THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
2018-2019 semester 1 MATH4060
Homework 2 solution

1.4 Compatibility with addition together with trichotomy implies exactly one of $\pm z$ is order-positive for nonzero z. Compatibility with multiplication then implies squares of nonzero numbers is order-positive, because $z^2 = (-z)^2$. However $1 = 1^2 \succ 0$ and $-1 = i^2 \succ 0$.

1.5 a. It suffices to show that for each $z \in \Omega_i$, $B(z,r) \subseteq \Omega_i$ if $B(z,r) \subseteq \Omega$. For $i = 1$, points in $B(z,r)$ can be connected to z via the radial path, and hence to w via concatenating with the path between w and z. For $i = 2$, if a point ζ in $B(z,r) \in \Omega_2^C = \Omega_1$, then the existence of the similar path from w through ζ to z implies $z \in \Omega^1$, contradiction.

b. Suppose Ω is pathwise connected but is disconnected by open sets Ω_1 and Ω_2. Let $w_i \in \Omega_i$, $i = 1, 2$ and $z : [0,1] \to \Omega$ such that $z(0) = w_1$ and $z(1) = w_2$. Let $t^* = \sup\{t \in [0,1] : z[0,t) \in \Omega_1\}$. The set where the supremum contains 0 and hence is nonempty; it is also bounded above by 1; so t^* is well defined. By continuity, $0 < t^* < 1$, because 0 and 1 have neighbourhoods mapped by z into the open sets Ω_1 and Ω_2 respectively. Since $z(t^*) \in \Omega$, $z(t^*) \in \Omega_i$ for $i = 1$ or $i = 2$. Again, by continuity, $z(t^* + r) \in \Omega_i$ for $|r| < \varepsilon_0$. Then if $i = 1$, $z[0, t + \varepsilon_0) \in \Omega_1$, contradictory to the maximality of t^*; if $i = 2$, then $z(t - \varepsilon/2) \notin \Omega_1$, contradictory to the minimality (as an upper bound) of t^*. The contradiction then follows.

1.26 By taking difference, it suffices to consider $f = 0$. Take a polygonal path between two points (whose existence is guaranteed by 1.5), and apply fundamental theorem of calculus.

2.4 The integrand is bounded by $e^{-\pi(x^2+2x\Im\xi)}$, whose integral wrt dx converges on $|\Im\xi| \le M$, and hence the integral converges and is holomorphic by Morera's theorem (and Fubini's theorem). By identity theorem, it suffices to show the equation holds for real ξ. Fix $\xi \in \mathbb{R}$.

$$\int_{\mathbb{R}} e^{-\pi x^2} e^{2\pi i x\xi} dx = \int_{\mathbb{R}} e^{-\pi x^2 + 2\pi i x\xi} dx = \int_{\mathbb{R}} e^{-\pi(x-i\xi)^2 - \pi\xi^2} dx = e^{-\pi\xi^2} \int_{\mathbb{R}-i\xi} e^{-\pi z^2} dz$$

The integral on $\mathbb{R} - i\xi$ converges because if $|\Im z| \le |\xi|$

$$|e^{-\pi z^2}| \le e^{\pi\xi^2} e^{\pi(\Re z)^2}. \tag{1}$$

By considering the contour integral on the rectangle with vertices $-R$, R, $R - i\xi$ and $-R - i\xi$, the integral on the vertical sides of which approach 0 as $R \to \infty$ because the of (1), the integral is equal to $\int_{\mathbb{R}} e^{-x^2} dx = 1$. The result follows.

2.5 Since $d(f(z)dz) = \partial_z f dz \wedge dz + \partial_{\bar{z}} f d\bar{z} \wedge dz$, where the first term vanishes because of antisymmetry of wedge product, and the second term vanishes because of complex differentiability. Then

$$\int_T f(z)dz = \int_{\partial(\text{interior of } T)} f(z)dz = \int_{\text{interior of } T} d(f(z)dz) = 0.$$

3.15 For (a), by Cauchy's inequality, $\sup_{B(0,R)} |f^{(k)}| \leq \frac{M2R}{R^k} \leq \frac{2^k AR^k + B}{R^k} \leq 2^k A + 1$ for large R. By Liouville's property, $f^{(k)} \equiv 0$, and hence f is a polynomial of degree at most k. For (d), use (a) on $\exp(f)$ and $k = 0$. Take log afterwards.

3.22 Suppose f is such a function, then the number of zeros in the disc is $\frac{1}{2\pi i} \int_{\partial B(0,1)} f'/f = -1$, which is impossible.

4.1
a. Let $g(x) = f(x+t)$. Then $\hat{g}(\xi) = e^{2\pi i \xi t} \hat{f}(\xi) = 0$. The result then follows from $(A - B)(\xi) = \hat{g}(\xi)$.

b. On the respective domain of A and B, $(x-t)\Im z \leq 0$, and hence $|e^{-2\pi i z(x-t)}| = e^{2\pi(x-t)\Im z} \leq 1$, so both functions are bounded by $\int_{\mathbb{R}} \frac{C}{1+x^2} dx$, and hence they are holomorphic by Morera's theorem and Fubini's theorem. Then F is holomorphic by Morera's theorem. Liouville's property implies F is constant. $|F(in)| \leq \|f\|_\infty/(2\pi n) \to 0$ as $n \to \infty$, hence $F \equiv 0$.

c. For the first equation, $A(0) = F(0) = 0$. For the second, the first implies $\int_a^b f = 0$ for all $a < b$. Since f is continuous, $f \equiv 0$.

