap l_{VV'}=V\in\mathbf{G}$, so immediately we come to a contraction to Lemma 3.8.

Case II: $l_{PQ}\in\mathcal{S}_{12}$. By definition, l_{PQ} has its two endpoints on two different component polygons of $\mathbf{D}$. Clearly, PQ lies on one edge of $\widetilde{\mathbf{D}}$, which we denote by ϵ_{PQ}. Let e_{PQ} be the connected component of $\epsilon_{PQ}\backslash\mathbf{D}$ containing PQ, we know $PQ\subset e_{PQ}\subset l_{PQ}$. Recalling that $PQ\subset\partial\Omega$, we next show that $e_{PQ}\subset\partial\Omega$. Firstly, we take an arbitrary point $x'\in\Omega$ which stays sufficiently close to PQ and lies on one side of l_{PQ}. By Lemma 3.11, we know that $\mathbf{G}\backslash l_{PQ}$ is connected, and thus for any point $x\in e_{PQ}$, there is a curve γ_1 which connects x and x' and lies entirely in $\mathbf{G}\backslash l_{PQ}$. Moreover, noting the component polygons of $\widetilde{\mathbf{D}}$ are compact and disjoint with each other, we can require that γ_1 is sufficiently close to e_{PQ} and avoids intersection with any other component polygon of $\widetilde{\mathbf{D}}$. Then, noting Ω is connected and unbounded, let γ_2 be a curve which connects x' to infinity and lies entirely in Ω. Let $\gamma = \gamma_1 \cup \gamma_2$, then γ connects x to infinity without any intersection with $\mathbf{D}$ and $\widetilde{\mathbf{D}}$. Since both $\mathbf{D}$ and $\widetilde{\mathbf{D}}$ are bounded, we know the unbounded connected component Ω of $\mathbb{R}^2\backslash(\mathbf{D}\cup\widetilde{\mathbf{D}})$ is unique. Now, we conclude that we must have $\gamma \subset \Omega$. In fact, noting $\gamma \subset \mathbb{R}^2\backslash(\mathbf{D}\cup\widetilde{\mathbf{D}})$ through our construction, if $\gamma \not\subset \Omega$, then there must exist a bounded component Ω' of $\mathbb{R}^2\backslash(\mathbf{D}\cup\widetilde{\mathbf{D}})$ such that $\gamma\cap\Omega' \neq \emptyset$. By Lemma 3.1, it is easy to deduce that $\gamma \cap \partial\Omega' \neq \emptyset$, that is, $\gamma \cap (\partial\mathbf{D} \cup \partial\widetilde{\mathbf{D}}) \neq \emptyset$, contradicting our construction. So, $\gamma \subset \Omega$, which implies that $e_{PQ} \subset \partial\Omega$. Hence without loss of generality, we may assume that $PQ = e_{PQ}$. We next still need to distinguish between two cases.

II(i) If either P or Q is not on $\partial\mathbf{D}$, then, $P \in \mathbf{G}$ or $Q \in \mathbf{G}$. Let us assume that $Q \in \mathbf{G}$, and clearly, Q is the vertex of a component polygon of $\widetilde{\mathbf{D}}$, which we denote by $\widetilde{D}_1$. We may assume that $\widetilde{D}_1$ is a sound-hard polygon, otherwise $\mathcal{G}_1$ is not empty by the first part of the proof. Another edge of $\widetilde{D}_1$ other than ϵ_{PQ}, having Q as the endpoint, is denoted by $\epsilon_{P'Q}$, see Fig. 1.

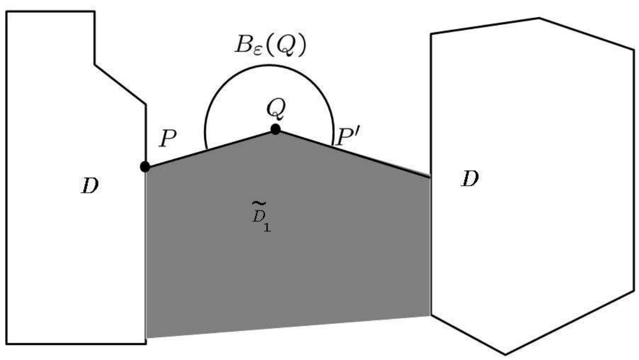

FIGURE 1. Illustration of the proof of Lemma 3.12

Noting the polygons forming $\widetilde{\mathbf{D}}$ are disjoint and compact, we can choose a sufficiently small ball $B_\varepsilon(Q)$ such that $B_\varepsilon(Q) \subset \mathbf{G}$ and $B_\varepsilon(Q) \cap (\widetilde{\mathbf{D}}\backslash\widetilde{D}_1) = \emptyset$ (see Fig. 1). Clearly, $B_\varepsilon(Q)\backslash\widetilde{D}_1 \subset \mathbb{R}^2\backslash(\mathbf{D}\cup\widetilde{\mathbf{D}})$ and $B_\varepsilon(Q)\cap\Omega \neq \emptyset$, so it is easy to see that $B_\varepsilon(Q)\backslash\widetilde{D}_1 \subset \Omega$. Letting $P'Q = \epsilon_{P'Q} \cap B_\varepsilon(Q)$, then we know $P'Q \subset \partial\Omega$, and $\partial_\nu u = \partial_\nu \tilde{u} = 0$ on $l_{P'Q}$, where $l_{P'Q}$ is the maximum extension of $P'Q$ in $\mathbf{G}$. Now, if $l_{P'Q}$ is infinite, i.e., $l_{P'Q} \in \mathcal{S}_2$, we can easily deduce by Lemmata 3.6 and 3.7 that $l_{PQ} \in \mathcal{G}_1$ by noting l_{PQ} and $l_{P'Q}$ are not parallel. If $l_{P'Q}$ is finite, we have either $l_{P'Q} \in \mathcal{S}_{11}$ or $l_{P'Q} \in \mathcal{S}_{12}$. For $l_{P'Q} \in \mathcal{S}_{11}$, we know $l_{P'Q} \in \mathcal{G}_1$ by Lemma 3.11; and if $l_{P'Q} \in \mathcal{S}_{12}$, then we have two non-parallel finite line segments l_{PQ} and $l_{PQ'}$, both belongs to $\mathcal{S}_{12}$. By Lemma 3.5, we know that at least one of l_{PQ} and $l_{P'Q}$ belongs to $\mathcal{G}_1$.

II(ii) Both P and Q are on $\partial\mathbf{D}$, i.e., $PQ = l_{PQ}$. If $PQ \in \mathcal{G}_1$, then we are done. We now assume that $PQ \notin \mathcal{G}_1$, i.e., $PQ \in \mathcal{G}_2$. In the following, we will call such line segments as PQ "non-extendable". That is, an open line segment $l \subset \partial\widetilde{\mathbf{D}}\backslash\mathbf{D}$ is "non-extendable" iff l is an open connected component of $\epsilon_l\backslash\mathbf{D}$, where ϵ_l is the edge of $\widetilde{\mathbf{D}}$ containing l, and the endpoints of l are on two different component polygons of $\mathbf{D}$; moreover, the connected set defined in (10) corresponding to l is unbounded. Let $\mathcal{X}$ be the set of all "non-extendable" line segments. Clearly, $\partial\widetilde{\mathbf{D}}\backslash\mathbf{D}$ is composed of finitely many open line segments and points, and we denote the set of those open line segments by $\mathcal{H}$. We refer to those open line segments in $\mathcal{Y} := \mathcal{H}\backslash\mathcal{X}$ as "extendable". Moreover, we define

$$\mathcal{X}_0 := \{l \in \mathcal{X};\ l \text{ is collinear to } PQ\}. \tag{14}$$

Next, we show a key observation that $\mathbf{G}\backslash\mathcal{X}_0$ is connected. If $\mathcal{X}_0$ contains only one line segment, i.e., PQ, then by Lemma 3.11, we obviously have that $\mathbf{G}\backslash\mathcal{X}_0 = \mathbf{G}\backslash PQ$ is connected. So, without loss of generality we may assume that $\mathcal{X}_0$ contains another line segment different from PQ. If $\mathbf{G}\backslash\mathcal{X}_0$ is not connected, then it must have bounded connected component. In fact, noting both $\mathcal{X}_0$ and $\mathbf{D}$ are bounded, the unbounded component of $\mathbf{G}\backslash\mathcal{X}_0$ is unique. Let E be one of the bounded connected component of $\mathbf{G}\backslash\mathcal{X}_0$ (see, e.g, Fig. 2). Clearly, ∂E must contain at least one line segment from $\mathcal{X}_0$, which we denote by l_1. Now, it is straightforward to check that the connected set defined in (10) corresponding to l_1 is contained in $R_1(E) \cup E$, where R_1 is the reflection with respect to L_1 with L_1 being the straight line in $\mathbb{R}^2$ containing l_1. Since E is bounded, this contradicts with our definition that l_1 is "non-extendable", which implies that $\mathbf{G}\backslash\mathcal{X}_0$ is connected.

