BASIC GEOMETRY OF HOLOMORPHIC MAPS

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In this short note, we will discuss some basic geometry of mappings defined by holomorphic functions. Write $B(z_0, r)$ for an open ball of radius r centered at z_0, i.e.

$$B(z_0, r) := \{z \in \mathbb{C} \colon |z - z_0| < r\}.$$

Proposition 1. *Suppose $f \colon \Omega \to \mathbb{C}$ is a non-constant holomorphic function on an open connected set $\Omega \subseteq \mathbb{C}$. If $z_0 \in \Omega$, $w_0 = f(z_0)$, and k is the order of vanishing of $f(z) - w_0$ at $z = z_0$, then there exists an open set $U \subset \Omega$, and an open set $V \subset \mathbb{C}$ containing w_0, such that for every $w \in V \setminus \{w_0\}$, there exists exactly k distinct $z_1, \dots, z_k \in U$ such that $f(z_1) = \dots = f(z_k) = w$. In short, we say that f is locally k to 1 near z_0.*

Proof. First notice that there exists a non-zero constant $a_k \in \mathbb{C}$ and some holomorphic function g on Ω, such that

$$f(z) - w_0 = a_k (z - z_0)^k + g(z)$$

for all $z \in \Omega$, and such that $g(z)$ vanishes at z_0 to order $\geq k + 1$. As a result,

$$\lim_{z \to z_0} \frac{g(z)}{a_k (z - z_0)^k} = 0.$$

Hence there exists $\delta > 0$, such that

$$|g(z)| \leq \frac{1}{2} |a_k| |z - z_0|^k$$

for all z with $0 < |z - z_0| \leq \delta$. Now for w close to w_0, we want to find solutions z to the equation $f(z) = w$ with z close to z_0. So we write

$$f(z) - w = a_k (z - z_0)^k + g(z) + (w - w_0).$$

We know how many roots $a_k (z - z_0)^k$ has, and we want to apply Rouché's theorem. So we would estimate the size of $|g(z) + (w - w_0)|$ on the boundary of some curve surrounding z_0, and compare it to $|a_k (z - z_0)^k|$ over that curve.

Let's take that curve to be a circle of radius δ around z_0, where δ is chosen as above. So let $U := B(z_0, \delta)$. Then for z on the boundary of U, we have

$$|g(z) + (w - w_0)| \leq |g(z)| + |w - w_0| \leq \frac{1}{2} |a_k (z - z_0)^k| + |w - w_0|$$

for all z on the boundary of U. This suggests that we take $V := B(w_0, \frac{1}{2} |a_k| \delta^k)$. Then for every $w \in V$, we have

$$|g(z) + (w - w_0)| < |a_k (z - z_0)^k|$$

for all z on the boundary of U. This allows us to apply Rouché's theorem on U to the functions $a_k (z - z_0)^k$ and $a_k (z - z_0)^k + g(z) + (w - w_0) = f(z) - w$: we conclude that for every $w \in V$, the function $f(z) - w$ has exactly k zeroes in U. If $w \neq w_0$, then $z \neq z_0$, and by shrinking δ if necessary, we may assume that $f'(z) \neq 0$ for all $z \in U \setminus \{z_0\}$. In that case, all zeroes of $f(z) - w$ are simple. This shows that for every $w \in V \setminus \{w_0\}$, the function $f(z) - w$ has exactly k distinct zeroes in U, which is our desired conclusion. □

We remark that if $k = 1$, and if $U_0 := U \cap f^{-1}(V)$, then f is a biholomorphism from U_0 to V. More generally, for a general k, if $U_0 := U \cap f^{-1}(V)$, then f is a k-to-1 covering of V by U_0.

