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Construction of explicit extension operators on general finite element grids

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Abstract

An energy-preserving explicit extension operator is proposed to extend finite element functions defined on the boundary of a star-shaped polygonal domain into its interior. The pre-assigned finite element triangulation in the interior of the domain needs not be multilevel-structured. The extension operator has wide applications in the construction of non-overlapping domain decomposition methods and fictitious domain methods.

Keywords: Extension operators; Finite element; Domain decomposition methods

1. Introduction

In the study of non-overlapping domain decomposition methods (DDMs), the so-called energy-preserving extension operators play important roles. Borrowing such operators, inexact subdomain solvers can be used to replace exact subdomain solvers in non-overlapping domain decomposition methods while keeping the same convergence rate achieved with exact solvers (cf. [13,22]). These extension operators are also essential in the construction of fictitious domain methods (cf. [16–20]). Since such extension operators are directly involved in the implementation of the domain decomposition and fictitious domain algorithms, an explicit and easy-to-implement extension operator will be significant in the reduction of the total computational complexities of the algorithms. The most natural way to construct such extension operators is to extend finite element functions on the boundary of a domain into the interior of the domain by the discrete harmonic extension. But this extension requires the exact solver on

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the solution domain and is very expensive in general. The other trivial one is the extension by some given constant (zero or average of the nodal values on the boundary) at interior nodes of the meshes. Though this construction is very simple, the energy norm of this operator is bounded with a constant heavily dependent on the mesh size, we refer to [3] and Section 8 of [22] for the details. The early attempt for an explicit construction of the energy-preserving extension operators on general grids can be found in [18], see also [19,20,17] for the constructions and applications in fictitious domain methods. For a disk domain, the construction of an explicit extension operator is simple and can be done in a computational cost of order $O(h^{-2})$, i.e., proportional to the number of the nodal points in the domain. Here h is the mesh size of the finite element grid. But, for a general domain with piecewise smooth boundary, the existing constructions are very technical, e.g., a coordinate system based on normals (or pseudo-normals) was used to deal with the non-smoothness of the boundary, and thus the actual implementations are much more complicated than that for the disk domain. For the meshes with hierarchical structures, we refer to [13,14] and the references therein for the construction of explicit extension operators.

In this paper, we present a new approach for constructing the energy-preserving explicit operators on quasi-uniform finite element grids (not multilevel-structured) defined in a general polygonal domain. The main idea of our construction is to make some special radial transformation to convert the study here to the case related to the disk domain, and then the explicit construction for a disk domain in [18] can be borrowed. This new explicit construction is nearly as simple as that for the disk domain, still preserving the optimal computational complexity of order $O(h^{-2})$.

The rest of this paper is organized as follows. In Section 2, we will present the explicit construction of a new energy-preserving extension operator. The rigorous theoretical analysis on the energy-preserving property of this operator will be carried out in Section 3. In Section 4, we will describe one application of our explicit extension operator in the construction of non-overlapping DDMs with inexact subdomain solvers.

2. Construction of the extension operator

Let Ω be a polygonal domain with boundary $\Gamma \equiv \partial\Omega$, which plays the role of a general *subdomain* in the applications for non-overlapping DDMs. The vertices of Ω are numbered in order as $P_1, P_2, \ldots, P_m$, respectively. Assume that there exists an interior point O such that Ω can be partitioned into m non-overlapping triangles $\{\triangle O P_i P_{i+1}\}_{i=1}^m$, where $P_{m+1} \equiv P_1$.

As we shall see, the generic constants which appear in many subsequent estimates on different norms of the discrete extension operator may depend on the above partition of Ω into $\{\triangle O P_i P_{i+1}\}_{i=1}^m$, and thus may depend on the location of the center O and the angles/sizes of the triangles $\triangle O P_i P_{i+1}$. Because of this, it would be more desirable if the subdomains in the domain decomposition are generated in a manner that they do not have large aspect ratios. In most applications for DDMs, we can choose O as the barycenter of each subdomain, for example. Clearly, the typical subregions such as triangles and quadrilaterals, which are frequently used in non-overlapping DDMs, satisfy this condition.

Let $\overline{\Omega} \equiv \bigcup_{K\in T_h} \overline{K}$ be a quasi-uniform triangulation of Ω, with each element K being an open triangle of size h. By quasi-uniformity we mean that there exist two positive constants C_0 and C_1 independent of h such that each triangle $K \in T_h$ contains (respectively is contained in) a disk of radius $C_0 h$ (respectively $C_1 h$). Here and in what follows, C (or c) (with or without subscript) always denotes a generic constant independent of the related parameters, e.g., the mesh size h. It should be emphasized that the finite element triangulation needs not be multilevel-structured. Based on this triangulation, we define the following finite element space:

$$S^h(\Omega) \equiv \{v \in C^0(\Omega);\ v|_K \in P_1(K),\ \forall K \in T_h\}, \tag{2.1}$$

where $P_1(K)$ denotes the space of linear polynomials on K. Moreover, when restricting the triangulation T_h onto the boundary Γ, we then obtain an induced subdivision of Γ, which we denote by T_h^Γ. It is easy to show that T_h^Γ is also quasi-uniform with mesh size h. For the need of further analysis, the nodes of T_h^Γ on the line segment $[P_i, P_{i+1}]$ are numbered in order as $P_{i,1} \equiv P_i, P_{i,2}, \ldots, P_{i,n_i} \equiv P_{i+1}$, respectively, $i = 1, 2, \ldots, m$. On the boundary Γ, we define the following finite element space:

$$S^h(\Gamma) \equiv \{v \in C^0(\Gamma);\ v|_e \in P_1(e),\ \forall e \in T_h^\Gamma\}, \tag{2.2}$$

which is the restriction of the finite element space $S^h(\Omega)$ on Γ.

As usual, let $H^1(\Omega)$ be the standard Sobolev space consisting of square integrable functions with square integrable first order weak derivatives, equipped with the standard semi-norm $|\cdot|_{1,\Omega}$ and the full norm $\|\cdot\|_{1,\Omega}$:

$$|u|_{1,\Omega}^2 \equiv \int_\Omega |\nabla u|^2 \,\mathrm{d}x, \qquad \|u\|_{1,\Omega}^2 \equiv \|u\|_{0,\Omega}^2 + |u|_{1,\Omega}^2,$$

where $\|u\|_{0,\Omega} \equiv (\int_\Omega u^2 \,\mathrm{d}x)^{1/2}$, $\nabla u \equiv (\partial_1 u, \partial_2 u)$ and $|\nabla u|$ is the Euclidean norm in R^2. In addition, $W^{1,\infty}(\Omega)$ denotes the Sobolev space consisting of essentially bounded functions with essentially bounded first order weak derivatives, equipped with the norm

$$\|u\|_{1,\infty,\Omega} \equiv \max(\|u\|_{0,\infty,\Omega}, |u|_{1,\infty,\Omega}), \qquad \|u\|_{0,\infty,\Omega} \equiv \operatorname*{ess\,sup}_{x\in\Omega} |u(x)|,$$
$$|u|_{1,\infty,\Omega} \equiv \|\nabla u\|_{0,\infty,\Omega}.$$

When s is a non-negative real number, $H^s(\Omega)$ and $\|\cdot\|_{s,\Omega}$ are defined by interpolation theory (cf. [2, 15]). Note that the Sobolev spaces and their norms and semi-norms associated with functions on the boundary Γ can be defined in the same manner.