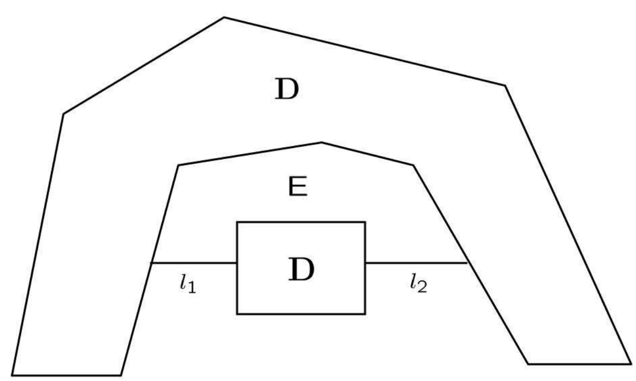

FIGURE 2. Illustration of the proof of Lemma 3.12

Noting $\widetilde{\mathbf{D}}$ is composed of solid polygons together with the fact that $\mathbf{G}\backslash\mathcal{X}_0$ is connected, we easily see that $\mathcal{H}\backslash\mathcal{X}_0 \neq \emptyset$. Again noting $\mathbf{G}\backslash\mathcal{X}_0$ is connected, there must be some line segment $l_0 \in \mathcal{H}\backslash\mathcal{X}_0$ which lies on the unbounded component of $\mathbf{G}\backslash(\mathbf{D}\cup\widetilde{\mathbf{D}})$, i.e., $\partial\Omega$. In fact, if there are no line segments from $\mathcal{H}\backslash\mathcal{X}_0$ which lies on $\partial\Omega$, then $\partial\Omega \subset \partial\mathbf{D}\cup\mathcal{X}_0$. We next show that $\partial\Omega = \partial\mathbf{D}\cup\mathcal{X}_0$, which then implies $\Omega = \mathbf{G}\backslash\mathcal{X}_0$. Firstly, since $\mathbf{D}$ and $\widetilde{\mathbf{D}}$ are bounded, we know the unbounded component of $\mathbf{R}^2\backslash(\mathbf{D}\cup\widetilde{\mathbf{D}})$, i.e., Ω is unique. Moreover, by noting $\partial\mathbf{D}$ and $\mathcal{X}_0$ are bounded, we know that both the unbounded connected sets Ω and $\mathbf{G}\backslash\mathcal{X}_0$ contain $\mathbb{R}^2\backslash B_{r_1}(0)$ with sufficiently large $r_1 > 0$. Next, for any $x \in \partial\mathbf{D}\cup\mathcal{X}_0$, since $\mathbf{G}\backslash\mathcal{X}_0$

is an unbounded connected set, there exists a curve γ_x which connects x to infinity and lies entirely in $\mathbf{G}\backslash\mathcal{X}_0$; that is, it avoids intersection with $\partial\mathbf{D}\cup\mathcal{X}_0$. Now, we conclude that $\gamma_x\subset\Omega$. In fact, if $\gamma_x\not\subset\Omega$, then $(\mathbb{R}^2\backslash\Omega)\cap\gamma_x\neq\emptyset$. But it is clear that γ_x intersect Ω outside of $B_{r_1}(0)$, i.e, $\Omega\cap\gamma_x\neq\emptyset$. Then, by Lemma 3.1 with $\mathcal{A}=\Omega$ and $\mathcal{B}=\gamma_x$, we easily get that $\partial\Omega\cap\gamma_x\neq\emptyset$. Noting $\partial\Omega\subset\partial\mathbf{D}\cup\mathcal{X}_0$, this certainly contradicts with our construction of γ_x. Hence, γ_x lies entirely in Ω, which implies that $x\in\partial\Omega$. That is, $\partial\mathbf{D}\cup\mathcal{X}_0\subset\partial\Omega$, thus we have shown that $\partial\Omega=\partial\mathbf{D}\cup\mathcal{X}_0$, which implies $\Omega=\mathbb{R}^2\backslash(\mathbf{D}\cup\mathcal{X}_0)=\mathbf{G}\backslash\mathcal{X}_0$. However, noting $\mathcal{H}\backslash\mathcal{X}_0\neq\emptyset$ and $\mathcal{H}\backslash\mathcal{X}_0\subset\mathbf{G}\backslash\mathcal{X}_0=\Omega$, there must be some line segment from $\mathcal{H}\backslash\mathcal{X}_0$ which lies on $\partial\Omega$, leading to a contradiction. Hence, there must be such $l_0\in\mathcal{H}\backslash\mathcal{X}_0$ which lies on $\partial\Omega$.

Depending on l_0 lying on a sound-soft or a sound-hard component polygon of $\widetilde{\mathbf{D}}$, we know have $u(l_0)=\tilde{u}(l_0)=0$ or $\partial_\nu u(l_0)=\partial_\nu\tilde{u}(l_0)=0$. If $u=0$ on l_0, as we did at the beginning of the proof, we conclude immediately that the maximum extension of l_0 belongs to $\mathcal{G}_1$. So we may assume that $\partial_\nu u=0$ on l_0. We first consider the case that l_0 is collinear to PQ. Noting $l_0\not\subseteq\mathcal{X}_0$, we know either l_0 can be extended to be an infinite half-line in $\mathbf{G}$, and this will lead us to ***Case I*** above; or the maximum extension of l_0 is finite which either is a line segment in $\mathcal{G}_1$ or, by our construction, a line segment of ***Case II(i)*** above where we can find another line segment belonging to $\mathcal{G}_1$. If l_0 is not collinear to PQ, then by Lemma 3.7, l_0 can not extend to infinity in $\mathbf{G}$, neither can the maximum extension of l_0 in $\mathbf{G}$ belong to $\mathcal{G}_2$. Hence, we have $l_0\in\mathcal{G}_1$, thus Lemma 3.12 is proved. □

By Lemma 3.12, $\mathcal{G}_1\neq\emptyset$, hence by Lemma 3.9, $\mathcal{G}_2\neq\emptyset$. Then by Lemma 3.10, we know that both $\mathcal{G}_1$ and $\mathcal{G}_2$ contain finitely many line segments. By Lemma 3.6 and Lemma 3.3, we may assume that $\partial_\nu u(l)=0$ for each $l\in\mathcal{G}_2$. Furthermore, by Lemma 3.5, the line segments in $\mathcal{G}_2$ are collinear to each other; that is, they lie on a same straight line. Following the proof of the connectedness of $\mathbf{G}\backslash\mathcal{X}_0$ in Lemma 3.12, where $\mathcal{X}_0$ is defined in (14), we still see that $\mathbf{G}\backslash\mathcal{G}_2$ is connected.

Step 2: Two "different" line segments in $\mathcal{G}_2$

The goal of this step is to find two line segments in $\mathcal{S}_2$, which are not parallel to each other, contradicting Lemma 3.3, thus completes the proof of $\mathbf{D}=\widetilde{\mathbf{D}}$.

We can write for some m' satisfying $1\leq m'<m$ that

$$\mathcal{S}_1=\mathcal{G}_1\cup\mathcal{G}_2 \tag{15}$$

with

$$\mathcal{G}_1=\bigcup_{j=1}^{m'}l_j,\qquad \mathcal{G}_2=\bigcup_{j=m'+1}^{m}l_j. \tag{16}$$

From the discussions in **Step 1**, we know both $\mathcal{G}_1$ and $\mathcal{G}_2$ are not empty, the line segments in $\mathcal{G}_2$ are collinear to each other, and the set $\mathbf{G}\backslash\mathcal{G}_2$ is connected.

Let $\mathbf{\Lambda}$ be the unbounded connected component of $\mathbf{G}\backslash\mathcal{S}_1=\mathbf{G}\backslash(\mathcal{G}_1\cup\mathcal{G}_2)$. Since both $\mathbf{D}$ and $\mathcal{S}_1$ are bounded, we know that $\mathbf{\Lambda}$ is unique. Obviously,

$$\mathbf{\Lambda}\cap\mathcal{S}_1=\emptyset. \tag{17}$$

Noting $\mathbf{G}\backslash\mathcal{G}_2$ is connected, there must exist a point $P\in\partial\mathbf{\Lambda}$ lying on a segment l of $\mathcal{G}_1$. Let E be the connected set defined in (10) corresponding to l. Since $l\in\mathcal{G}_1$, we know that E is bounded. Next, we show that

$$\partial E\cap\mathbf{\Lambda}\neq\emptyset. \tag{18}$$

In fact, we first see that $B_\varepsilon(P) \subset E$ for sufficiently small $\varepsilon > 0$. Then, because $P \in \partial\mathbf{\Lambda}$, we know that $B_\varepsilon(P) \cap \mathbf{\Lambda} \neq \emptyset$. That is, $E \cap \mathbf{\Lambda} \neq \emptyset$. Furthermore, it is obvious that $(\mathbb{R}^2 \backslash E) \cap \mathbf{\Lambda} \neq \emptyset$. Hence, by Lemma 3.1, we immediately get (18).

