A GLIMPSE OF THE SYZ CONJECTURE AND RELATED DEVELOPMENTS

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1. Mirror symmetry before SYZ

Calabi-Yau manifolds are important in mathematics for many reasons: they give examples of Einstein manifolds whose metrics are Ricci-flat but not flat, and they occupy a special position in algebraic geometry as a class of varieties with Kodaira dimension 0. But when Yau [177, 178] gave his celebrated solution of the Calabi conjecture in 1976, no one had expected that these manifolds were also going to play such an indispensable role in physics, or more precisely, in *string theory* [13] – a candidate for unifying general relativity and quantum field theory, whose development in turn lead to the discovery of a mathematical phenomenon called *mirror symmetry* that has generated a huge amount of research and drastically influenced many branches of mathematics.

The story began in the late 1980's when Dixon [44] and Lerche-Vafa-Warner [119] discovered that string theory might not distinguish Calabi-Yau manifolds, or more precisely, that two *different* Calabi-Yau manifolds, when used as compactifications of the hidden extra dimensions of spacetime, could give rise to *equivalent* string theories. This surprising idea was soon realized by Greene-Plesser [67] and Candelas-Lynker-Schimmrigk [14] who independently found nontrivial examples of such pairs of Calabi-Yau manifolds; these are called *mirror pairs* because two such manifolds exhibit an interchange of Hodge numbers.

Mathematicians became really interested in mirror symmetry when it was exploited by Candelas, de la Ossa, Green and Parkes in a remarkable calculation of the numbers of rational curves on a quintic 3-fold in $\mathbb{P}^4$ [12], which solved a long-standing problem in enumerative geometry and caught much attention from algebraic geometers at that time.

This triggered the development of important subjects like Gromov-Witten theory, and eventually lead to independent proofs of the *mirror theorems* by Givental [63, 64] and Lian-Liu-Yau [125, 126, 127, 128], which in particular showed that the calculations by Candelas et al. are mathematically correct. This marked the first milestone in the mathematical study of mirror symmetry. The next question is how to understand mirror symmetry in an intrinsic and mathematical way.

The first such formulation was Kontsevich's *Homological Mirror Symmetry* (HMS) conjecture, proposed in his 1994 ICM address [111]. In physics terminology, a Calabi-Yau manifold X determines two topological string theories: the *A-model* and *B-model*, which are controlled by the symplectic and complex geometry of X respectively [169, 175]. From this perspective, mirror symmetry can be understood as an isomorphism between the A-model (symplectic geometry) of X and the B-model (complex geometry) of its mirror $\check{X}$, and vice versa. The above enumerative predictions about the quintic 3-fold is one of many interesting manifestations of this bigger picture.

Kontsevich's HMS conjecture formulates mirror symmetry succinctly as an equivalence between the Fukaya category of Lagrangian submanifolds in X (A-model) and the derived category of coherent sheaves on the mirror $\check{X}$ (B-model). His conjecture is both deep and elegant, and is expected to imply the enumerative predictions by mirror symmetry. Nevertheless, it does not indicate how such an equivalence can be found, nor does it tell us how to construct the mirror of a given Calabi-Yau manifold.

2. Formulation of the SYZ conjecture

In 1996, Strominger, Yau and Zaslow [163] made a ground-breaking proposal which revealed the intimate relation between a pair of mirror Calabi-Yau manifolds in a geometric manner:

Conjecture 2.1 (The SYZ conjecture [163])**.** *Suppose that X and $\check{X}$ are Calabi-Yau manifolds mirror to each other. Then*

(i) *both X and $\check{X}$ admit special Lagrangian torus fibrations with sections $\mu : X \to B$ and $\check{\mu} : \check{X} \to B$ over the same base:*

$$\begin{tikzcd} X \arrow[dr, "\mu"'] & & \check{X} \arrow[dl, "\check{\mu}"] \\ & B & \end{tikzcd}$$

which are fiberwise dual to each other in the sense that if the fibers $\mu^{-1}(b) \subset X$ and $\check{\mu}^{-1}(b) \subset \check{X}$ over $b \in B$ are nonsingular, then they are dual tori; and

(ii) *there exist fiberwise Fourier–type transforms responsible for the interchange between symplectic-geometric (resp. complex-geometric) data on X and complex-geometric (resp. symplectic-geometric) data on $\check{X}$.*

In a nutshell, this says that the mysterious mirror phenomenon can be understood simply as a Fourier transform, known as *T-duality*. This remarkable and far-reaching conjecture not only provides a beautiful geometric explanation to mirror symmetry, but also suggests a concrete mirror construction, namely, a mirror of any given Calabi-Yau manifold X is given by fiberwise dualizing a special Lagrangian torus fibration on X. It immediately became a major approach in the mathematical study of mirror symmetry (the other being Kontsevich's HMS conjecture), and has attracted a lot of attention from both mathematicians and physicists.

Let us briefly review the heuristic reasoning behind the SYZ conjecture. First of all, a key idea in string theory is the existence of *Dirichlet branes*, or D-branes. Physical arguments suggest that D-branes in the B-model (or simply *B-branes*) are coherent sheaves over complex subvarieties while those in the A-model (*A-branes*) are special Lagrangian submanifolds equipped with flat connections. As mirror symmetry predicts an equivalence between the A-model of X and the B-model of $\check{X}$, the moduli space of an A-brane on X should be identified with the moduli space of the mirror B-brane on $\check{X}$.