In this paper, we seek to construct an explicit extension operator $E_h : S^h(\Gamma) \to S^h(\Omega)$ such that $(E_h v)|_\Gamma = v$ and

$$\|E_h v\|_{1,\Omega} \lesssim \|v\|_{1/2,\Gamma}, \quad \forall v \in S^h(\Gamma). \tag{2.3}$$

Such an operator is called an energy-preserving extension operator. In what follows, following [22], for any two non-negative numbers x and y, $x \lesssim y$ means that $x \leqslant Cy$ for some constant C independent of the mesh size h, and $x \eqsim y$ means $x \lesssim y$ and $y \lesssim x$.

To make our description clearer, we divide our construction of the extension operator E_h into three steps.

Step 1. Construct a transfer operator $E_{1,h} : S^h(\Gamma) \to S^h(\partial D)$.

We first draw a disk D with point O as its center and some positive number R as its radius respectively, such that the domain Ω is contained in D. Typically we choose $R = 2\max_{1\leqslant i\leqslant m} |OP_i|$. We then draw a line from O to $P_{i,j}$ and denote the intersection of the line with the boundary ∂D as $Q_{i,j}$, see Fig. 1: Left. Let $Q_i \equiv Q_{i-1,n_{i-1}} = Q_{i,1}$. Thus, an induced subdivision of the circle ∂D is obtained, which we denote

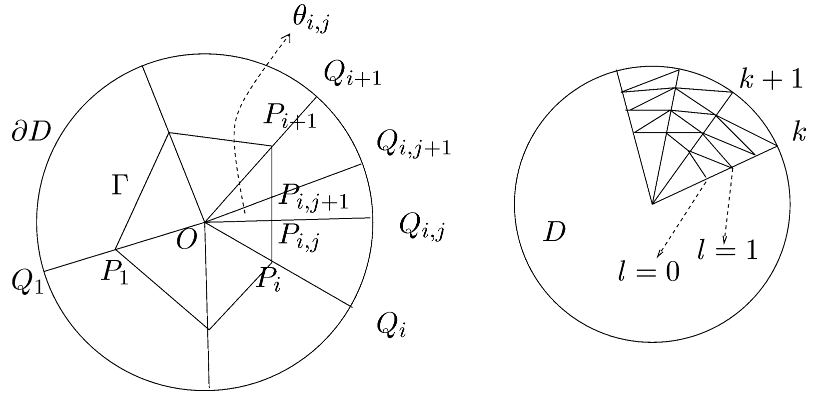

Fig. 1. Left: the auxiliary circle with the nodes induced from the nodes on Γ. Right: auxiliary meshes on the circle D.

by $T_h^{\partial D}$, with the nodes $\{Q_{i,j}\}_{1\leqslant i\leqslant m,\ 1\leqslant j\leqslant n_i-1}$. Based on this subdivision, we define a finite element space on the circular boundary ∂D:

$$S^h(\partial D)\equiv\left\{v\in C^0(\partial D);\ v|_{\widehat{Q_{i,j}Q}_{i,j+1}}\in P_1\big(\widehat{Q_{i,j}Q}_{i,j+1}\big)\right\},\tag{2.4}$$

where $1\leqslant i\leqslant m,\ 1\leqslant j\leqslant n_i-1$, and $\widehat{Q_{i,j}Q}_{i,j+1}$ denotes the arc on the circle ∂D with endpoints $Q_{i,j}$ and $Q_{i,j+1}$, $P_1(\widehat{Q_{i,j}Q}_{i,j+1})$ denotes the space of linear polynomials (according to the arc length parameter) on the arc $\widehat{Q_{i,j}Q}_{i,j+1}$. Then the transfer operator $E_{1,h}: S^h(\Gamma)\to S^h(\partial D)$ is simply defined by

$$(E_{1,h}v)(Q_{i,j})=v(P_{i,j}),\quad 1\leqslant i\leqslant m,\ 1\leqslant j\leqslant n_i,\ \forall v\in S^h(\Gamma).\tag{2.5}$$

Step 2. Construct an extension operator $E_{2,h}: S^h(\partial D)\to C^0(\overline{D})$.

For ease of exposition, we re-label the nodes on ∂D such that $Q_{1,1}$ is with index 1, $Q_{1,2}$ is with index $2,\ldots,Q_{1,n_1}$ is with index n_1, $Q_{2,2}$ is with index $n_1+1,\ldots$. Thus, each function $v\in S^h(\partial D)$ is uniquely determined by its nodal values $v(1),\ldots,v(T)$, where $v(k)\equiv v(Q_{i,j})$ with $k=\sum_{t=1}^{i-1}n_t+i-1+j$ and $T=\sum_{t=1}^{m}n_t-m$. For convenience we will extend $v(k)$ periodically for all integers, i.e., $v(k)=v(k+T),\ k=0,\pm1,\pm2,\ldots$.

As in [17,18], we then construct an auxiliary radial-annular mesh $D_{1,h}$ in a ring with the outer and inner radii being R and $\frac{R}{2}$, respectively. This is done by drawing the radius from each node on ∂D and selecting $N+1=\mathrm{O}(h^{-1})$ points on it, with step size $h_1=\frac{R}{2N}$, taking them to be ordered sequentially along the radius starting from the interior boundary (see Fig. 1: Right). Thus, each node of $D_{1,h}$ is characterized by the index pair (k,l), where k is the number of the boundary node (radius), and l the number of the point on the radius (the ring). In $D_{1,h}$ we define a discrete function $v(k,l)$ by

$$v(k,l)=\frac{2lh_1}{R(2(N-l)+1)}\sum_{t=-(N-l)}^{N-l}v(k+t),\quad k=1,\ldots,T,\ l=0,1,\ldots,N.\tag{2.6}$$

Next divide $D_{1,h}$ into small quadrilateral cells by connecting the corresponding nodes on each two neighboring radii, and denote by $D_{l,k}$ the cell with four nodes (k,l), $(k,l+1)$, $(k+1,l)$, $(k+1,l+1)$. We further partition each cell of $D_{1,h}$ into two triangles by connecting the left bottom and right upper corners, the induced triangulation still denoted by $D_{1,h}$. Obviously, the triangulation $D_{1,h}$ is quasi-uniform with mesh size h. We then obtain a function $\bar{v}$ which is linear on each triangle in $D_{1,h}$, with its nodal values given by the corresponding values $v(k,l)$. Note that $\bar{v}$ takes zero values on the interior boundary of $D_{1,h}$, thus it can be viewed as the function in $C^0(\overline{D})$ by zero extension. Then the extension operator $E_{2,h}$ is defined by $\bar{v} = E_{2,h}v$.

Step 3. Construct a pull-back operator $E_{3,h}: C^0(\overline{D}) \to S^h(\Omega)$.

We first make a polar coordinate system with O as the center. For any point $P(r,\theta)$ in $\overline{D}$, draw a line from O to P and denote by P' its intersection with the boundary Γ (possibly after extending the line OP). We then define $\rho(\theta) \equiv |OP'|$, and introduce an one-to-one map $Z: \overline{\Omega} \to \overline{D}$ by

$$Z(r,\theta) = \left(\frac{Rr}{\rho(\theta)}, \theta\right), \tag{2.7}$$

which has the important property that it maps Γ onto ∂D. Based on this mapping, we define a pull-back operator $\Phi: C^0(\overline{D}) \to C^0(\overline{\Omega})$ by

$$(\Phi v)(r,\theta) = v\big(Z(r,\theta)\big) = v\left(\frac{Rr}{\rho(\theta)}, \theta\right), \quad \forall v \in C^0\big(\overline{D}\big) \text{ and } (r,\theta) \in \overline{\Omega}. \tag{2.8}$$

With this, the required pull-back operator $E_{3,h}: C^0(\overline{D}) \to S^h(\overline{\Omega})$ is defined by

$$E_{3,h}v = I_h(\Phi v), \quad \forall v \in C^0\big(\overline{D}\big), \tag{2.9}$$

where I_h is the standard piecewise linear finite element interpolation from $C^0(\overline{\Omega})$ onto $S^h(\Omega)$ [9].