Now, let L be the straight line in $\mathbb{R}^2$ containing l, by the reflection principle in Lemma 3.4, u is either even symmetric with respect to L in E if $\partial_\nu u(l) = 0$ or odd symmetric with respect to L in E if $u(l) = 0$. Since ∂E is composed of some points and open line segments lying on $\partial\mathbf{D}$ and $\partial R(\mathbf{D})$, where R is the reflection with respect to L, we see that there must exist an open line segment $\tilde{l} \subset \mathbf{\Lambda} \cap \partial E$ such that $u(\tilde{l}) = 0$ or $\partial_\nu u(\tilde{l}) = 0$. Now, by Lemma 3.2 and (17), it is seen that $\tilde{l}$ can be extended to an infinite line segment $\tilde{L} \in \mathcal{S}_2$. In fact, assume contrarily that the maximum extension of $\tilde{L}$ in $\mathbf{G}$ belongs to $\mathcal{S}_1$, i.e., $\tilde{L} \in \mathcal{S}_1$, then $\tilde{l} \subset \mathbf{\Lambda} \cap \tilde{L} \subset \mathbf{\Lambda} \cap \mathcal{S}_1$, which contradicts (17).

If $u = 0$ on l, then we readily get a contradiction to Lemma 3.3, which concludes Theorem 2.2. Now we assume that $\partial_\nu u = 0$ on $\tilde{L}$. Next, we show another crucial fact that $\tilde{L} \subset \mathbf{\Lambda}$. Indeed, if $\tilde{L} \not\subset \mathbf{\Lambda}$, then apparently, there must exist some intersection points of $\tilde{L}$ with $\mathcal{S}_1$, which readily yields a contradiction to Lemma 3.8.

Now, since ∂E forms the boundary of a bounded polygonal domain and $\tilde{L}$ lies entirely in $\mathbf{\Lambda}$, there exists a point $V_0 \in \mathbf{\Lambda}$, which is a vertex of the polygonal domain, and an open line segment $\tilde{l}' \subset \mathbf{\Lambda} \cap \partial E$ starting at V_0, which is not parallel to $\tilde{L}$. Again by Lemma 3.2 and (17), the maximum extension of $\tilde{l}'$ in $\mathbf{G}$ belongs to $\mathcal{S}_2$. Clearly, we have obtained a contradiction to Lemma 3.3.

The proof of Theorem 2.2 is completed. □

4. **Proofs of Lemmata 3.5–3.11.**

Proof of Lemma 3.5. Assume contrarily that there exists $l_0 \in \mathcal{G}_2$ such that $u(l_0) = 0$. Let E_0 be the connected set constructed in (10) corresponding to l_0. We know from the earlier discussion that E_0 contains $\mathbb{R}^2 \backslash B_r(x_0)$ with any fixed $x_0 \in l_0$ and $r > 0$ sufficiently large. Since $u(l_0) = 0$, by the reflection principle in Lemma 3.4, we know $u(x) = -R_0 u(x)$ in E_0; namely, $u(x)$ is odd symmetric with respect to L_0 in E_0. So, let $\widetilde{L}_0$ be one of the two infinite portions of L_0 outside of $B_r(x_0)$ (see Fig. 3), then $u(x) = 0$ on $\widetilde{L}_0$, which contradicts Lemma 3.3. This proves the first statement of the lemma.

Next, we again use the contradiction argument to prove the second statement. Assume contrarily that $l_1, l_2 \in \mathcal{G}_2$ are two "different" line segments. Obviously, from the first result of the lemma and the definition of $\mathcal{G}_2$, we know that $\partial_\nu u = 0$ on $l_1 \cup l_2$. Let E_j $(j = 1, 2)$ be the unbounded connected sets constructed in (10) corresponding to l_j $(j = 1, 2)$. Moreover, we let $B_r(x_j)$ $(j = 1, 2)$ with arbitrarily fixed $x_j \in l_j$ $(j = 1, 2)$ and sufficiently large $r > 0$, be the ball such that $\mathbb{R}^2 \backslash B_r(x_j) \subset E_j$ $(j = 1, 2)$. Since $\partial_\nu u(l_1) = \partial_\nu u(l_2) = 0$, it is seen by the reflection principle in Lemma 3.4 that $u(x)$ is even symmetric with respect to L_j $(j = 1, 2)$ in E_j. Hence we have $\partial_\nu u(x) = 0$ on $\widetilde{L}_1$ and $\widetilde{L}_2$, where $\widetilde{L}_1$ and $\widetilde{L}_2$ are respectively one of the infinite portions of $L_1 \backslash B_r(x_1)$ and $L_2 \backslash B_r(x_2)$ (see Fig. 3 for a similar illustration). Since $L_1 \neq L_2$, it follows that $\widetilde{L}_1 \nparallel \widetilde{L}_2$ or $\widetilde{L}_1 \parallel \widetilde{L}_2$. But if $\widetilde{L}_1 \nparallel \widetilde{L}_2$, we immediately get a contradiction to Lemma 3.3. So we next assume $\widetilde{L}_1 \parallel \widetilde{L}_2$.

Let $\bar{B}_r(x_j) \subset T_{r'}(0)$ $(j = 1, 2)$ with sufficiently large $r' > r$ (see Fig. 4). Moreover, we choose $T_{r'}(0)$ in a way such that both $\widetilde{L}_1$ and $\widetilde{L}_2$ are perpendicular to one of the edges of $T_{r'}(0)$ (see Fig. 4). For convenience, we will still denote by $\widetilde{L}_1$ and $\widetilde{L}_2$,

