MATH 1010A/K 2017-18

University Mathematics

Tutorial Notes II

Ng Hoi Dong

Question

(Q1) A sequence $\{x_n\}$ is defined by $x_0 = 1,\ x_1 = 2$ and $x_n = \dfrac{x_{n-1} + x_{n-2}}{2}$ for $n \geq 2$.

- **(a)** Write down the values of $x_2 - x_1$, $x_3 - x_2$ and $x_4 - x_3$.
- **(b)** For $n = 1, 2, 3, ...$, guess an expression for $x_n - x_{n-1}$ in terms of n and prove it.
- **(c)** Hence find $\lim\limits_{n\to\infty} x_n$.

(Q2) Let $p > 0$ and $p \neq 1$,

$\{a_n\}$ is a sequence of positive numbers defined by $\begin{cases} a_0 = 2 \\ a_n = \dfrac{1}{\sqrt[p]{n}} + \dfrac{1}{p} a_{n-1}, \quad n = 1, 2, 3, ... \end{cases}$.

- **(a)** Prove that $\lim\limits_{n\to\infty} a_n = 0$ if the limit exists.
- **(b)** Using (a), or otherwise,
  - **(i)** if $2 = a_0 < a_1 < a_2 < ...$, show that $\lim\limits_{n\to\infty} a_n$ does not exist.
  - **(ii)** if $a_{k-1} \geq a_k$ for some $k > 1$, show that $a_{n-1} \geq a_n$ for $n \geq k$ and deduce that $\lim\limits_{n\to\infty} a_n = 0$.
- **(c)** Using (a),(b), or otherwise,
  - **(i)** if $0 < p < 1$, show that $\lim\limits_{n\to\infty} a_n$ does not exist.
  - **(ii)** if $p \geq 2$, show that $\lim\limits_{n\to\infty} a_n = 0$.
- **(d)** Using (a),(b), or otherwise,
  - **(i)** Suppose $1 < p < 2$. Prove by mathematical induction that $a_n < \dfrac{2}{p-1}$ for $n \geq 0$.
  - **(ii)** Suppose $1 < p < 2$. Prove that $\lim\limits_{n\to\infty} a_n = 0$.

(Q3) Let $f(x) = \sqrt{\dfrac{x + |x|}{x + 2}}$, $g(x) = \sqrt{x^2 - |x| - 2}$.

Find the maximal domain of f, g (in $\mathbb{R}$).

(Q4) Find values of a and b such that

$$f(x) = \begin{cases} ax + 2b, & x \leq 0, \\ x^2 + 3a - b, & 0 < x \leq 2, \\ 4x - 2b, & x > 2 \end{cases}$$

is continuous at every $x \in \mathbb{R}$.

Answer

(A1) Let $x_0 = 1,\ x_1 = 2$ and $x_n = \dfrac{x_{n-1} + x_{n-2}}{2}$ for $n \geq 2$.

(a) $x_2 = \dfrac{3}{2}, x_3 = \dfrac{7}{4}, x_4 = \dfrac{13}{8}$. Then $x_2 - x_1 = -\dfrac{1}{2}, x_3 - x_2 = \dfrac{1}{4}, x_4 - x_3 = -\dfrac{1}{8}$.

(b) Guess $x_n - x_{n-1} = (-1)^{n-1}\dfrac{1}{2^{n-1}}$.

Let $P(n)$ be the statement that "$x_n - x_{n-1} = (-1)^{n-1}\dfrac{1}{2^{n-1}}$" .

Note that $P(1)$ is true since $x_1 - x_0 = 1 = (-1)^0\dfrac{1}{2^0}$.

Let $k \in \mathbb{Z}$ and $k \geq 2$, assume $P(k)$ is true, i.e. $x_k - x_{k-1} = (-1)^{k-1}\dfrac{1}{2^{k-1}}$.

Consider $n = k + 1$,

$$\begin{aligned} x_{k+1} - x_k &= \frac{x_k + x_{k_1}}{2} - x_k \\ &= -\frac{x_k - x_{k-1}}{2} \\ &= \frac{(-1)^k}{2^k}. \end{aligned}$$

so $P(k + 1)$ is true.