Now, the explicit extension operator $E_h: S^h(\Gamma) \to S^h(\Omega)$ is obtained simply by composing three operators given above:

$$E_h v = E_{3,h} \circ E_{2,h} \circ E_{1,h} v, \quad \forall v \in S^h(\Gamma). \tag{2.10}$$

From the above construction process, it is easy to see that the extension operator E_h is well-defined. For the sake of our later analysis, its construction was presented above in a relatively comprehensive but hopefully better understandable manner. In fact, the implementation of the construction can be done easily and it is no need to really construct the radial-annual mesh $D_{1,h}$ and the auxiliary triangulation on $D_{1,h}$ as described in step 2, they are introduced just for the convenience of exposition and the later analysis of the algorithm. Let us see how this can be done. For any $v \in S^h(\Gamma)$, we can first obtain the nodal values of the grid function $E_{1,h}v$ on ∂D directly, then compute the discrete function $v(k,l)$ using the formula (2.6). Now it suffices to evaluate the nodal values of $E_h v$ on Ω in order to determine the extension function $E_h v$. To do this, for any nodal point $P(r,\theta)$ in Ω, by (2.8) and (2.9), we need only to determine which triangle of $D_{1,h}$ the point $(\frac{Rr}{\rho(\theta)}, \theta)$ belongs to. For this, we first locate a quadrilateral $D_{l,k}$ with $(\frac{Rr}{\rho(\theta)}, \theta) \in D_{l,k}$. k can be easily determined using θ. To determine l, let m_1 be the largest of those integers which are less than or equal to $\frac{Rr}{\rho(\theta)h_1}$. Then $m_1 h_1 - \frac{R}{2} = (m_1 - N)h_1$, so we know $l = m_1 - N$.

In summary, the construction can be implemented as follows:

Algorithm 1 (*Construction algorithm for E_h*). Given any $v \in S^h(\Gamma)$.

Step A. Compute the nodal values on the circle ∂D: for $1 \leqslant i \leqslant m$, $1 \leqslant j \leqslant n_i - 1$, set $k := \sum_{t=1}^{i-1} n_t + (i-1) + j$, compute $v_1(k) := v(P_{i,j})$.

Step B. For $k=1,2,\dots,T$, $l=0,1,\dots,N$, compute

$$v(k,l) := \frac{2lh_1}{R(2(N-l)+1)} \sum_{t=-(N-l)}^{N-l} v_1(k+t).$$

Step C. For any node $(r,\theta)\in T_h$, set $E_h v(r,\theta)=0$ if $r<\frac{\rho(\theta)}{2}$. If $r\geqslant \frac{\rho(\theta)}{2}$, compute the index (l,k) with $(\frac{Rr}{\rho(\theta)},\theta)\in D_{l,k}$, then compute

$$E_h v(r,\theta) := \sum_{i=0}^{1}\sum_{j=0}^{1} v(k+i,l+j)a_{ij}(k,l). \tag{2.11}$$

Remark 1. At least one of the four coefficients $a_{ij}(k,l)$ in (2.11) vanishes, the other three are the barycentric coordinates of the point $(\frac{Rr}{\rho(\theta)},\theta)$, which belongs to one of two triangles from $D_{l,k}$.

Remark 2. The above construction of the extension operator E_h can be easily generalized to the three-dimensional domain Ω. And the subsequent results are still true with some technical modifications of the proofs in this paper.

Next we give the construction of the transpose of the extension operator E_h, since it is needed when applied to non-overlapping DDMs and fictitious domain methods (cf. [13,17,22]). Let $(\cdot,\cdot)_{0,h}$ and $\langle\cdot,\cdot\rangle_{0,h}$ be the standard discrete L^2-inner products in $S^h(\Omega)$ and $S^h(\Gamma)$, respectively (cf. [22]). Then the transpose $E_h^t : S^h(\Omega)\to S^h(\Gamma)$ of E_h is defined as follows:

$$\langle v, E_h^t w\rangle_{0,h} = (E_h v, w)_{0,h}, \quad \forall v\in S^h(\Gamma),\ w\in S^h(\Omega). \tag{2.12}$$

For $v\in S^h(\Omega)$ (respectively $S^h(\Gamma)$), we denote by $\tilde{v}$ the column vector with the components being the nodal values of v. Then the formula (2.12) can be written as

$$h^2\big(\widetilde{E_h v}\big)^t \tilde{w} = h\tilde{v}^t\big(\widetilde{E_h^t w}\big). \tag{2.13}$$

Moreover, let

$$\widetilde{E}_{2,h} : \big\{v_1(k)\big\}_{1\leqslant k\leqslant T} \to \big\{v(k,l)\big\}_{\substack{1\leqslant k\leqslant T,\\ 0\leqslant l\leqslant N}}$$

and

$$\widetilde{E}_{3,h} : \big\{v(k,l)\big\}_{\substack{1\leqslant k\leqslant T,\\ 0\leqslant l\leqslant N}} \to \big\{E_h v(r,\theta)\big\}_{(r,\theta)\in\mathcal{N}_h}$$

be the matrix (tensor) representations of those linear operators realized by step B and step C of Algorithm 1, respectively. Then we easily have

$$\big(\widetilde{E_h v}\big) = \widetilde{E}_{3,h}\widetilde{E}_{2,h}\tilde{v},$$

which together with (2.13) yields

$$\big(\widetilde{E_h^t w}\big) = h\widetilde{E}_{2,h}^t\widetilde{E}_{3,h}^t,$$

where $\widetilde{E}_{2,h}^t$ and $\widetilde{E}_{3,h}^t$ denote the conventional transposes of the tensors $\widetilde{E}_{2,h}$ and $\widetilde{E}_{3,h}$, respectively. By a straightforward computation we then have the following construction algorithm for E_h^t:

Algorithm 2 (*Construction algorithm for* E_h^t). Given $w \in S^h(\Omega)$.

Step A. Set $w_2(k,l) = 0,\ k = 1,\dots,T,\ l = 0,1,\dots,N$. For each nodal point (r,θ) with $r > \frac{\rho(\theta)}{2}$, compute the index (k,l) with $(\frac{Rr}{\rho(\theta)},\theta) \in D_{l,k}$, and then compute

$$w_2(k+i,l+j) := w_2(k+i,l+j) + a_{i,j}(k,l)w(r,\theta).$$

Set $w_2(k,l) := \frac{2lh_1}{R(2(N-l)+1)} w_2(k,l)$, $k = 1,\dots,T$, $l = 0,1,\dots,N$. Finally $w_2(k,l)$ is defined for all integers k by periodic extension: $w_2(k,l) = w_2(k+T,l)$, $k = 0,\pm1,\pm2,\dots.$

Step B. Compute $w_1(k),\ k = 1,2,\dots,T$ by

$$w_1(k) := \sum_{l=0}^{N-1}\sum_{j=0}^{N-l}\bigl\{w_2(k+l,j) + w_2(k-l,j)\bigr\} + \sum_{j=0}^{N} w_2(k,j).$$

Step C. Compute the nodal values of $E_h^t w$: for $1 \leqslant i \leqslant m,\ 1 \leqslant j \leqslant n_i - 1$, set $k = \sum_{t=1}^{i-1} n_t + (i-1) + j$ and compute

$$\bigl(E_h^t w\bigr)(P_{i,j}) = h w_1(k).$$

Remark 3. The arithmetic operations of Algorithm 1 is proportional to the number of nodes in Ω, i.e., $O(1/h^2)$, or $O(H^2/h^2)$ if the diameter of Ω is of order $O(H)$. To see this, we need only to check the cost of step B of the algorithm, the other two steps are just the realization of some one-to-one mappings or the local average of nodal values, thus require at most $O(1/h^2)$ arithmetic operations. It is presented in [18,17] that, by writing (2.6) in recursive form, only $O(1/h^2)$ arithmetic operations are needed to implement step B of Algorithm 1. Therefore, we need only $O(1/h^2)$ arithmetic operations to compute the extension $E_h v$ for any $v \in S^h(\Gamma)$. Similarly, only $O(1/h^2)$ arithmetic operations are needed to compute $E_h^t v$ for $v \in S^h(\Omega)$ by noting that only $O(1/h^2)$ arithmetic operations are needed to implement step B of Algorithm 2 (cf. [17]). Thus, from the viewpoint of arithmetic operations, the explicit extension operator E_h constructed here is "optimal" like that for the disk case given in [17,18,20].