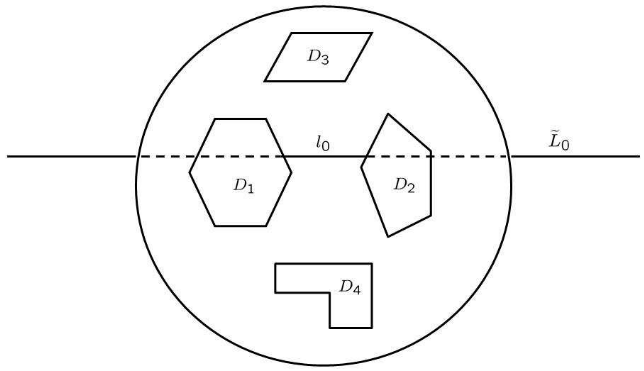

FIGURE 3. Illustration of the proof of Lemma 3.5

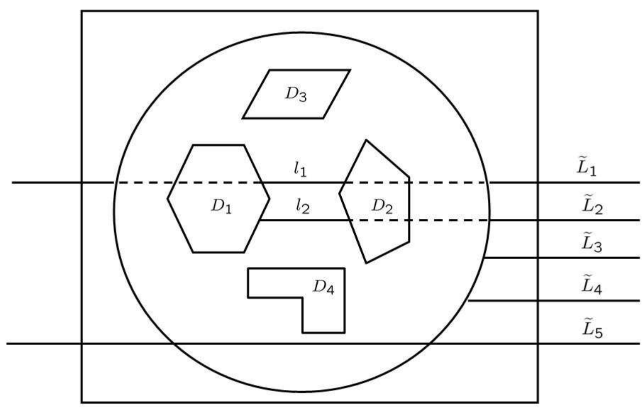

FIGURE 4. Illustration of the proof of Lemma 3.5

respectively, the infinite half lines $\widetilde{L}_1\backslash T_{r'}(0)$ and $\widetilde{L}_2\backslash T_{r'}(0)$. Let $\mathbf{G}' = \mathbb{R}^2\backslash T_{r'}(0)$. Since $\partial_\nu u = 0$ on $\widetilde{L}_2$, using a similar reflection argument as above, we see that u is even symmetric with respect to L_2 in $(R_2(\mathbf{G}'\backslash\widetilde{L}_2) \cap \mathbf{G}') \cup \widetilde{L}_2$. It is apparent that $\widetilde{L}_1, R_2(\widetilde{L}_1) \subset R_2(\mathbf{G}'\backslash\widetilde{L}_2) \cap \mathbf{G}'$ and hence by noting $\partial_\nu u = 0$ on $\widetilde{L}_1$, we have $\partial_\nu u = 0$ on $R_2(\widetilde{L}_1)$ by Lemma 3.4. Set $\widetilde{L}_3 = R_2(\widetilde{L}_1)$ and let L_3 be the straight line in $\mathbb{R}^2$ containing $\widetilde{L}_3$. If the infinite half-line $\widetilde{L}_3$ has its endpoint on $\partial T_{r'}(0)$, we repeat the above argument with respect to L_3 to find $\partial_\nu u = 0$ on $\widetilde{L}_4 := R_3(\widetilde{L}_2)$. Continuing with this procedure, we can get a family of parallel infinite half-lines, $\widetilde{L}_j$, $j = 1, 2, \ldots$, such that $\partial_\nu u = 0$ on $\widetilde{L}_j$. Since the distance between each pair of $\widetilde{L}_j$ and $\widetilde{L}_{j+1}$, $j = 1, 2, \ldots$ is $\mathbf{d}(\widetilde{L}_1, \widetilde{L}_2) > 0$ being fixed, we see that there is a $\widetilde{L}_M$, $M \in \mathbb{N}$, such that L_M, the straight line in $\mathbb{R}^2$ containing $\widetilde{L}_M$, lies entirely in $\mathbf{G}'$. That is, $T_{r'}(0)$ lies entirely on one side of L_M, and the same holds for $\mathbf{D}$. Obviously, by Lemma 3.2, we have $\partial_\nu u = 0$ on L_M. Let D_1 be one sound-hard component polygon of $\mathbf{D}$ and hence $\partial_\nu u = 0$ on $l_{D_1} \cup l_{D_1'}$, where l_{D_1} and l'_{D_1} are two edges of D_1 possessing a common vertex. Then, again using the reflection argument as above, it is easily seen that u is even symmetric with respect to L_M in $\mathbb{R}^2 \backslash (\mathbf{D} \cup R_M(\mathbf{D}))$, hence $\partial_\nu u = 0$ on $R_M(l_{D_1}) \cup R_M(l'_{D_1})$. Noting $R_M(l_{D_1})$ and $R_M(l'_{D_1})$ lying on the other side of L_M, both can be extended to infinite half-lines in G. Therefore, we have established a contradiction to Lemma 3.3, which completes the proof. □

Proof of Lemma 3.6. This can be readily seen from the proof of Lemma 3.5. □

Proof of Lemma 3.7. Combining the proof of Lemma 3.5 and the results in Lemmata 3.3 and 3.6 gives the proof. □

Proof of Lemma 3.8. Assume contrarily that there exist $l_0 \in \mathcal{S}_1$ and $l'_0 \in \mathcal{S}_2$ such that $l_0 \cap l'_0 = P \in \mathbf{G}$. Clearly, we know $\partial_\nu u = 0$ on l'_0 by Lemma 3.5 and $l_0 \in \mathcal{G}_1$ by Lemma 3.7. Now, we take $\gamma(t) \subset l'_0 (t \geq 0)$ such that $\gamma(0) = P$ and $\lim_{t\to\infty} |\gamma(t)| = +\infty$ (see Fig. 5). Obviously, $\gamma(t)(t \geq 0) \subset \mathbf{G}$ is a regular curve. Now, we set

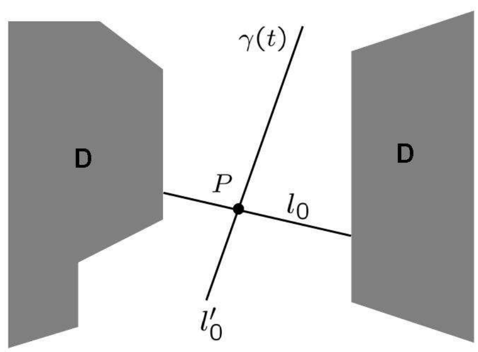

FIGURE 5. Illustration of the proof of Lemma 3.8

$$\rho = \mathbf{d}(\gamma, \mathbf{D}) \quad \text{and} \quad r_0 = \frac{1}{2}\rho.$$

Since γ is a closed set in $\mathbf{G}$ and $\mathbf{D}$ is compact, we see that $\rho > 0$ is attainable. Hence, $r_0 > 0$ and for any point $x \in \gamma(t)$, we have $\bar{B}_{r_0}(x) \subset \mathbf{G}$. Next, we set $x_0 := \gamma(t_0)$ with $t_0 = 0$.

Let $\tilde{x}_1^+ = \gamma(\tilde{t}_1) \in S_{r_0}(x_0) \cap \gamma$, and $\tilde{x}_1^- \in S_{r_0}(x_0)$ be the symmetric point of $\tilde{x}_1^+$ with respect to L_0. It is remarked that by Lemma 3.1, γ must intersect $S_{r_0}(x_0)$, and by noting that γ is an infinite half-line, the intersection must be a unique point. Now, let $\mathbf{G}_0^+$ be the connected component of $\mathbf{G}\backslash l_0$ containing $\tilde{x}_1^+$ and $\mathbf{G}_0^-$ be the connected component of $\mathbf{G}\backslash l_0$ containing $\tilde{x}_1^-$. Let E_0^+ be the connected component of $\mathbf{G}_0^+ \cap R_0(\mathbf{G}_0^-)$ containing $\tilde{x}_1^+$ and E_0^- be the connected component of $\mathbf{G}_0^- \cap R_0(\mathbf{G}_0^+)$ containing $\tilde{x}_1^-$. Observe that $E_0^+ = R_0(E_0^-)$, and if we set E_0 to be the symmetric set with respect to L_0, namely, $E_0 = E_0^+ \cup l_0 \cup E_0^-$, then E_0 contains the closed ball $\bar{B}_{r_0}(x_0)$. In fact, E_0 is the same connected set defined in (10) corresponding to l_0 and hence is bounded by noting $l_0 \in \mathcal{G}_1$. Clearly, ∂E_0

is composed of some line segments lying on $\partial\mathbf{D}$ and $R_0(\partial\mathbf{D})$. Moreover, by the reflection principle in Lemma 3.4, one sees that $u(x) = -R_0 u(x)$ in E_0 if $u(l_0) = 0$ and $u(x) = R_0 u(x)$ in E_0 if $\partial_\nu u(l_0) = 0$. Now, by the unboundedness of γ and the boundedness of E_0, one can easily derive by Lemma 3.1 that there must exist a $t_1 > \tilde{t}_1$, such that $x_1 = \gamma(t_1) \in \partial E_0$. For definiteness, we take x_1 be the 'last' point on γ to intersect ∂E_0, that is, $t_1 = \max\{t > 0; \gamma(t) \in \partial E_0\} < \infty$. Let $\widetilde{l}_1$ be the open line segment on ∂E_0 whose closure containing x_1. Since γ is an infinite half-line, it may happen that $\widetilde{l}_1 \subset \gamma$, and if this happens, by noting ∂E_0 forms the boundary of a polygonal domain, we let $\widetilde{l}_1$ be the other line segment on ∂E_0 which has x_1 as a vertex. That is, without loss of generality, we may assume that $\widetilde{l}_1 \nparallel \gamma$. Since ∂E_0 is composed of some line segments lying on $\partial\mathbf{D}$ and $R_0(\partial\mathbf{D})$ and $u(x)$ is either even or odd symmetric with respect to L_0 in E_0, we have either $u(\widetilde{l}_1) = 0$ or $\partial_\nu u(\widetilde{l}_1) = 0$. Let l_1 be the maximum extension of $\widetilde{l}_1$ in $\mathbf{G}$, then by the analytic continuation we know that either $u(l_1) = 0$ or $\partial_\nu u(l_1) = 0$. If $u(l_1) = 0$, then by Lemma 3.3, it cannot be an infinite line segment, and further by Lemma 3.5, $l_1 \notin \mathcal{G}_2$ and therefore, we have $l_1 \in \mathcal{G}_1$. Whereas if $\partial_\nu u(l_1) = 0$, by noting $\partial_\nu u(\gamma) = 0$, we easily deduce from Lemma 3.7 that $l_1 \in \mathcal{G}_1$. That is, we have $l_1 \in \mathcal{G}_1$. Furthermore, our construction ensures that l_1 is different from l_0 and the length of $\gamma(t)$ from t_0 to t_1 is larger than r_0, i.e.,