Corollary 2. *If f is an injective holomorphic function on an open set $\Omega \subset \mathbb{C}$, then $f'(z) \neq 0$ for every $z \in \Omega$.*

Proof. If $f'(z_0) = 0$ for some $z_0 \in \Omega$, then from the previous proposition, f is locally k to 1 for some $k \geq 1$. So f cannot be injective. □

Proposition 3. *If f is holomorphic on an open set containing a point z_0 and $f'(z_0) \neq 0$, then f preserves angles at z_0.*

Proof. Suppose γ_1 and γ_2 are two curves in the open set with $\gamma_1(0) = \gamma_2(0) = z_0$, and $\gamma_1'(t) \neq 0$, $\gamma_2'(t) \neq 0$ for all t. Let θ be the angle from γ_1 to γ_2 at z_0. Then

$$\frac{\gamma_2'(0)}{|\gamma_2'(0)|} = e^{i\theta} \frac{\gamma_1'(0)}{|\gamma_1'(0)|}.$$

Since $(f \circ \gamma_1)'(0) = f'(z_0)\gamma_1'(0)$, and similarly $(f \circ \gamma_2)'(0) = f'(z_0)\gamma_2'(0)$, from $f'(z_0) \neq 0$, we also get

$$\frac{(f \circ \gamma_2)'(0)}{|(f \circ \gamma_2)'(0)|} = e^{i\theta} \frac{(f \circ \gamma_1)'(0)}{|(f \circ \gamma_1)'(0)|}.$$

Hence the angle from $f \circ \gamma_1$ to $f \circ \gamma_2$ at $f(z_0)$ is also θ. □

In particular, we also have

Proposition 4. *If f is holomorphic on an open set containing a point z_0 and $f'(z_0) \neq 0$, then f preserves orientation at z_0.*

Let's give a direct proof of this proposition using Jacobian determinants (which extends to higher dimensions).

Proof. Let $f(z) = u(x, y) + iv(x, y)$ where $u(x, y) = \operatorname{Re} f(x + iy)$ and $v(x, y) = \operatorname{Im} f(x + iy)$. Then the Jacobian matrix is

$$\begin{pmatrix} \partial_x u & \partial_y u \\ \partial_x v & \partial_y v \end{pmatrix} = \begin{pmatrix} \partial_x u & -\partial_x v \\ \partial_x v & \partial_x u \end{pmatrix}$$

by the Cauchy–Riemann equations. Thus, the determinant of the Jacobian matrix is

$$(\partial_x u)^2 + (\partial_x v)^2 = |f'(z)|^2,$$

which is positive by assumption. This shows that f preserves orientation at z_0. □

Finally, we have the following inverse function theorem in one complex variable:

Proposition 5. *If $f\colon U \to V$ is bijective and holomorphic, then its inverse $g := f^{-1}\colon V \to U$ is also holomorphic.*

In particular, being biholomorphic defines an equivalence relation between domains in $\mathbb{C}$.

Proof. Let $w_0 \in V$, and $z_0 = g(w_0)$. We claim that $g(w)$ is continuous at w_0. Indeed, for any $\varepsilon > 0$, the set $f(B(z_0, \varepsilon))$ is an open set containing w_0, by the open mapping theorem. Hence there exists some $\delta > 0$ such that $B(w_0, \delta) \subseteq f(B(z_0, \varepsilon))$, i.e. $g(B(w_0, \delta)) \subseteq B(z_0, \varepsilon)$. This shows that for any w with $|w - w_0| < \delta$, we have $|g(w) - z_0| < \varepsilon$, i.e. $|g(w) - g(w_0)| < \varepsilon$. Hence $g(w)$ is continuous at w_0.

Now we have shown that $f'(z_0) \neq 0$. Thus

$$\lim_{z \to z_0} \frac{z - z_0}{f(z) - f(z_0)} = \frac{1}{f'(z_0)}.$$

This says for any $\varepsilon > 0$, there exists $\delta > 0$, such that

$$\left| \frac{z - z_0}{f(z) - f(z_0)} - \frac{1}{f'(z_0)} \right| < \varepsilon \tag{1}$$

whenever $0 < |z - z_0| < \delta$. By continuity of $g(w)$ at w_0, there exists $\delta' > 0$ such that $|g(w) - g(w_0)| < \delta$ whenever $|w - w_0| < \delta'$. As a result, whenever $0 < |w - w_0| < \delta'$, we have $0 < |g(w) - z_0| < \delta$, and from the choice of δ as in (1) we have

$$\left| \frac{g(w) - z_0}{f(g(w)) - f(z_0)} - \frac{1}{f'(z_0)} \right| < \varepsilon,$$

i.e.

$$\left| \frac{g(w) - g(w_0)}{w - w_0} - \frac{1}{f'(z_0)} \right| < \varepsilon.$$

This proves that g is differentiable at w_0, and that

$$g'(w_0) = \frac{1}{f'(z_0)}.$$

□