3. Energy-preserving properties of the extension operator

In this section, we will show that the explicit extension operator E_h constructed in Section 2 satisfies the estimate (2.3). Let us first give some useful lemmas.

Lemma 1. *The induced subdivision $T_h^{\partial D}$ given in step 1 of Section 2 is quasi-uniform with mesh size h, that is,*

$$|\widehat{Q_{i,j}Q}_{i,j+1}| \eqsim h, \quad \textit{for any arc } \widehat{Q_{i,j}Q}_{i,j+1} \in T_h^{\partial D}.$$

Proof. Let $|\widehat{Q_{i,j}Q}_{i,j+1}| \equiv R\theta_{i,j}$, i.e., the circular arc $\widehat{Q_{i,j}Q}_{i,j+1}$ is subtended by the central angle $\theta_{i,j}$. Considering the triangle $\triangle OP_{i,j}P_{i,j+1}$ and using the cosine-theorem, we have

$$|P_{i,j}P_{i,j+1}|^2 = |OP_{i,j}|^2 + |OP_{i,j+1}|^2 - 2|OP_{i,j}||OP_{i,j+1}|\cos\theta_{i,j},$$

or equivalently,

$$\cos\theta_{i,j} = \frac{1}{2}\left(\frac{|OP_{i,j}|}{|OP_{i,j+1}|} + \frac{|OP_{i,j+1}|}{|OP_{i,j}|}\right) - \frac{|P_{i,j}P_{i,j+1}|^2}{2|OP_{i,j}||OP_{i,j+1}|}. \tag{3.1}$$

It follows from the quasi-uniformity of the triangulation T_h^Γ that there exist two positive constants c_2 and C_2, such that

$$c_2 h \leqslant |P_{i,j}P_{i,j+1}| \leqslant C_2 h. \tag{3.2}$$

It is also very clear that there exist two positive constants A and B independent of h such that

$$0 < A \leqslant |OP_{i,j}| \leqslant B < +\infty, \quad i = 1, \dots, m, \ j = 1, \dots, n_i. \tag{3.3}$$

Therefore, by the Cauchy–Schwartz inequality, (3.2) and (3.3), we see

$$\cos\theta_{i,j} \geqslant 1 - \frac{(C_2 h)^2}{2A^2}. \tag{3.4}$$

Furthermore, we know that $\cos\theta_{i,j} = 1 - 2\sin^2\frac{\theta_{i,j}}{2} \leqslant 1 - 2(\frac{\theta_{i,j}}{\pi})^2$, which together with (3.4) immediately gives

$$\theta_{i,j} \leqslant \frac{\pi C_2}{2A} h. \tag{3.5}$$

On the other hand, from the sine-theorem, we have

$$\frac{\sin\theta_{i,j}}{|P_{i,j}P_{i,j+1}|} = \frac{\sin\angle OP_{i,j}P_{i,j+1}}{|OP_{i,j+1}|},$$

which, with (3.2) and (3.3), yields

$$\begin{aligned}\sin\theta_{i,j} &= \frac{\sin\angle OP_{i,j}P_{i,j+1}}{|OP_{i,j+1}|}|P_{i,j}P_{i,j+1}| \geqslant \frac{c_2 h}{B}\sin\angle OP_{i,j}P_{i,j+1} \\ &\geqslant \frac{c_2 h}{B}\min(\sin\angle OP_iP_{i+1}, \sin\angle OP_{i+1}P_i) \geqslant \frac{c_2 C_3 h}{B},\end{aligned} \tag{3.6}$$

where $C_3 \equiv \min_{1\leqslant i\leqslant m}\sin\angle OP_iP_{i+1}$, which is a constant independent of h. The desired result then follows directly from the fact $\sin\theta_{i,j} \leqslant \theta_{i,j}$, and (3.5), (3.6). □

Lemma 2. *For the transfer operator $E_{1,h}$ defined in step* 1 *of Section* 2, *the following estimates hold for any $v \in S^h(\Gamma)$:*

$$\|E_{1,h}v\|_{0,\partial D} \lesssim \|v\|_{0,\Gamma}, \qquad \|E_{1,h}v\|_{1,\partial D} \lesssim \|v\|_{1,\Gamma}, \qquad \|E_{1,h}v\|_{1/2,\partial D} \lesssim \|v\|_{1/2,\Gamma}. \tag{3.7}$$

Proof. By Lemma 1, the induced subdivision $T_h^{\partial D}$ is quasi-uniform with mesh size h. Then the first two inequalities follow from the conventional scaling arguments (cf. [4,9,22]) and the definition of $E_{1,h}$

directly. We now verify the last inequality using the interpolation theory in discrete form (cf. [10]). To do so, let us first construct a L^2-orthogonal projection operator Q_h^Γ from $L^2(\Gamma)$ onto $S^h(\Gamma)$ as follows:

$$\left\langle Q_h^\Gamma v, w\right\rangle_{0,\Gamma} = \langle v, w\rangle_{0,\Gamma}, \quad \forall w \in S^h(\Gamma),\ v \in L^2(\Gamma), \tag{3.8}$$

where $\langle\cdot,\cdot\rangle_{0,\Gamma}$ means the conventional $L^2(\Gamma)$-inner product, i.e., $\langle v, w\rangle_{0,\Gamma} = \int_\Gamma vw\,\mathrm{d}s$. For this projection operator, the following estimates hold (cf. [21]):

$$\left\| Q_h^\Gamma v\right\|_{s,\Gamma} \lesssim \|v\|_{s,\Gamma}, \quad \forall v \in H^s(\Gamma),\ s = 0, 1. \tag{3.9}$$

On the other hand, by the interpolation theory of Sobolev spaces (cf. [2,15]),

$$\|v\|_{1/2,\Gamma}^2 \equiv \|v\|_{0,\Gamma}^2 + \int_0^{+\infty} t^{-2} K_\Gamma(t, v)^2\,\mathrm{d}t, \tag{3.10}$$

where

$$K_\Gamma(t, v) \equiv \inf_{v=v_0+v_1} \left(\|v_0\|_{0,\Gamma}^2 + t^2\|v_1\|_{1,\Gamma}^2\right)^{1/2}$$

with $v_0 \in L^2(\Gamma)$ and $v_1 \in H^1(\Gamma)$. It is well known (cf. [12,15]) that the definition (3.10) is equivalent to the following intrinsic definition of $\|\cdot\|_{1/2,\Gamma}$:

$$\|v\|_{1/2,\Gamma}^2 \eqsim \|v\|_{0,\Gamma}^2 + \int_\Gamma\int_\Gamma (t-s)^{-2}\left|v(t) - v(s)\right|^2\,\mathrm{d}t\,\mathrm{d}s. \tag{3.11}$$