$$|\gamma(t_0 \le t \le t_1)| \ge |\gamma(t_0 \le t \le \tilde{t}_1)| \ge r_0.$$

Next, let $\tilde{x}_2^+ = \gamma(\tilde{t}_2) \in S_{r_0}(x_1) \cap \gamma$, and $\tilde{x}_2^-$ be the symmetric point of $\tilde{x}_2^+$ with respect to L_1, then let $\mathbf{G}_1^+$ be the connected component of $\mathbf{G}\backslash l_1$ containing $\tilde{x}_2^+$ and $\mathbf{G}_1^-$ be the connected component of $\mathbf{G}\backslash l_1$ containing $\tilde{x}_2^-$. Let E_1^+ be the connected component of $\mathbf{G}_1^+ \cap R_1(\mathbf{G}_1^-)$ containing $\tilde{x}_2^+$ and E_1^- be the connected component of $\mathbf{G}_2^- \cap R_2(\mathbf{G}_2^+)$ containing $\tilde{x}_2^-$. Set $E_1 = E_1^+ \cup l_1 \cup E_1^-$, then we see that E_1 contains the closed ball $\bar{B}_{r_0}(x_1)$ and its boundary is composed of some line segments lying on $\partial\mathbf{D}$ and $R_1(\partial\mathbf{D})$. By a similar argument as used earlier for deriving $x_1 = \gamma(t_1)$ and l_1, there exists a point $x_2 = \gamma(t_2)(t_2 > \tilde{t}_2)$ and open line segment $\widetilde{l}_2 \subset \partial E_1$ whose closure containing x_2. Furthermore, we may assume that x_2 is the 'last' point on γ to intersect ∂E_1 and $\widetilde{l}_2 \nparallel \gamma$. Let l_2 be the maximum extension of $\widetilde{l}_2$ in $\mathbf{G}$, then clearly, we still have either $u(l_2) = 0$ or $\partial_\nu u(l_2) = 0$. The same as above for treating l_1, we can show that $l_2 \in \mathcal{G}_1$. Moreover, we see that l_2 is different from l_1 and l_0 and the length of $\gamma(t)$ from t_1 to t_2 is larger than r_0, i.e.,

$$|\gamma(t_1 \le t \le t_2)| \ge |\gamma(t_1 \le t \le \tilde{t}_2)| \ge r_0.$$

Continuing with this procedure, we can construct a strictly increasing sequence $\{t_n\}_{n=0}^\infty$ such that for any n, $x_n = \gamma(t_n) \in l_n$ with $l_n \in \mathcal{G}_1$. Moreover, the line segments l_n, $n \in \mathbb{N}$, are different from each other, and the length of $\gamma(t)$ from t_n to t_{n+1} is not less than r_0, i.e.,

$$|\gamma(t_n \le t \le t_{n+1})| \ge r_0. \tag{19}$$

Since every finite line segment in $\mathcal{G}_1$ has its two endpoints on $\partial\mathbf{D}$ and $\mathbf{D}$ is bounded, we see that $\mathcal{G}_1$ is bounded. In fact, $\mathcal{G}_1 \subset \overline{\mathrm{co}(\mathbf{D})}$, where $\mathrm{co}(\mathbf{D})$ is the convex hull of $\mathbf{D}$. By further noting $\lim_{t\to\infty} |\gamma(t)| = +\infty$, we must have $\lim_{n\to\infty} t_n = T$ for some finite T. Otherwise, we would have $\lim_{n\to\infty} t_n = +\infty$ due to the fact that t_n is strictly increasing and this further implies $\lim_{n\to\infty} |\gamma(t_n)| = +\infty$, contradicting the fact that $\gamma(t_n) = x_n \in \mathcal{G}_1$ for each n and the boundedness of $\mathcal{G}_1$. Then, because

$\gamma(t)$ is C^1-smooth curve, we must have that

$$\lim_{n\to\infty} |\gamma(t_n \le t \le t_{n+1})| = \lim \int_{t_n}^{t_{n+1}} |\gamma'(t)| = 0, \tag{20}$$

which contradicts the inequality (19). The proof is completed. □

Proof of Lemma 3.9. The lemma can be shown by using a similar path argument as that in the proof of Lemma 3.8. For the convenience of readers, we would give some details in the following and put emphasis on the necessary modifications.

Let $\mathbf{G}'$ be the unbounded connected component of $\mathbf{G}\backslash l_0$. Next, we fix an arbitrary point $x_0 \in l_0$. Let $\gamma = \gamma(t)$ $(t \ge 0)$ be a regular curve such that $\gamma(0) = x_0$, $\gamma(t)$ $(t > 0)$ lies entirely in $\mathbf{G}'$ and $\lim_{t\to\infty} |\gamma(t)| = +\infty$. It is clear that $\gamma(t) \in l_0$ iff $t = 0$, and we set $t_0 = 0$. Let

$$\rho = \mathbf{d}(\gamma, \mathbf{D}) \quad \text{and} \quad r_0 = \frac{1}{2}\rho,$$

and it is noted that $\rho > 0$ is attainable. Hence, $r_0 > 0$ and for any point $x \in \gamma(t)$, we have $\bar{B}_{r_0}(x) \subset \mathbf{G}$.

Let $\tilde{x}_1^+ = \gamma(\tilde{t}_1) \in S_{r_0}(x_0) \cap \gamma$, and $\tilde{x}_1^- \in S_{r_0}(x_0)$ be the symmetric point of $\tilde{x}_1^+$ with respect to L_0. Furthermore, we have taken $\tilde{t}_1 = \max\{t > 0; \gamma(t) \in S_{r_0}(x_0)\} < +\infty$. Now, let $\mathbf{G}_0^+$ be the connected component of $\mathbf{G}\backslash l_0$ containing $\tilde{x}_1^+$ and $\mathbf{G}_0^-$ be the connected component of $\mathbf{G}\backslash l_0$ containing $\tilde{x}_1^-$. Let E_0^+ be the connected component of $\mathbf{G}_0^+ \cap R_0(\mathbf{G}_\mathbf{0}^-)$ containing $\tilde{x}_1^+$ and E_0^- be the connected component of $\mathbf{G}_0^- \cap R_0(\mathbf{G}_0^+)$ containing $\tilde{x}_1^-$. Set $E_0 = E_0^+ \cup l_0 \cup E_0^-$, which is exactly the connected set defined in (10) corresponding to l_0 and hence is bounded by noting $l_0 \in \mathcal{G}_1$. As in the proof of Lemma 3.8, we know that ∂E_0 is composed of some line segments lying on $\partial\mathbf{D}$ and $R_0(\partial\mathbf{D})$ and $u(x)$ is either even or odd symmetric with respect to L_0 in E_0. Now, by the unboundedness of γ and the boundedness of E_0, one can easily derive by Lemma 3.1 that there must exist a $t_1 > \tilde{t}_1$, such that $x_1 = \gamma(t_1) \in \partial E_0$. Let $\widetilde{l}_1$ be the open line segment on ∂E_0 whose closure containing x_1. Clearly, we have either $u(\widetilde{l}_1) = 0$ or $\partial_\nu u(\widetilde{l}_1) = 0$. Let l_1 be the maximum extension of $\widetilde{l}_1$ in $\mathbf{G}$, then by the analytic continuation we know that either $u(l_1) = 0$ or $\partial_\nu u(l_1) = 0$. If $u(l_1) = 0$, then by Lemma 3.3, it cannot be an infinite line segment, and further by Lemma 3.5, $l_1 \notin \mathcal{G}_2$ and therefore, we have $l_1 \in \mathcal{G}_1$. Whereas if $\partial_\nu u(l_1) = 0$, we first show that $l_1 \notin \mathcal{S}_2$, that is, l_1 cannot extend to infinity in $\mathbf{G}$. In fact, if l_1 can extend to infinity in $\mathbf{G}$, since l_1 is extended from an open line segment lying on ∂E_0 and ∂E_0 forms the boundary of a polygonal domain, we know that l_1 has to be separated from ∂E_0 at some point $V \in \mathbf{G}$. Then V is a vertex of ∂E_0. Hence, there is another point $V' \in \mathbf{G}$ such that $V'V \subset \partial E_0 \cap \mathbf{G}$. Let l_1' be the maximum extension of VV' in $\mathbf{G}$ and clearly, $u = 0$ or $\partial_\nu u = 0$ on l_1'. Noting $l_1 \nparallel l_1'$, we have by Lemma 3.3 that $l_1' \in \mathcal{S}_1$. But then we have $l_1 \in \mathcal{S}_2$, $l_1' \in \mathcal{S}_1$ and $l_1 \cap l_1' = V$, and this obviously contradicts Lemma 3.8. Hence, we must have that $l_1 \notin \mathcal{S}_2$, that is, $l_1 \in \mathcal{S}_1$. That is, for both cases of $u(l_1) = 0$ or $\partial_\nu u(l_1) = 0$, we have either $l_1 \in \mathcal{G}_1$ or $l_1 \in \mathcal{G}_2$. If $l_1 \in \mathcal{G}_2$, then we are done. To proceed further, we may assume that $l_1 \in \mathcal{G}_1$. We may further assume that $x_1 = \gamma(t_1)$ is the 'last' point on γ to intersect l_1, that is,

$$t_1 = \max\{t > 0; \gamma(t) \in l_1\} < \infty.$$

Then, we note the following two crucial facts: l_1 is different from l_0, since l_0 intersect γ only at x_0; the length of $\gamma(t)$ from t_0 to t_1 is larger than r_0, i.e.,

$$|\gamma(t_0 \le t \le t_1)| \ge |\gamma(t_0 \le t \le \tilde{t}_1)| \ge r_0.$$

Next, with the line segment l_1 and the point $x_1 = \gamma(t_1)$, we can perform a similar reflection argument as above to derive another point $x_2 = \gamma(t_2)(t_2 > \tilde{t}_2)$ and another open line segment $\widetilde{l_2}$ whose closure contains x_2. Let l_2 be the maximum extension of $\widetilde{l_2}$ in $\mathbf{G}$, then clearly, we still have either $u(l_2) = 0$ or $\partial_\nu u(l_2) = 0$. Then using the same argument as above for treating l_1, we can see that either $l_2 \in \mathcal{G}_2$ and we are done, or $l_2 \in \mathcal{G}_1$. If we still have $l_2 \in \mathcal{G}_1$, we may further assume that $x_2 = \gamma(t_2)$ is the 'last' point on γ to intersect l_2. Then, we see that l_2 is different from l_1 and l_0, since $x_0 = \gamma(t_0)$ and $x_1 = \gamma(t_1)$ are, respectively, the last point to pass through l_0 and l_1, and the length of $\gamma(t)$ from t_1 to t_2 is larger than r_0, i.e.,

$$|\gamma(t_1 \leq t \leq t_2)| \geq |\gamma(t_1 \leq t \leq \tilde{t}_2)| \geq r_0.$$