By principal of mathematical induction, $P(n)$ is true for any $n = 1, 2, 3, ...,$

i.e. $x_n - x_{n-1} = (-1)^{n-1}\dfrac{1}{2^{n-1}}$ for any $n = 1, 2, 3, ...$.

(c) Note that $x_n - x_0 = \displaystyle\sum_{i=1}^{n}(x_i - x_{i-1}) = \sum_{i=1}^{n}(-1)^{i-1}\frac{1}{2^{i-1}}$

and $\displaystyle\sum_{i=0}^{\infty}(-1)^{i-1}\frac{1}{2^{i-1}} = \frac{1}{1 - \left(-\frac{1}{2}\right)} = \frac{2}{3}$.

Therefore, $\lim\limits_{n\to\infty} x_n$ exist and $\lim\limits_{n\to\infty} x_n = x_0 + \dfrac{2}{3} = \dfrac{5}{3}$.

(A2) Let $p,\ a_n$ are defined as the question.

(a) If the limit exists, by the definition of a_n,

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{\sqrt[p]{n}} + \frac{1}{p}\lim_{n\to\infty} a_{n-1} = \frac{1}{p}\lim_{n\to\infty} a_n$$
$$\lim_{n\to\infty} a_n = 0 \quad \text{since } p \neq 1$$

(b) Using (a),

(i) Suppose it were true that $\lim\limits_{n\to\infty} a_n$ exists,

Since $\{a_n\}$ is increasing sequence, so (Why?)

$$2 = a_0 \leq a_n \leq \lim_{n\to\infty} a_n = 0 \text{ for any } n = 0, 1, 2, ...$$

Contradiction arises, hence $\lim\limits_{n\to\infty} a_n$ does not exist.

(ii) Let $P(n)$ be the statement that "$a_{k+n-1} \geq a_{k+n}$" .

Since $a_{k-1} \geq a_k$, $P(0)$ is true.

Let l be a nonnegative integer, assume $P(l)$ is true, i.e. $a_{k+l-1} \geq a_{k+l}$, then

$$a_{k+l} - a_{k+l+1} = \left(\frac{1}{\sqrt[p]{k+l}} - \frac{1}{\sqrt[p]{k+l+1}}\right) + \frac{1}{p}(a_{k+l-1} - a_{k+l}) \geq 0$$

Hence, $a_{k+l} \geq a_{k+l+1}$, i.e. $P(l + 1)$ is true.

By the principal of mathematical induction, $P(n)$ is true for any $n = 0, 1, 2, \dots$.
Since $\{a_{k+n-1}\}_{n=1}^{\infty}$ is monotone decreasing and bounded below (by 0),
by Monotone Convergent Theorem, $\lim_{n\to\infty} a_n = \lim_{n\to\infty} a_{n+k-1}$ exist.
Apply (a), $\lim_{n\to\infty} a_n = 0$.

(c) Using (a),(b),

(i) For any $n = 1, 2, 3, \dots$, since $0 < p < 1$, we have

$$a_n = \frac{1}{\sqrt[p]{n}} + \frac{1}{p}a_{n-1} > \frac{1}{p}a_{n-1} > a_{n-1}.$$

Hence, $2 = a_0 < a_1 < a_2 < \dots$, by (b)(i), $\lim_{n\to\infty} a_n$ does not exist.

(ii) Since $p \geq 2$, we have

$$a_1 = \frac{1}{\sqrt[p]{1}} + \frac{1}{p}a_0 = 1 + \frac{2}{p} \leq 2 = a_0.$$

By (b)(ii), $\lim_{n\to\infty} a_n = 0$.