The norm $\|\cdot\|_{1/2,\partial D}$ and $K_{\partial D}(t, v)$ are defined in the same manner. Given any $v \in S^h(\Gamma)$, let $v = v_0 + v_1$ be an arbitrary function decomposition such that $v_0 \in L^2(\Gamma)$ and $v_1 \in H^1(\Gamma)$. Hence, $E_{1,h}v = E_{1,h}Q_h^\Gamma v_0 + E_{1,h}Q_h^\Gamma v_1$ which is just a function decomposition for defining the norm $\|E_{1,h}v\|_{1/2,\partial D}$. Therefore, from the first two inequalities of (3.7) and (3.9) we have

$$K_{\partial D}(t, E_{1,h}v) \leqslant \left(\left\|E_{1,h}Q_h^\Gamma v_0\right\|_{0,\partial D}^2 + t^2\left\|E_{1,h}Q_h^\Gamma v_1\right\|_{1,\partial D}^2\right)^{1/2} \lesssim \left(\|v_0\|_{0,\Gamma}^2 + t^2\|v_1\|_{1,\Gamma}^2\right)^{1/2},$$

which directly leads to

$$K_{\partial D}(t, E_{1,h}v) \lesssim K_\Gamma(t, v).$$

This inequality together with (3.10) proves the last inequality of (3.7). □

For simplicity, we will denote by $\triangle_i$ below the triangle $\triangle OP_iP_{i+1}$ and by $\widetilde{\triangle}_i$ the circular sector formed by two radii OQ_i, OQ_{i+1} and the arc $\widehat{Q_iQ}_{i+1}$.

Lemma 3. *The one-to-one map Z in* (2.7) *is a smooth diffeomorphism from $\triangle_i$ on to $\widetilde{\triangle}_i$. Moreover, for any $v \in W^{t,p}(\widetilde{\triangle}_i)$, $0 \leqslant t < +\infty$, $1 \leqslant p \leqslant +\infty$, the pull-back $\Phi v \in W^{t,p}(\triangle_i)$, and the following estimate holds*:

$$\|\Phi v\|_{t,p,\triangle_i} \lesssim \|v\|_{t,p,\widetilde{\triangle}_i}, \quad \forall v \in W^{t,p}\big(\widetilde{\triangle}_i\big),\ i = 1, \ldots, m, \tag{3.12}$$

where $W^{s,p}(\triangle_i)$ denotes the conventional Sobolev space with $W^{s,2}(\triangle_i) = H^s(\triangle_i)$ (*cf.* [12,15]).

Proof. Since $\rho(\theta)$ is a smooth function when restricted on $\widetilde{\triangle}_i$ and bounded positively away from below by the constant $c = \mathrm{dist}(O, \Gamma)$, it is easy to derive the first conclusion. The estimate (3.12) follows from the first conclusion directly (cf. [12]). □

Remark 4. It should be noted that the mapping Z is not a globally smooth (even not C^1) diffeomorphism from $\overline{\Omega}$ onto $\overline{D}$, and thus more efforts have to be made in order to obtain the main result of this paper.

The next result can be found in [17,18]:

Lemma 4. *For the extension operator $E_{2,h}$ defined in step 2 of Section 2, the following estimate holds*:

$$\|E_{2,h}v\|_{1,D} \lesssim \|v\|_{1/2,\partial D}, \quad \forall v \in S^h(\partial D).$$

After the above preparations, we are ready to present the main results of this paper. For clarity, we divide our consideration into two cases. The first case assumes the finite element triangulation T_h is aligned with the line segments OP_i, $i = 1, \ldots, m$, that is, each edge of a triangle $K \in T_h$ either lies on some OP_i completely or has only one vertex on OP_i or does not intersect any OP_i.

Theorem 1. *Assume that the finite element triangulation T_h is aligned with the line segments OP_i, $i = 1, \ldots, m$. Then, for the explicit extension operator E_h constructed in Section 2, we have*

$$\|E_h v\|_{1,\Omega} \lesssim \|v\|_{1/2,\Gamma}, \quad \forall v \in S^h(\Gamma). \tag{3.13}$$

Proof. Using the estimates for the interpolation operator I_h (cf. [4,9]) and Lemma 3, we immediately have for any $v \in S^h(\Gamma)$,

$$\begin{aligned}
\|E_h v\|^2_{1,\triangle_i} &= \left\| I_h(\Phi E_{2,h} E_{1,h} v)\right\|^2_{1,\triangle_i} \\
&\lesssim \|\Phi E_{2,h} E_{1,h} v\|^2_{1,\triangle_i} + \left\|\Phi E_{2,h} E_{1,h} v - I_h(\Phi E_{2,h} E_{1,h} v)\right\|^2_{1,\triangle_i} \\
&\lesssim \|E_{2,h} E_{1,h} v\|^2_{1,\widetilde{\triangle}_i} + h^{2\varepsilon} \|\Phi E_{2,h} E_{1,h} v\|^2_{1+\varepsilon,\triangle_i} \\
&\lesssim \|E_{2,h} E_{1,h} v\|^2_{1,\widetilde{\triangle}_i} + h^{2\varepsilon} \|E_{2,h} E_{1,h} v\|^2_{1+\varepsilon,\widetilde{\triangle}_i},
\end{aligned}$$

where $\varepsilon \in (0, 0.5)$ is any fixed number. Summing up the inequalities over $i = 1, \ldots, m$ and using the definition of the norm $\|\cdot\|_{1+\varepsilon}$ (cf. [11, p. 17]) we obtain

$$\begin{aligned}
\|E_h v\|^2_{1,\Omega} &= \sum_{i=1}^{m} \|E_h v\|^2_{1,\triangle_i} \lesssim \sum_{i=1}^{m} \left(\|E_{2,h} E_{1,h} v\|^2_{1,\widetilde{\triangle}_i} + h^{2\varepsilon} \|E_{2,h} E_{1,h} v\|^2_{1+\varepsilon,\widetilde{\triangle}_i}\right) \\
&\lesssim \|E_{2,h} E_{1,h} v\|^2_{1,D} + h^{2\varepsilon} \|E_{2,h} E_{1,h} v\|^2_{1+\varepsilon,D},
\end{aligned}$$

which, with the inverse inequality (cf. [1]) and Lemmata 2 and 4, leads to

$$\|E_h v\|^2_{1,\Omega} \lesssim \|E_{2,h} E_{1,h} v\|^2_{1,D} \lesssim \|v\|_{1/2,\Gamma}. \qquad \square$$

Now we proceed to consider the more general case, i.e., the finite element triangulation T_h is not necessarily aligned with the line segments OP_i, $i = 1, \ldots, m$. Hence, there exist some triangles $K \in T_h$

each of which is separated by OP_i's into two or more parts belonging to different coarse triangles $\triangle_i$, $i = 1, \ldots, m$. In reality, such cases are often encountered in the study of the domain decomposition methods for unstructured meshes (cf. [6–8]), where one needs to construct a proper transfer operator from a unstructured coarse space into a finer finite element space such that the operator keeps the H^1-stability and L^2-norm optimal error estimates. Thus, some additional assumptions on the finite elements (fine triangles) intersecting with coarse triangles are needed. In our current situation, we need only the h-independent norm estimate for the explicit extension operator E_h from $S^h(\Gamma)$ into $S^h(\Omega)$. To our surprise, no additional assumptions are needed on the triangulation. To show this, we first introduce some notations. Define

$$T_h^{\triangle_i} \equiv \{K \in T_h;\ K \subset \triangle_i\}, \qquad T_h^{OP_i} \equiv \{K \in T_h;\ K \cap OP_i \neq \emptyset\},$$
$$\widetilde{T}_h^{OP_i} \equiv \left\{K \in T_h^{OP_i};\ \exists j \neq i \text{ such that } K \in T_h^{OP_j}\right\}.$$

Geometrically, $\widetilde{T}_h^{OP_i}$ consists of those triangles which are "close" to the center O and are divided by the edges of coarse triangles $\triangle_i$ into at least three parts. These elements are most difficult to deal with for deriving our required estimates. But, fortunately, the number of such kind of "bad" triangles is bounded above by a natural number N_2 independent of h, as indicated in the next lemma.