With this $l_2 \in \mathcal{G}_1$, we can continue with the above procedure to find $t_3 > t_2$ such that $x_3 = \gamma(t_3) \in l_3$ with either $l_3 \in \mathcal{G}_2$ then we are done, or $l_3 \in \mathcal{G}_1$. Moreover, through our construction, we still have $|\gamma(t_2 \leq t \leq t_3)| \geq r_0$ if $l_3 \in \mathcal{G}_1$. Continuing with this procedure, eventually, we are led to: either there is a line segment $l_m \in \mathcal{G}_2, m \geq 1$, then we are done, or there are countably many different line segments $\{l_n\}_{n=0}^\infty \subset \mathcal{G}_1$ and a strictly increasing sequence $\{t_n\}_{n=0}^\infty$ such that for each n, $x_n = \gamma(t_n) \in l_n$ and $|\gamma(t_n \leq t \leq t_{n+1})| \geq r_0$. If we assume the latter case, then it is easy to derive the relationship (20) which gives a similar contradiction to that for Lemma 3.8. The proof is completed. □

Next, we treat the proof of Lemma 3.10. To this end, we need some auxiliary results as follows.

Lemma 4.1. *For any $\varepsilon > 0$ and $0 < \theta < 2\pi$, consider the sectorial domain $E = \{x \in \mathbb{R}^2; 0 < \arg x < \theta, |x| < \varepsilon\}$ and the three points $A = (\varepsilon, 0), O = (0,0), B = (\varepsilon\cos\theta, \varepsilon\sin\theta)$. Take a point $P \in E$ such that $\phi = \angle AOP \in (0,\theta)$ and $\phi/\theta \notin \mathbb{Q}$, where $\mathbb{Q}$ is the set of rational numbers. Let $\widehat{E}$ be a connected unbounded domain containing E and assume that $v \in H^1_{loc}(\widehat{E})$ such that*

$$\Delta v + k^2 v = 0 \quad in \ \ \widehat{E},$$

and satisfies either of the following conditions

(i) $v = 0$ on $OA \cup OB \cup OP$;
(ii) $\partial_\nu v = 0$ on $OA \cup OB \cup OP$;
(iii) $v = 0$ on $OA \cup OB$ and $\partial_\nu v = 0$ on OP;
(iv) $\partial_\nu v = 0$ on $OA \cup OB$ and $v = 0$ on OP;

then $v(x) - \exp\{ikx \cdot d\}$, for $x \in \widehat{E}$, does not satisfy the Sommerfeld radiation condition (2).

Proof. Case (i) and case (ii) are verified respectively in [6] (Lemma 2) and [8] (Lemma 4). For case (iii), since $\phi \in (0,\theta)$ and $\phi/\theta \notin \mathbb{Q}$, we can choose $m \in \mathbb{N}$ and $0 < \phi_1 < \phi$ such that

$$\theta = m\phi + \phi_1, \quad \frac{\phi_1}{\phi} \notin \mathbb{Q}.$$

Then, if $m \geq 2$ (see the left figure in Fig. 6), we set $\varphi = 2\phi$ and

$$P' = (\varepsilon\cos\varphi, \varepsilon\sin\varphi).$$

Clearly, $\angle AOP' = \varphi \in (0,\theta)$ and $\varphi/\theta = 2\phi/\theta \notin \mathbb{Q}$. Furthermore, noting $v = 0$ on OA and $\partial_\nu v = 0$ on OP, we have by Lemma 3.4 that $v = 0$ on OP', and this reduces case (iii) to case (i). On the other hand, if $m = 1$ (see the right figure in Fig. 6), then $\phi_1 = \theta - \phi$. We set $\varphi = \phi - \phi_1 = \theta - 2\phi_1 = 2\phi - \theta$ and $P' = (\varepsilon\cos\varphi, \varepsilon\sin\varphi)$.

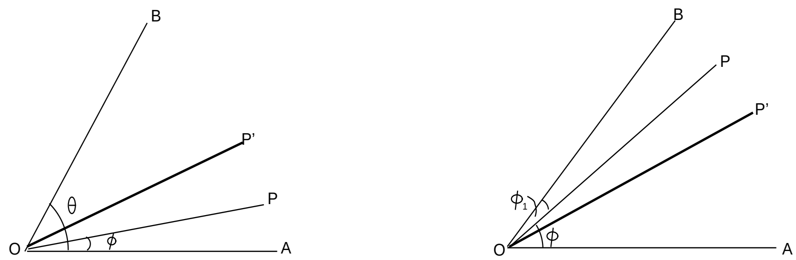

FIGURE 6. Illustration of case (iii), Lemma 4.1

Apparently, $\varphi \in (0, \theta)$ and $\varphi/\theta = (2\phi - \theta)/\theta = (2\phi/\theta - 1) \notin \mathbb{Q}$. Noting $v = 0$ on OB and $\partial_\nu v = 0$ on OP, we have by Lemma 3.4 that $v = 0$ on OP', and this also reduces case (iii) to case (i). In a like manner, we can prove case (iv). □

Lemma 4.2. *Let the sector E and the points A, B, O be defined as in Lemma 4.1, and $P \in E$ be a point such that $\phi = \angle AOP \in (0, \theta)$ and*

$$\frac{\phi}{\theta} = \frac{n}{m} \in \mathbb{Q}, \tag{21}$$

where $m, n \in \mathbb{N}$, $1 \leq n \leq m-1$ and the greatest common divisor of m and n is one. Suppose $v \in H^1(E)$ satisfies

$$\Delta v + k^2 v = 0 \quad in \ \ E,$$

and one of the following conditions:

(i) $v = 0$ on $OA \cup OB$ and $v = 0$ on OP;
(ii) $\partial_\nu v = 0$ on $OA \cup OB$ and $\partial_\nu v = 0$ on OP;
(iii) $v = 0$ on $OA \cup OB$ and $\partial_\nu v = 0$ on OP;
(iv) $\partial_\nu v = 0$ on $OA \cup OB$ and $v = 0$ on OP.

Then there exist $m - 1$ points $P^j \in E$, $1 \leq j \leq m-1$, such that $\angle AOP^j = \frac{j}{m}\theta$ and $v = 0$ on OP^j for case (i); $\partial_\nu v = 0$ on OP^j for case (ii); $v = 0$ or $\partial_\nu v = 0$ on OP^j for both cases (iii) and (iv).

Proof. Case (i) is Lemma 4 in [6], while by making use of the reflection principle of Lemma 3.4, cases (ii)-(iv) can be proved quite similarly to case (i). □

The following are several useful corollaries of Lemma 4.2.

Corollary 1. *For case (iii) in Lemma 4.2, if m is an odd number, then $v = 0$ in E.*

Proof. Let $m = 2z + 1$ with $z \in \mathbb{N}$. By Lemma 4.2, we know that there exist $2z$ points $P^j \in E$, $1 \leq j \leq 2z$ such that

$$\angle AOP^j = \frac{j}{2z+1}\theta, \quad \text{and } v = 0 \text{ or } \partial_\nu v = 0 \text{ on } OP^j, \ 1 \leq j \leq 2z. \tag{22}$$

Clearly, we have $P^n = P$. We next distinguish two cases:
Case 1. n is even. We let $n = 2z'$ with $1 \leq z' \leq z$. Since $v = 0$ on OA and $\partial_\nu v = 0$ or $v = 0$ on OP^1, by Lemma 3.4, we have $v = 0$ on OP^2. But noting $v = 0$ or $\partial_\nu v = 0$ on OP^3, we have $v = 0$ on OP^4 by using Lemma 3.4. Continuing with this procedure, we can eventually see that $v = 0$ on $OP^{2z'} = OP$. But by assumptions $\partial_\nu v = 0$ on OP. Then we can apply Holmgren's theorem (see Theorem 6.12 in [4])

to conclude that $v = 0$ in E.