(d) Using (a),(b),

(i) Suppose $1 < p < 2$. Let $P(n)$ be the statement that "$a_n < \frac{2}{p-1}$".
Note that $a_0 = 2 < \frac{2}{p-1}$ since $p - 1 < 1$, hence $P(0)$ is true.
Let k be a nonnegative integer, assume $P(k)$ is true, i.e. $a_k < \frac{2}{p-1}$, then

$$\begin{aligned} a_{k+1} &= \frac{1}{\sqrt[p]{n+1}} + \frac{1}{p}a_k < 1 + \frac{1}{p}\frac{2}{p-1} \\ &= \frac{p^2 - p + 2}{p(p-1)} = \frac{(p-2)(p-1) + 2p}{p(p-1)} \\ &= \frac{p-2}{p} + \frac{2p}{p(p-1)} < \frac{2p}{p(p-1)} \\ &= \frac{2}{p-1}. \end{aligned}$$

Hence $P(k+1)$ is true, by the principal of mathematical induction,
$P(n)$ is true for any $n = 0, 1, 2, \dots$, i.e. $a_n < \frac{2}{p-1}$ for any $n = 0, 1, 2, \dots$.

(ii) By (d)(i), $\{a_n\}$ bounded above.
Suppose it were true that $\{a_n\}$ is strictly increasing, by (b)(i), $\lim_{n\to\infty} a_n$ does not exist,
which is a contradiction with Monotone Convergent Theorem,
hence $\{a_n\}$ is not strictly increasing, by (b)(ii), $\lim_{n\to\infty} a_n = 0$.

(A3) For $f(x) = \sqrt{\frac{x + |x|}{x + 2}}$, since the denominator cannot be 0.

Hence, -2 NOT belongs to the domain of f.

Note $$\frac{x + |x|}{x + 2} = \begin{cases} \frac{x + x}{x + 2} = \frac{2x}{x + 2} & \text{if } x \geq 0 \\ \frac{x - x}{x + 2} = 0 & \text{if } x < 0,\ x \neq -2 \end{cases}.$$

Note $\frac{2x}{x+2}$ always non-negative for any $x \geq 0$.

So f well-defined for any $x \geq 0$. (The expression inside square root need to be non-negative.)

Maximum domain of f is $\mathbb{R} \setminus \{-2\}$.

For $g(x)=\sqrt{x^2-|x|-2}$, Note $x^2-|x|-2=\begin{cases} x^2-x-2=(x-2)(x+1) & \text{if } x\geq 0 \\ x^2+x-2=(x+2)(x-1) & \text{if } x<0 \end{cases}$.

x	$x<-2$	$x=-2$	$-2<x<0$	$x=0$	$0<x<2$	$x=2$	$x>2$
$x^2-\lvert x\rvert-2$	$+$	0	$-$	$-$	$-$	0	$+$

Since the expression inside the square need to be non-negative,

the number between -2 and 2 (not include ± 2) NOT belongs to the domain of g.

Hence, the maximal domain of g is $\mathbb{R}\setminus(-2,2)$. (Or you can write $(-\infty,-2]\cup[2,+\infty)$)

(A4) Note that f is a polynomial when $x<0$, $0<x<2$ or $x>2$.

Hence f is obviously continuous on $\mathbb{R}\setminus\{0,2\}$.

Suppose f is continuous in $\mathbb{R}$ everywhere.

Then $\lim_{x\to 0} f(x)$, $\lim_{x\to 2} f(x)$ exist and $\lim_{x\to 0} f(x)=f(0)$, $\lim_{x\to 2} f(x)=f(2)$.

That means $\lim_{x\to 0^+} f(x)=f(0)=\lim_{x\to 0^-} f(x)$, $\lim_{x\to 2^+} f(x)=f(2)=\lim_{x\to 2^-} f(x)$. Note that

$$\begin{aligned}
\lim_{x\to 0^-} f(x) &= \lim_{x\to 0^-} ax+2b=2b \\
\lim_{x\to 0^+} f(x) &= \lim_{x\to 0^+} x^2+3a-b=3a-b \\
\lim_{x\to 2^-} f(x) &= \lim_{x\to 2^-} x^2+3a-b=4+3a-b \\
\lim_{x\to 2^+} f(x) &= \lim_{x\to 2^+} 4x-2b=8-2b.
\end{aligned}$$

Therefore, we have $\begin{cases} 2b & =3a-b \\ 4+3a-b & =8-2b \end{cases}$. Hence, $a=b=1$.