Lemma 5. *There exists some positive constant C_4, independent of the mesh size h, such that each $K \in \widetilde{T}_h^{OP_i}$ belongs to the disk $B_O(C_4h)$, where $B_O(C_4h)$ denotes the disk centered at O and with a radius C_4h.*

Proof. Let $K \in \widetilde{T}_h^{OP_i}$. Clearly K must be be in a small neighborhood of the point O. Let r denote the distance from O to K. According to the definition of $\widetilde{T}_h^{OP_i}$, there exist two points A_1 and A_2 on K, such that $A_1 \in OP_i$, $A_2 \in OP_{i+1}$ (or OP_{i-1}). Without loss of generality, we assume that $A_2 \in OP_{i+1}$ (see Fig. 2: Left). Then from the cosine-theorem and the quasi-uniformity of T_h, we see

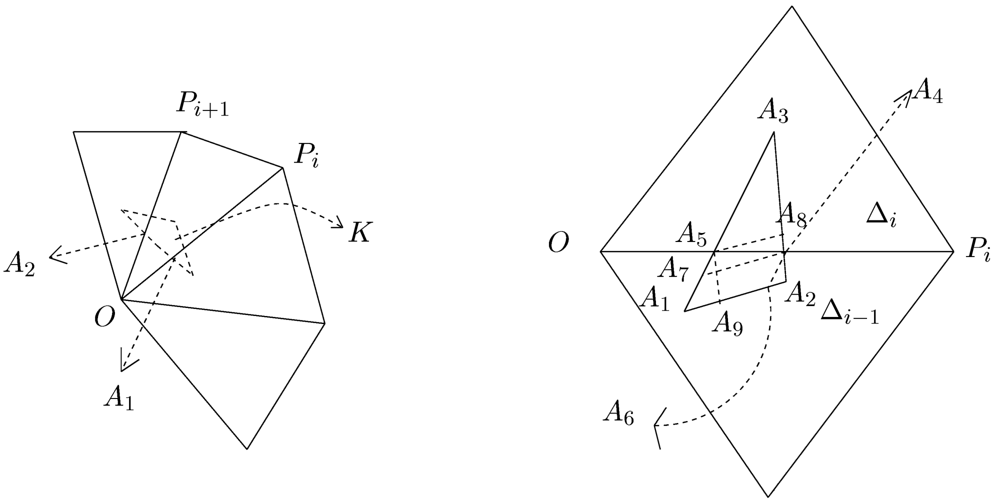

Fig. 2. Left: the figure used to show the proof of Lemma 5. Right: the figure used to show the third case in the proof of Theorem 2.

$$(C_1h)^2 \geqslant |A_1A_2|^2 = |OA_1|^2 + |OA_2|^2 - 2\cos\angle P_iOP_{i+1}|OA_1||OA_2| \geqslant 2\big(1 - |\cos\angle P_iOP_{i+1}|\big)r^2,$$

which leads to

$$r \leqslant \frac{C_1h}{\sqrt{2(1 - |\cos\angle P_iOP_{i+1}|)}},$$

and thus by the triangle inequality, K is contained in the disk with center O and radius

$$C_1h + r \leqslant \left(1 + \frac{1}{\sqrt{2(1 - |\cos\angle P_iOP_{i+1}|)}}\right)C_1h.$$

The desired result then follows. □

Theorem 2. *For any quasi-uniform triangulation T_h, the estimate* (2.3) *holds.*

Proof. First by the Poincaré inequality and the fact that $(E_hv)|_\Gamma = v$ for any $v \in S^h(\Gamma)$ we have

$$\|E_hv\|_{0,\Omega}^2 \lesssim \left(|E_hv|_{1,\Omega} + \left|\int_\Gamma E_hv\,\mathrm{d}s\right|\right)^2 \lesssim |E_hv|_{1,\Omega}^2 + \|v\|_{1/2,\Gamma}^2. \tag{3.14}$$

Using (3.14) it suffices to show $|E_hv|_{1,\Omega} \lesssim |v|_{1/2,\Gamma}$ for the required estimate (2.3). The proof is divided into three cases.

We first consider those elements in $T_h^{\triangle_i}$. Repeating the same process as used in proving Theorem 1, we immediately have for $i = 1, \ldots, m$,

$$\sum_{K\in T_h^{\triangle_i}} \|E_hv\|_{1,K}^2 \lesssim \|E_{2,h}E_{1,h}v\|_{1,\widetilde{\triangle}_i}^2 + h^{2\varepsilon}\|E_{2,h}E_{1,h}v\|_{1+\varepsilon,\widetilde{\triangle}_i}^2,$$

which gives

$$\sum_{i=1}^m \sum_{K\in T_h^{\triangle_i}} \|E_hv\|_{1,K}^2 \lesssim \sum_{i=1}^m \big\{\|E_{2,h}E_{1,h}v\|_{1,\widetilde{\triangle}_i}^2 + h^{2\varepsilon}\|E_{2,h}E_{1,h}v\|_{1+\varepsilon,\widetilde{\triangle}_i}^2\big\} \lesssim \|v\|_{1/2,\Gamma}^2. \tag{3.15}$$

Secondly, we consider the elements in $\widetilde{T}_h^{OP_i}$. For any $K \in \widetilde{T}_h^{OP_i}$, we denote by A_1, A_2, A_3 its three vertices. From the standard scaling argument we have

$$|E_hv|_{1,K}^2 \lesssim \big|w(A_1) - w(A_2)\big|^2 + \big|w(A_1) - w(A_3)\big|^2 + \big|w(A_2) - w(A_3)\big|^2,$$

where $w \equiv \Phi(E_{2,h}E_{1,h}v)$. Assume that the line segment A_kA_l ($1 \leqslant k, l \leqslant 3$) is cut into q pieces by the coarse triangles $\triangle_{j_s}$, $s = 1, \ldots, q$. Then, by the mean value theorem and Lemma 3 we have

$$\big|w(A_k) - w(A_l)\big| \leqslant h \max_{1\leqslant s\leqslant q} |w|_{1,\infty,\triangle_{j_s}} \lesssim h|E_{2,h}E_{1,h}v|_{1,\infty,D}. \tag{3.16}$$

Lemma 5 implies the number of $K \in \widetilde{T}_h^{OP_i}$ is bounded above by a natural number N_2 independent of h. Then by Lemmata 2 and 4, (3.16) and the inverse inequality we have

$$\sum_{i=1}^m \sum_{K\in \widetilde{T}_h^{OP_i}} |E_hv|_{1,K}^2 \lesssim h^2|E_{2,h}E_{1,h}v|_{1,\infty,D}^2 \lesssim \|E_{2,h}E_{1,h}v\|_{1,D}^2 \lesssim \|v\|_{1/2,\Gamma}^2. \tag{3.17}$$

It remains to consider those elements in $T_h^{OP_i}$ but not in $\widetilde{T}_h^{OP_i}$. Consider such an element $K = \triangle A_1A_2A_3$ (see Fig. 2: Right). From the standard scaling argument and the triangle inequality we see

$$\begin{aligned} |E_h v|_{1,K}^2 &\lesssim \left|w(A_2) - w(A_3)\right|^2 + \left|w(A_3) - w(A_1)\right|^2 \\ &\lesssim \left|w(A_2) - w(A_4)\right|^2 + \left|w(A_4) - w(A_3)\right|^2 \\ &\quad + \left|w(A_3) - w(A_5)\right|^2 + \left|w(A_5) - w(A_1)\right|^2, \end{aligned} \tag{3.18}$$

where A_4 is the intersecting point of the line segments OP_i and A_2A_3 while A_5 is the intersecting point of the line segments OP_i and A_1A_3.