Case 2. n is odd. We let $n = 2z' - 1$ with $1 \le z' \le z$. Since $v = 0$ on OB and $\partial_\nu v = 0$ or $v = 0$ on OP^{2z}, by Lemma 3.4, we know $v = 0$ on OP^{2z-1}. But noting $v = 0$ or $\partial_\nu v = 0$ on OP^{2z-2}, we have $v = 0$ on OP^{2z-3} again by Lemma 3.4. Continuing with this procedure, we can eventually see that $v = 0$ on $OP^{2z'-1} = OP$, which together with the assumption that $\partial_\nu v = 0$ on OP readily shows $v = 0$ in E by Holmgren's theorem. □

Similar to the proof of Corollary 1, we can demonstrate

Corollary 2. *For case (iv) in Lemma 4.2, if m is an odd number, then $v = 0$ in E.*

Corollary 3. *For case (iii) and case (iv) in Lemma 4.2, if $m = 2z$ with $z \in \mathbb{N}$, then we have $u = 0$ or $\partial_\nu u = 0$ on OP^z, which implies that $u = 0$ or $\partial_\nu u = 0$ on OP', where $P' \in E$ such that*

$$\frac{\angle AOP'}{\theta} = \frac{1}{2}.$$

Proof of Lemma 3.10. We prove by contradiction. Assume that $\mathcal{S}_1$ contains infinitely many line segments $P_jQ_j,\ j \in \mathbb{N}$. Since $\mathbf{D}$ is composed of finitely many polygons, without loss of generality, we may assume that $P_j \in \partial D_1$ and $Q_j \in \partial D_1'$ for $j \in \mathbb{N}$, where D_1 and D_1' are component polygons of $\mathbf{D}$. We remark that it may happen that $D_1 = D_1'$, i.e., P_j and Q_j lie on a same polygons. By passing to a subsequence, we may further assume that

1. $P_i \ne P_j$ if $i \ne j$;
2. $\lim_{j\to\infty} P_j = P_\infty$ and $\lim_{j\to\infty} Q_j = Q_\infty$;
3. $P_j, Q_j,\ j \in \mathbb{N}$ are respectively located at one side of P_∞, Q_∞, and P_j's are not vertices of D_1;
4. $P_jP_{j+1} \subset \partial D_1$ and $Q_jQ_{j+1} \subset \partial D_1'$;
5. either $u(P_jQ_j) = 0$ for all $j \in \mathbb{N}$ or $\partial_\nu u(P_jQ_j) = 0$ for all $j \in \mathbb{N}$.

It is noted that the endpoints $\{Q_j\}_{j\in\mathbb{N}}$ may not necessarily be mutually distinct. Apparently, there are four cases to consider:

(a). D_1 is a sound-soft polygon and $u(P_jQ_j) = 0$ for all $j \in \mathbb{N}$;
(b). D_1 is a sound-hard polygon and $\partial_\nu u(P_jQ_j) = 0$ for all $j \in \mathbb{N}$;
(c). D_1 is a sound-soft polygon and $\partial_\nu u(P_jQ_j) = 0$ for all $j \in \mathbb{N}$;
(d). D_1 is a sound-hard polygon and $u(P_jQ_j) = 0$ for all $j \in \mathbb{N}$.

Case (a) and Case (b) can be shown to lead to a contradiction completely similar to that in [8] (see the proof of Lemma 7 in [8], which is also applicable to the sound-soft case). It is emphasized that the scatterer considered in [8] is a single sound-hard polygon. However, to show the finiteness of the line segments in Case (a) and Case (b), one only needs to study the local behaviors of those line segments near D_1. So the arguments in [8] are still applicable to derive a similar contradiction with Cases (a) and (b), even the current scatterer composed of finitely many sound-hard and sound-soft polygons.

In the following, we only consider Case (c). And Case (d) can be treated in a like manner. By Lemma 4.1, it is readily seen that

$$\frac{\angle(Q_jP_j, \partial D_1)}{\pi} = \frac{n_j}{m_j} \in \mathbb{Q} \quad \forall j \in \mathbb{N}, \tag{23}$$

where $m_j, n_j \in \mathbb{N}$ for all $j \in \mathbb{N}$, and the greatest common divisor for m_j and n_j is one. Furthermore, by Corollary 1 we have that m_j for $j \in \mathbb{N}$ is even. Otherwise, by the unique continuation, we would get $u = 0$ in $\mathbf{G}$, which obviously contradicts Lemma 3.3. So we may let $m_j = 2z_j$, with $z_j \in \mathbb{N}$ for $j \in \mathbb{N}$. Then, by noting $\partial_\nu u(Q_jP_j) = 0$ and using Corollary 3, we have $\partial_\nu u(Q_j'P_j) = 0$ or $u(Q_j'P_j) = 0$, where $Q_j'P_j \subset \mathbf{G}$ is such that

$$\frac{\angle(Q_j'P_j, \partial D_1)}{\pi} = \frac{1}{2}, \quad j \in \mathbb{N}. \tag{24}$$

Next, we first assume that there are infinitely many $j \in \mathbb{N}$ such that $\partial_\nu u(Q_j'P_j) = 0$. Then, by extracting a subsequence, we may assume that $\partial_\nu u(Q_j'P_j) = 0$ for all $j \in \mathbb{N}$. Since $\lim_{j\to\infty} |P_jP_{j+1}| = 0$, by repeatedly applying Lemma 3.4 to the quadrilateral domains $P_jQ_j'Q_{j+m}'P_{j+m}$ with respect to the symmetry axes P_jQ_j', $j, m \in \mathbb{N}$, we can show the following: there is a family $\{l_m\}_{m\in\mathbb{N}}$ of line segments with $\partial_\nu u(l_m) = 0$, $l_m \parallel P_jQ_j'$ for all $j, m \in \mathbb{N}$ and $\cup_{m\in\mathbb{N}} l_m$ is dense in the set $U := \{P; |PP_\infty| < \delta\} \cap \mathbf{G}$ with sufficiently small $\delta > 0$. Since the Laplace operator is invariant with respect to rigid motions, we can assume that P_jQ_j' are parallel to the x_2-axis and ∂D_1 is on the x_1-axis near P_∞. Hence, from $\partial u/\partial\nu(l_m) = 0$, we have $\partial_{x_1} u(l_m) = 0$ in $\mathbf{G}$, which readily gives that $\partial_{x_1} u = 0$ in U by the continuity of $\partial_{x_1} u$ in $\mathbf{G}$. Furthermore, by noting that $\partial_{x_1} u$ also satisfies the Helmholtz equation, we have by analytic continuation that $u(x_1, x_2) = \mathbf{u}(x_2)$ for all $(x_1, x_2) \in \mathbf{G}$. Next, since $\mathbf{u}$ satisfies the equation $\Delta\mathbf{u} + k^2\mathbf{u} = 0$ in $\mathbf{G}$, i.e. $\mathbf{u}_{x_2x_2} + k^2\mathbf{u} = 0$ in $\mathbf{G}$, we can derive from the boundary condition $\mathbf{u}(0) = 0$ (because $u(\partial D_1) = 0$) that $\mathbf{u}(x_2) = \mathbf{c}\sin kx_2$ for some constant $\mathbf{c} \in \mathbb{C}$. Then, from the Sommerfeld radiation condition, we have $\lim_{|x|\to\infty} |\mathbf{c}\sin kx_2 - \exp\{\mathrm{i}kx\cdot d\}| = 0$. Particularly, by taking $\tilde{x} = (x_1, \pi/k)$ and letting $x_1 \to \infty$, we have $\lim_{x_1\to\infty} |\exp\{\mathrm{i}k\tilde{x}\cdot d\}| = 0$, which is certainly not true. On the other hand, if there are infinitely many $j \in \mathbb{N}$ such that $u(Q_j'P_j) = 0$, by extracting a subsequence if necessary, we may again assume that $u(Q_j'P_j) = 0$ for all $j \in \mathbb{N}$. Then, by a similar argument as above and repeated application of Lemma 3.4 to the quadrilateral domains $P_jQ_j'Q_{j+m}'P_{j+m}$ with respect to the symmetry axes P_jQ_j', for $j, m \in \mathbb{N}$, eventually gives that $u = 0$ in $\mathbf{G}$, contradicting Lemma 3.3. The proof is completed. □

