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ERRATUM TO "SYZ MIRROR SYMMETRY FOR TORIC CALABI-YAU MANIFOLDS"

Kwokwai Chan, Siu-Cheong Lau & Naichung Conan Leung

Abstract

In this erratum, we give corrected versions of Conjecture 1.1 and Proposition 5.3 of our paper [**1**].

1. A corrected version of [1, Conjecture 1.1]

Conjecture 1.1 (which is the same as Conjecture 5.1) in [**1**] cannot be true as stated because the Picard number l of the toric Calabi-Yau manifold X is strictly less than the dimension of the middle homology $H_n(\check{X}, \mathbb{Z})$ of the mirror $\check{X}$ and hence the cycles $\gamma_1, \ldots, \gamma_l$ cannot form a basis of $H_n(\check{X}, \mathbb{Z})$. The correct conjecture was stated as [**2**, Conjecture 2]:

Conjecture 1 (Replacement of Conjecture 1.1 in [**1**])**.** *The SYZ map is inverse to a mirror map. In other words, there exist integral cycles $\gamma_1, \ldots, \gamma_l$ forming part of an integral basis of $H_n(\check{X}, \mathbb{Z})$ such that*

$$q_a = \exp\left(-\int_{\gamma_a} \check{\Omega}_{\check{q}}\right)$$

for $a = 1, \ldots, l$, where $\check{q} = \phi(q)$ is the SYZ map defined by generating functions of genus zero open Gromov-Witten invariants $n_{\beta_i+\alpha}$.

2. A corrected version of [1, Proposition 5.3]

Proposition 5.3 in [**1**] is not true because the hypothesis in [**1**, Lemma 5.2] is not satisfied for some noncompact divisors $\mathscr{D}_i$ of a toric Calabi-Yau manifold X. This result and its proof should be replaced by the following:

Proposition 2 (Replacement of Proposition 5.3 in [**1**])**.** *Suppose that the support $|\Sigma|$ of the fan defining X is a cone over a convex polytope $\mathcal{P}$ in the affine hyperplane $\{\underline{\nu} = 1\} \subset N_{\mathbb{R}}$. Then for any primitive generator $v_i \in N$ which is a corner of the convex polytope, we have $n_{\beta_i+\alpha} = 0$ for all $\alpha \in H_2^{\text{eff}}(X, \mathbb{Z}) - \{0\}$.*

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Proof. Suppose that the moduli space $\mathcal{M}_1(\beta_i + \alpha)$ of disks is non-empty for some $\alpha \in H_2^{\mathrm{eff}}(X, \mathbb{Z}) - \{0\}$. Then $\alpha \neq 0$ is realized by a non-constant rational curve Q in X. Since X is Calabi-Yau, [**1**, Lemma 5.2] says that Q must lie inside the toric divisors $\bigcup_{i=0}^{m-1} \mathscr{D}_i$. Now Q has non-empty intersection with the holomorphic disk representing $\beta_i \in \pi_2(X, \mathbf{T})$ for a generic toric fiber $\mathbf{T}$. This implies Q has at least one sphere component C lying inside the divisor $\mathscr{D}_i$, which has non-empty intersection with the open toric orbit $(\mathbb{C}^\times)^{n-1} \subset \mathscr{D}_i$ of the divisor. We will prove that the divisor $\mathscr{D}_i$ cannot support such a holomorphic sphere C.

The toric divisor itself $\mathscr{D}_i \subset X$ is a toric manifold, whose fan $\Sigma_{\mathscr{D}_i}$ is given by the quotient of Σ by the v_i-direction. If v_i is a corner of the polytope $\mathcal{P}$, then the support $|\Sigma_{\mathscr{D}_i}|$ is a convex polytope with one of its corners being the origin. Thus there exists a half space defined by $\{\nu > 0\} \subset (N/\langle v_i \rangle)_{\mathbb{R}}$ for some $\nu \in M^{\perp v_i}$ containing $|\Sigma_{\mathscr{D}_i}|$. Then the function on $\mathscr{D}_i$ corresponding to ν is holomorphic and takes zeroes on all the toric divisors of $\mathscr{D}_i$. Thus the hypothesis of [**1**, Lemma 5.2] is satisfied, which implies that any non-constant holomorphic sphere in $\mathscr{D}_i$ must lie entirely in the toric divisors of $\mathscr{D}_i$. This contradicts the requirement that the holomorphic sphere C has non-empty intersection with the open toric orbit $(\mathbb{C}^\times)^{n-1} \subset \mathscr{D}_i$. q.e.d.

Example 3. Let $X = K_Y$ be the total space of the canonical line bundle K_Y over a toric Fano manifold Y. The fan of X is the cone over the fan of Y. The polytope $\mathcal{P}$ in this case contains only one interior lattice point, and all boundary lattice points are vertices of the polytope. Thus all non-compact divisors $\mathscr{D}_i$ correspond to vertices of $\mathcal{P}$, and so the open Gromov-Witten invariant $n_{\beta_i+\alpha} = 0$ for all $\alpha \in H_2^{\mathrm{eff}}(X, \mathbb{Z}) - \{0\}$. In particular, all other results in [**1**] and those in [**2**] are unaffected by the above replacement of [**1**, Proposition 5.3].

Example 4. Consider $X = K_{\mathbb{F}_2}$, the total space of the canonical line bundle over the Hirzebruch surface $\mathbb{F}_2$, which is not Fano. The fan of $\mathbb{F}_2$ consists of primitive generators $v_1 = (1,1)$, $v_2 = (0,1)$, $v_3 = (-1,1)$, and $v_4 = (0,-1)$. The fan of $X = K_{\mathbb{F}_2}$ is the cone over the fan of $\mathbb{F}_2$. In this case v_2 is a boundary lattice point of $\mathcal{P}$ which is not a vertex. Thus there can be stable disks whose disk components are the basic disk β_2: n_{β_2+ke} is non-trivial for $k \in \mathbb{N}$, where e is the (-2)-curve of $\mathbb{F}_2$ embedded into X.

References

[1] K. Chan, S.-C. Lau & N.C. Leung, *SYZ mirror symmetry for toric Calabi-Yau manifolds*, J. Differential Geom. **90** (2012), no. 2, 177–250, MR 2899874, Zbl pre06051255.

[2] K. Chan, S.-C. Lau & H.-H. Tseng, *Enumerative meaning of mirror maps for toric Calabi-Yau manifolds*, Adv. Math. **244** (2013), 605–625, MR 3077883, Zbl pre06264346.

Department of Mathematics
The Chinese University of Hong Kong
Shatin, Hong Kong
E-mail address: kwchan@math.cuhk.edu.hk

Department of Mathematics
Harvard University
One Oxford Street
Cambridge, MA 02138
E-mail address: s.lau@math.harvard.edu

Department of Mathematics and
The Institute of Mathematical Sciences and
The Chinese University of Hong Kong
Shatin, Hong Kong
E-mail address: leung@math.cuhk.edu.hk