In order to estimate the four terms in (3.18), draw a line which passes A_4 (respectively A_5) and is parallel to the line A_1A_3 (respectively A_2A_3), this line intersects the line segment A_1A_2 at A_6 (respectively A_9); in the same manner, draw a line which passes the point A_4 (respectively A_5) and is parallel to the line A_1A_2, this line intersects the line segment A_1A_3 (respectively A_2A_3) at A_7 (respectively A_8). Since the triangulation T_h is quasi-uniform, it is also shape-regular (cf. [9,23]), that means, there exists some $\theta_0 > 0$, such that each interior angle of $K \in T_h$ is not less than θ_0 (which is also the equivalent definition of the shape-regular triangulation). It is easy to see that the triangles $\triangle A_2A_4A_6$, $\triangle A_1A_9A_5$, $\triangle A_3A_5A_8$ are all the shape-regular triangles with the same parameter θ_0 as that of T_h. Moreover, if $\angle A_3A_4A_5 \geqslant \theta_0/2$, $\triangle A_3A_4A_5$ is a shape-regular triangle with the parameter $\theta_0/2$; otherwise, $\triangle A_4A_5A_7$ is a shape-regular triangle with the parameter $\theta_0/2$. Furthermore, we construct an interpolation operator $\tilde{I}_h$ on the triangles $\triangle A_2A_4A_6$, $\triangle A_3A_5A_4$ and $\triangle A_1A_9A_5$ for the case that $\angle A_3A_4A_5 \geqslant \theta_0/2$, or the one on the triangles $\triangle A_2A_4A_6$, $\triangle A_3A_5A_8$, $\triangle A_1A_5A_9$ and $\triangle A_4A_5A_7$ for the case that $\angle A_3A_4A_5 < \theta_0/2$, using the values of an interpolated function at the vertices of these triangles. Without loss of generality, we consider only the second case. Then it follows from the standard scaling argument and the standard technique for deriving the finite element error estimates (cf. [4,9]) that

$$\left|w(A_2) - w(A_4)\right|^2 \lesssim \left|\tilde{I}_h w\right|_{1,\triangle A_2A_4A_6}^2 \lesssim |w|_{1,\triangle A_2A_4A_6}^2 + h_*^{2\varepsilon}|w|_{1+\varepsilon,\triangle A_2A_4A_6}^2, \tag{3.19}$$

where $h_* = \mathrm{diam}(\triangle A_2A_4A_6) \leqslant h$. Similar results hold for $|w(A_3) - w(A_5)|$ and $|w(A_5) - w(A_1)|$ with triangles $\triangle A_3A_5A_8$ and $\triangle A_1A_9A_5$, respectively, and more,

$$\begin{aligned} \left|w(A_3) - w(A_4)\right|^2 &\lesssim \left|w(A_3) - w(A_5)\right|^2 + \left|w(A_5) - w(A_4)\right|^2 \\ &\lesssim |w|_{1,\triangle A_3A_5A_8}^2 + |w|_{1,\triangle A_4A_5A_7}^2 + (h^*)^{2\varepsilon}\big(|w|_{1+\varepsilon,\triangle A_3A_5A_8}^2 + |w|_{1+\varepsilon,\triangle A_4A_5A_7}^2\big), \end{aligned} \tag{3.20}$$

where $h^* = \max\{\mathrm{diam}(\triangle A_3A_5A_8),\ \mathrm{diam}(\triangle A_4A_5A_7)\} \leqslant h$.
Note that all the triangles appearing in (3.19) and (3.20) lie within the coarse triangles $\triangle_i$ or $\triangle_{i-1}$, thus it follows from (3.18)–(3.20) and Lemmata 2, 3 and 4 that

$$\begin{aligned} \sum_{i=1}^{m} \sum_{K \in T_h^{OP_i} \setminus \widetilde{T}_h^{OP_i}} |E_h v|_{1,K}^2 &\lesssim \sum_{i=1}^{m} \big\{\|w\|_{1,\triangle_i}^2 + h^{2\varepsilon}\|w\|_{1+\varepsilon,\triangle_i}^2\big\} \\ &\lesssim \sum_{i=1}^{m} \big\{\|E_{2,h}E_{1,h}v\|_{1,\widetilde{\triangle}_i}^2 + h^{2\varepsilon}\|E_{2,h}E_{1,h}v\|_{1+\varepsilon,\widetilde{\triangle}_i}^2\big\} \lesssim \|v\|_{1/2,\Gamma}^2. \end{aligned} \tag{3.21}$$

Now Theorem 2 follows directly from (3.15), (3.14), (3.17) and (3.21). □

4. An application

Energy-preserving explicit extension operators have wide applications in the construction of non-overlapping DDMs and fictitious domain methods, see for example [3,17,20,22]. In this section we present only one application of the operator in the construction of non-overlapping DDMs with inexact subdomain solvers. Consider the model problem:

$$\begin{aligned} -\nabla\cdot\big(\rho(x)\nabla U(x)\big) &= F(x) \quad \text{in } \Omega, \\ U(x) &= 0 \qquad\;\; \text{on } \partial\Omega, \end{aligned} \tag{4.1}$$

where $\Omega \subset R^2$ is a polygon and $\rho(x)$ is piecewise constant in Ω or $\rho(x) \equiv 1$. Without loss of generality, we assume that Ω is of unit diameter (cf. [22]).

First, we decompose Ω into some mutually disjoint open subdomains Ω_i such that

$$\overline{\Omega} = \bigcup_{i=1}^{J} \overline{\Omega}_i. \tag{4.2}$$

When the coefficient $\rho(x)$ is piecewise constant, each subdomain Ω_i is chosen in such a way that $\rho(x)$ equals to the constant ρ_i in Ω_i. Then we refine each Ω_i into smaller triangular elements K of size h such that the union of all elements K in Ω_i, $i = 1, 2, \ldots, J$, forms a quasi-uniform triangulation T_h of Ω. Let V^h be the piecewise linear finite element space defined on T_h:

$$V^h = \big\{ v \in H_0^1(\Omega);\ v|_K \in P_1(K),\ \forall K \in T_h \big\}.$$

Then the finite element approximation for (4.1) is to find $u \in V^h$ such that

$$A(u, v) = (F, v), \quad \forall\, v \in V^h, \tag{4.3}$$

where $(\cdot,\cdot)$ is the L^2-inner product in $L^2(\Omega)$, and

$$A(u, v) = \int_{\Omega} \rho(x) \nabla u \cdot \nabla v \, \mathrm{d}x.$$

Furthermore, we assume that for each subdomain Ω_i there exists an interior point O_i in Ω_i such that Ω_i can be divided into m_i quasi-uniform non-overlapping triangles $\{\tau_{i,j}\}_{j=1}^{m_i}$ of size H, namely, there exist positive constants c and C independent of i, j, h, H, such that each $\tau_{i,j}$ contains (respectively is contained in) a disk of radius cH (respectively CH). In most applications for non-overlapping DDMs, we have $m_i \leqslant 5$ or 6. It is important to note that this partition of Ω_i into $\{\tau_{i,j}\}_{j=1}^{m_i}$ is independent of the triangulation T_h of Ω.