Proof of Lemma 3.11. Let $AB := l \in \mathcal{S}_{11}$. Without loss of generality, we may assume that $A, B \in \partial D_1$ with D_1 being a component polygon of $\mathbf{D}$. Clearly, the connected path ∂D_1 is divided by the two points A and B into two parts, and one of which, along with AB, forms a (nonempty) polygon. We denote by $\mathbf{G}^+$ the interior of the polygon. Now, we shall show that $\mathbf{G}^+$ must lie entirely in $\mathbf{G}$; i.e., it is a bounded connected component of $\mathbf{G}$. If this is not true, then $\mathbf{G}^+ \cap \mathbf{D} \neq \emptyset$. On the other hand, for any point $x_0' \in AB$, noting $x_0' \in \mathbf{G} := \mathbb{R}^2\backslash\mathbf{D}$, we have a sufficiently small ball $B_\varepsilon(x_0')$ such that $B_\varepsilon(x_0') \subset \mathbf{G}$. Clearly $B_\varepsilon(x_0') \cap \mathbf{G}^+ \neq \emptyset$, so we know $\mathbf{G} \cap \mathbf{G}^+ = (\mathbb{R}^2\backslash\mathbf{D}) \cap \mathbf{G}^+ \neq \emptyset$. Then by Lemma 3.1 with $\mathcal{A} = \mathbf{D}$ and $\mathcal{B} = \mathbf{G}^+$, we have $\partial\mathbf{D} \cap \mathbf{G}^+ \neq \emptyset$, which contradicts with our definition of $\mathbf{G}^+$. Hence $\mathbf{G}^+$ is a bounded connected component of $\mathbf{G}\backslash AB$. Since the open connected set $\mathbf{G}$ cannot be divided into more than two connected components by the line segment AB (see e.g. the first step in the proof of Jordan's curve theorem in Appendix 4, Chapter 9, [7]), and noting $\mathbf{G}$ is unbounded, we therefore have another (unique) unbounded connected component of $\mathbf{G}\backslash AB$, namely, $\mathbf{G}^- = (\mathbf{G}\backslash AB)\backslash\mathbf{G}^+$.

We consider the case $AB := l \in \mathcal{S}_{12}$. Noting the facts that A, B lying on two different component polygons and the component polygons of $\mathbf{D}$ are disjoint, it is apparent to see that $\mathbf{G}\backslash l$ is connected. □

5. Concluding remarks.

5.1. Remove the disjoint condition. Our uniqueness argument equally applies to the case without the disjoint condition, namely, condition (ii) in Definition 2.1, for the mixed-type multiple polygonal scatterers discussed in Theorem 2.2. First, let us investigate what will happen if two component polygons of $\mathbf{D}$ intersect with each other. Let $D_1, D_2 \subset \mathbf{D}$ and $D_1 \cap D_2 \neq \emptyset$. If $D_1 \cap D_2$ contains an open portion of $\partial D_1 \cup \partial D_2$, then noting D_1 and D_2 are compact polygons, we see $D_1 \cup D_2$ must be a single component polygon in $\mathbf{D}$ (see, e.g., Fig. 7, the two scatterers are equivalent to each other).

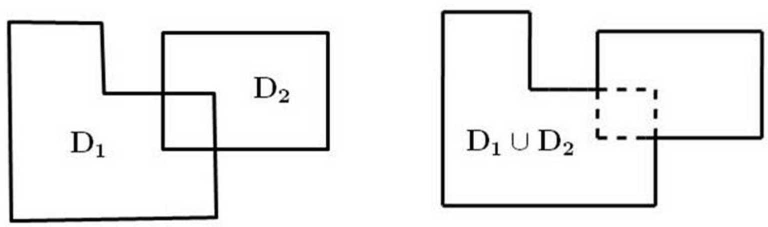

FIGURE 7. Illustration of the equivalence of two scatterers

Hence, if D_1 intersects with D_2, then they intersect only at some points. In fact, we can show that the intersection point is unique. In fact, if D_1 intersects D_2 at more than one point, e.g., let $\{P, P'\} = D_1 \cap D_2$, then ∂D_1 and ∂D_2 are, respectively divided by P and P' into two parts. Obviously, one of the two parts of ∂D_1 forms with another part on ∂D_2 a polygonal domain which disconnects to infinity; that is, $\mathbb{R}^2 \backslash (D_1 \cup D_2)$ is not connected since it has a bounded connected component, which contradicts to our assumption that $\mathbb{R}^2 \backslash \mathbf{D}$ is connected. Now, we define for each $j = 1, 2, \ldots, m$,

$$D'_j = \{D_i \subset \mathbf{D};\ D_i \cap D_j \neq \emptyset,\ i = 1, 2, \ldots, m\}, \tag{25}$$

and we call D'_j a "polygon". The expression of $\mathbf{D}$ can be reformulated as

$$\mathbf{D} = \bigcup_{j=1}^{m'} D'_j$$

with each D'_j being a "polygon", and $D'_i \cap D'_j = \emptyset$ for $i \neq j$. We would like to remark that for this $\mathbf{D}$, the connected complement $\mathbf{G} = \mathbb{R}^2 \backslash \mathbf{D}$ is not necessary a Lipschitz domain. But in the current work, we are mainly concerned with the uniqueness in the inverse problem, so we would still assume that there exists a unique solution $u \in H^1_{loc}(\mathbf{G})$ for the forward scattering problem.

With the above preparations, it is now straightforward to modify our proof of Theorem 2.2 for mixed-type multiple polygonal scatterers without the disjoint condition, and obtain

Theorem 5.1. *Assume that $\mathbf{D}$ and $\widetilde{\mathbf{D}}$ are two mixed-type multiple polygonal scatterers as described in Definition 2.1, but without the disjoint condition in (ii), with respective boundary conditions $\mathfrak{B}$ and $\widetilde{\mathfrak{B}}$. If the far-field patterns for $\mathbf{D}$ and $\widetilde{\mathbf{D}}$ coincide for a single incident plane wave at arbitrarily fixed incident direction and wave number, then $\mathbf{D} = \widetilde{\mathbf{D}}$ and $\mathfrak{B} = \widetilde{\mathfrak{B}}$.*

5.2. **Uniqueness in high-dimensions.** As for uniqueness for polyhedral scatterers in higher dimensions of $\mathbb{R}^N$ $(N \geq 3)$, the situation becomes much more sophisticated. The uniqueness with far-field data from one single incident wave is still open. However, by modifying the proof in [10], we can show the uniqueness for polyhedral scatterers without knowing their a prior physical properties by using far-field data from N different plane waves. In fact, the scatterers could be much more general in this case, which can admit the simultaneous presence of both solid and crack-type components. Let us first follow [10] to prescribe exactly the terminology *a mixed-type multiple polyhedral scatterer*. In the following, a cell is defined to be the closure of an open subset of an $(N-1)$-dimensional $(N \geq 2)$ hyperplane. And an obstacle $\mathbf{D}$ is said to be a *multiple polyhedral scatterer in* $\mathbb{R}^N$ $(N \geq 2)$ if it is a compact subset of $\mathbb{R}^N$ with connected complement $\mathbf{G} = \mathbb{R}^N \backslash \mathbf{D}$, and the boundary of $\mathbf{G}$ is composed of a finite union of cells, i.e.,

$$\partial \mathbf{G} = \bigcup_{j=1}^{m} C_j, \tag{26}$$

where each C_j is a cell. Furthermore, if $\mathbf{D}$ is called to be of mixed type if the physical properties of each cell is unknown a priori; namely, they may be either lie on some sound-hard or sound-soft solid polyhedra, or they themselves are sound-hard or sound-soft screens. Now, on $\partial\mathbf{G}$, we will be associated with the following mixed boundary condition

$$u = 0 \quad \text{on} \bigcup_{j=1}^{m'} C_j, \quad \text{and} \quad \partial_\nu u = 0 \quad \text{on} \bigcup_{j=m'+1}^{m} C_j, \tag{27}$$

where $0 \leq m' \leq m$, corresponding to the case that $C_1, \ldots, C_{m'}$ are sound-soft, while $C_{m'+1}, \ldots, C_m$ are sound-hard. Again denoting by $\mathfrak{B}u = 0$ the above boundary condition, we can show that

Theorem 5.2. *Assume that* $\mathbf{D}$ *and* $\widetilde{\mathbf{D}}$ *are two mixed-type multiple polyhedral scatterers in* $\mathbb{R}^N$ $(N \geq 2)$ *with respective boundary conditions* $\mathfrak{B}$ *and* $\widetilde{\mathfrak{B}}$*. If the far-field patterns for* $\mathbf{D}$ *and* $\widetilde{\mathbf{D}}$ *coincide for* N *incident plane waves at a fixed wave number and* N *linearly independent incident directions, then* $\mathbf{D} = \widetilde{\mathbf{D}}$ *and* $\mathfrak{B} = \widetilde{\mathfrak{B}}$.

The proof of Theorem 5.2 follows basically from that of [10] with some modifications and we refer to [11] for a detailed exposition.

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