4.2 Let $\varepsilon = a - b$. By Cauchy's inequality, for $|x| > 2\varepsilon$, $|x - \operatorname{sgn}(x)\varepsilon| \geq |x|/2$, and hence

$$|f^{(n)}(x+iy)| \leq \frac{1}{2\pi} \frac{[1 + (x - \operatorname{sgn}(x)\varepsilon)]^{-1}}{\varepsilon^n} \leq \left(\frac{4}{2\pi\varepsilon^n}\right) \frac{1}{1+x^2}.$$

4.3 For the first equation, fix $a > 0$ and $\xi \in \mathbb{R}$. Consider the semicircle centred at 0 through R, $-R$ and $-\operatorname{sgn}(\xi) iR$. The integral of $\frac{a}{a^2+x^2} e^{-2\pi i z \xi}$ on the arc converges to 0 because $|e^{-2\pi i z \xi}| \leq 1$ by choice, and $\int \frac{a}{a^2+z^2} dz = O(1/R)$. The residue at $-\operatorname{sgn}(\xi) ia$ is $-\frac{\operatorname{sgn}(\xi)}{2i}$. The result then follows. For the second equation, split the domain of integration to $(-\infty, 0)$ and $(0, \infty)$ and integrate.

4.6 The equation may be derived by applying Poisson summation formula on $f(z) = e^{-2\pi a|z|}$, and by 4.3, $\hat{f}(\xi) = \frac{a}{a^2+\xi^2}$.

$$\begin{aligned}
\sum_{n \in \mathbb{Z}} e^{-2\pi a|n|} &= 1 + 2 \sum_{n \geq 1} e^{-2\pi a n} \\
&= 1 + \frac{2e^{-2\pi a}}{1 - e^{-2\pi a}} \\
&= \frac{1 + e^{-2\pi a}}{1 - e^{-2\pi a}} \\
&= \coth(\pi a)
\end{aligned}$$

4.7 a. Let $f(z) = (\tau + z)^{-k}$. Its Fourier transform may be calculated by considering the integral of $f(z)e^{-2\pi i z\xi}$ on the semicircle through $-R$, R and $-(\mathrm{sgn}\xi)iR$. On the contour, $2\pi(\Im z)\xi \leq 0$ by choice, and the exponential is bounded by 1, and hence the integral on the circular arc is $O(R^{-(k-1)})$. Precisely when $\xi \geq 0$ does the contour encloses the pole $-\tau$, whose residue is $\frac{1}{(k-1)!}\frac{d^{k-1}}{dz^{k-1}}\Big|_{z=-\tau} e^{-2\pi i z\xi} = \frac{(2\pi i)^{k-1}}{(k-1)!}\xi^{k-1}e^{2\pi i\tau\xi}$. Therefore, for $\xi \in \mathbb{R}$, $\hat{f}(\xi) = \chi_{[0,\infty)}(\xi)\frac{(-2\pi i)^k}{(k-1)!}\xi^{k-1}e^{2\pi i\tau\xi}$. The equation then follows from Poisson summation formula.

b. Putting $k = 2$ gives

$$\begin{aligned}
\sum_{n\in\mathbb{Z}} \frac{1}{(\tau+n)^2} &= (-2\pi i)^2 \sum_{n\geq 0} n e^{2\pi i\tau n} \\
&= -2\pi i \frac{d}{dz}\Big|_{z=\tau} \sum_{n\geq 0} e^{2\pi i z n} \\
&= -2\pi i \frac{d}{dz}\Big|_{z=\tau} \frac{1}{1-e^{2\pi i z}} \\
&= (-2\pi i)^2 \frac{e^{2\pi i\tau}}{(1-e^{2\pi i\tau})^2} \\
&= \frac{\pi^2}{\sin^2(\pi\tau)}
\end{aligned}$$

c. Yes, by identity theorem.

additional The change of variable $v = e^u$ gives $I = \int_0^\infty \frac{v^a}{1+v}\frac{dv}{v} = \int_{\mathbb{R}} \frac{e^{ua}}{1+e^u} du$. Consider the integral of $\frac{e^{ua}}{1+e^u}$ on the rectangular contour with vertices $R, R+2\pi i, -R+2\pi i, -R$. The integral on the upper horizontal segment is $-e^{2\pi i a}I$. Since $0 < a < 1$, the integrals on the vertical segments on the right and left are $O(e^{R(a-1)})$ and $O(e^{-aR})$ respectively as $R \to +\infty$, and hence converge to 0. The only pole in the contour is $i\pi$ with residue $\lim_{z\to i\pi} \frac{(z-i\pi)e^{az}}{1+e^z} = -e^{i\pi a}$. Therefore, $I = -\frac{2\pi i e^{i\pi a}}{1-e^{2\pi i a}} = \frac{\pi}{\sin \pi a}$.