Let the operator A_h on V^h be defined by

$$(A_h u, v)_{0,h,\Omega} = A(u, v), \quad \forall u, v \in V^h,$$

where $(\cdot,\cdot)_{0,h,\Omega}$ denotes the standard L^2-discrete inner product (cf. [22]). For the matrix representation $\widetilde{A}_h$ of A_h (cf. Section 3) and the stiffness matrix $\mathcal{A}_h = (A(\phi_i, \phi_j))$, where $\{\phi_i\}$ are the nodal basis functions of V^h, by the direct calculation we have $\widetilde{A}_h = \frac{1}{h^2}\mathcal{A}_h$.

Corresponding to each subdomain Ω_i, define

$$A_i(u, v) = \int_{\Omega_i} \rho_i \nabla u \cdot \nabla v \, \mathrm{d}x, \quad \forall u, v \in S_0^h(\Omega_i),$$

where

$$S_0^h(\Omega_i)=\{v\in H_0^1(\Omega_i);\ v|_K\in P_1(K),\ \forall K\in\Omega_i\cap T_h\}.$$

As for A_h, the subdomain operator $A_{i,h}$ in $S_0^h(\Omega_i)$ is defined by

$$(A_{i,h}u,v)_{0,h,\Omega_i}=A_i(u,v),\quad \forall u,v\in S_0^h(\Omega_i).$$

Also, we have $\widetilde{A}_{i,h}=\frac{1}{h^2}\mathcal{A}_{i,h}$, where $\widetilde{A}_{i,h}$ denotes the matrix representation of $A_{i,h}$ and $\mathcal{A}_{i,h}$ the stiffness matrix associated with the bilinear form $A_i(\cdot,\cdot)$.

Let Γ be the interface among all the subdomains $\{\Omega_i\}_{i=1}^p$, i.e., $\Gamma=\bigcup_{i=1}^p\Gamma_i$ with $\Gamma_i=\partial\Omega_i\setminus\partial\Omega$, and the restriction of V^h on Γ be denoted by $V^h(\Gamma)$. For $u,v\in V^h(\Gamma)$, let u_H, v_H be the discrete harmonic extensions of u, v (cf. [22]). Define the discrete interface operator S_h on $V^h(\Gamma)$ by

$$\langle S_hu,v\rangle_{0,h,\Gamma}=A(u_H,v_H),\quad \forall u,v\in V^h(\Gamma).$$

We have $\widetilde{S}_h=\frac{1}{h}\mathcal{S}_h$, where $\mathcal{S}_h$ is the usual Schur complement matrix (cf. [22]).

Based on the explicit extension operator of Section 2, we define a global extension operator

$$\mathbf{E}_h: V^h(\Gamma)\to V^h$$

as follows. For any $v\in V^h(\Gamma)$,

$$(\mathbf{E}_hv)(x)=(E_{i,h}v)(x),\quad \forall x\in\overline{\Omega}_i, \tag{4.4}$$

where $E_{i,h}: S^h(\partial\Omega_i)\to S^h(\Omega_i)$ is the explicit extension operator E_h of Section 2 with the domain Ω there replaced by the subdomain Ω_i. Here $S^h(\partial\Omega_i)$ and $S^h(\Omega_i)$ are the restrictions of V^h on $\partial\Omega_i$ and Ω_i, respectively.

Using the assumptions on $\{\Omega_i\}$ and the standard scaling argument, we can easily obtain from the proof of Theorem 2 that

$$|\mathbf{E}_hv|_{1,\Omega_i}\lesssim H^{-1/2}\|v\|_{0,\partial\Omega_i}+|v|_{1/2,\partial\Omega_i}\lesssim|v|_{1/2,\partial\Omega_i},\quad \forall v\in V^h, \tag{4.5}$$

where we have used the basic fact (cf. [4,22]):

$$\inf_{c\in R^1}\|v+c\|_{0,\partial\Omega_i}\lesssim H^{1/2}|v|_{1/2,\partial\Omega_i}.$$

For any $v\in V^h$, we have (cf. [22])

$$\langle S_hv,v\rangle_{0,h,\Gamma}\eqsim\sum_{i=1}^p\rho_i|v|^2_{1/2,\partial\Omega_i},$$

this with (4.5) yields

$$A(\mathbf{E}_hv,\mathbf{E}_hv)\lesssim\langle S_hv,v\rangle_{0,h,\Gamma},\quad \forall v\in V^h(\Gamma).$$

This energy-preserving property of $\mathbf{E}_h$ implies by the fictitious space lemma (cf. Section 8, [22]; or [5, 20]) that if $B_{i,h}$ is a good preconditioner of $A_{i,h}$ and W_h is a good preconditioner of S_h, then

$$B_h=\sum_{i=1}^p I_iB_{i,h}I_i^t+\mathbf{E}_hW_h\mathbf{E}_h^t \tag{4.6}$$

is a good preconditioner for the original operator A_h. Here I_i denotes the natural extension from $S_0^h(\Omega_i)$ into V^h by zero. In fact, the condition number of $B_h A_h$ has the following bound:

$$\mathrm{Cond}(B_h A_h) \lesssim \max\Big\{\max_{1\leqslant i\leqslant p} \mathrm{Cond}(B_{i,h}A_{i,h}),\ \mathrm{Cond}(W_h S_h)\Big\}. \tag{4.7}$$

We refer to [22] for the construction of many interface preconditioners which do not involve any subproblem solvers on subdomains, all these interface preconditioners can be lifted to the preconditioners for the stiffness matrix A_h defined in the global domain Ω, using (4.6). The implementation of $\mathbf{E}_h v$ for $v \in V^h$ can be obtained by (4.4) and Algorithm 1 of Section 2. We now derive the transpose of $\mathbf{E}_h^t : V^h \to V^h(\Gamma)$. For any $v \in V^h$ and $w \in V^h(\Gamma)$, we have

$$\langle \mathbf{E}_h^t v, w\rangle_{0,h,\Gamma} = (v, \mathbf{E}_h w)_{0,h,\Omega}. \tag{4.8}$$

Let $\mu \in V^h$ be a finite element function such that $\mu(x_j) = 1$ if x_j is a node lying in the interior of some subdomain, and $\mu(x_j) = \frac{1}{k}$ if x_j is a common boundary node of k different subdomains from $\{\Omega_i\}_{i=1}^m$. Then we can write using (4.8)

$$\begin{aligned}\langle \mathbf{E}_h^t v, w\rangle_{0,h,\Gamma} &= h^2 \sum_{x_i\in\mathcal{N}_h} v(x_i)\mathbf{E}_h w(x_i) = h^2 \sum_{i=1}^m \sum_{x_j\in\overline{\Omega}_i\cap\mathcal{N}_h} \mu(x_j)v(x_j)\mathbf{E}_h w(x_j)\\ &= \sum_{i=1}^m (\mu v, \mathbf{E}_h w)_{0,h,\Omega} = \sum_{i=1}^m \langle \mathbf{E}_h^t(\mu v), w\rangle_{0,h,\Gamma_i}.\end{aligned} \tag{4.9}$$

Taking $w \in V^h(\Gamma)$ in (4.9) to be a function which vanishes at all the nodes except at the node x_k, where $w(x_k) = 1$, we derive for any node $x_k \in \mathcal{N}_h$,

$$\big(\mathbf{E}_h^t v\big)(x_k) = \sum_i \big(\mathbf{E}_{i,h}^t(\mu v)\big)(x_k),$$

where the summation is taken over all the indices i such that $x_k \in \Gamma_i$. Then the action of $\mathbf{E}_h^t$ can be implemented according to this formula.

Remark 5. In many existing methods (cf. [22]), the discrete harmonic extension operators are used as the extension operators $\mathbf{E}_h$ in (4.6). Thus the action of the preconditioner B_h needs to solve a subproblem on each subdomain exactly, and so is expensive in general. With the explicit extension operator $\mathbf{E}_h$, the action of the preconditioner B_h is much less expensive.

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