Note that a point in $\check{X}$ is a B-brane and the moduli space is $\check{X}$ itself, so it should be identified with the moduli space of a certain A-brane (L, ∇) on X, where

$L \subset X$ is a special Lagrangian submanifold and ∇ is a flat $U(1)$-connection on L. Since $\check{X}$ is swept by its points, X should be swept by deformations of L as well. Now McLean's theorem [136] tells us that the moduli space of a special Lagrangian submanifold $L \subset X$ is unobstructed and modeled on $H^1(L;\mathbb{R})$, while the moduli space of flat $U(1)$-connections (modulo gauge) on L is given by $H^1(L;\mathbb{R})/H^1(L;\mathbb{Z})$. So in order to match the dimensions, we should have $\dim H^1(L;\mathbb{R}) = \dim_{\mathbb{C}} \check{X} = n$, and hence X should admit a special Lagrangian torus fibration

$$\mu : X \to B.$$

Furthermore, the manifold $\check{X}$ itself may be viewed as a B-brane whose moduli space is a singleton, and it intersects each point in $\check{X}$ once, so the corresponding A-brane (L, ∇) should give a special Lagrangian section σ to μ with $H^1(\sigma;\mathbb{R}) = 0$. In particular, the base B should have first Betti number $b_1 = 0$.

Applying the same argument to $\check{X}$ yields a special Lagrangian torus fibration with section

$$\check{\mu} : \check{X} \to \check{B}.$$

For a torus fiber $L_b := \mu^{-1}(b) \subset X$, its dual $L_b^\vee$ can be viewed as the moduli space of flat $U(1)$-connections on L which, under mirror symmetry, correspond to points in $\check{X}$. This shows that $L_b^\vee$ is a submanifold in $\check{X}$. More elaborated arguments show that $L_b^\vee$ should be identified with a special Lagrangian torus fiber of $\check{\mu}$, and we deduce that μ and $\check{\mu}$ are fibrations over the same base which are fiberwise dual to each other.

Notice that the above gives a transform carrying the A-branes (L, ∇) where L is a fiber of μ to points (as B-branes) in $\check{X}$. This is an instance of so-called *SYZ transforms*, which are Fourier–type transforms mapping symplectic-geometric data on X to complex-geometric data on $\check{X}$.

In the original SYZ paper [163], it was observed that Ricci-flat metrics on the mirror $\check{X}$ should behave differently from the *semi-flat Calabi-Yau metrics*, constructed earlier by Greene-Shapere-Vafa-Yau [68]. Therefore, the SYZ mirror construction should in general be modified by *instanton or quantum corrections*. As we shall see, a key step in the investigation of mirror symmetry is to understand these corrections, which should come from higher Fourier modes of the SYZ transforms.

3. Semi-flat SYZ

When there are no singular fibers in the special Lagrangian torus fibrations, the SYZ construction can be worked out nicely. Firstly, McLean's classic results [136] give us two naturally defined *integral affine structures*[1] on the base manifold B of a special Lagrangian torus fibration $\mu : X \to B$: the *symplectic* and *complex* affine structures. More specifically, a normal vector field v to a fiber $L_b := \mu^{-1}(b)$ determines a 1-form $\alpha := -\iota_v \omega \in \Omega^1(L_b;\mathbb{R})$ and an $(n-1)$-form $\beta := \iota_v \mathrm{Im}\,\Omega \in \Omega^{n-1}(L_b;\mathbb{R})$, where ω and Ω are the Kähler and holomorphic volume form on X respectively. McLean [136] proved that the corresponding deformation is special Lagrangian if and only if both α and β are closed. By identifying TB with $H^1(L_b;\mathbb{R})$ (resp. $H^{n-1}(L_b;\mathbb{R})$) using the cohomology class of α (resp. β), we get the symplectic (resp. complex) affine structure on B. This also gives us the *McLean metric* $g(v_1, v_2) := -\int_{L_b} \iota_{v_1}\omega \wedge \iota_{v_2}\mathrm{Im}\,\Omega$ on B.

[1] An integral affine structure on a manifold is an atlas of charts whose transition maps are all integral affine linear transformations.

Hitchin used these structures and the *Legendre transform* to illustrate the SYZ conjecture in his beautiful paper [95]. Let $x_1, \ldots, x_n$ be local coordinates on B with respect to the symplectic affine structure. Then the McLean metric can be written as the Hessian of a convex function ϕ on B, i.e. $g\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right) = \frac{\partial^2 \phi}{\partial x_i \partial x_j}$. Setting $\check{x}_i := \partial\phi/\partial x_i$ $(i = 1, \ldots, n)$ gives precisely the coordinates on B with respect to the complex affine structure, and if

$$\check{\phi} := \sum_{i=1}^{n} \check{x}_i x_i - \phi(x_1, \ldots, x_n)$$

is the Legendre transform of ϕ, then we have $x_i = \partial\check{\phi}/\partial\check{x}_i$ and $g\left(\frac{\partial}{\partial \check{x}_i}, \frac{\partial}{\partial \check{x}_j}\right) = \frac{\partial^2 \check{\phi}}{\partial \check{x}_i \partial \check{x}_j}$.

If the fibration $\mu : X \to B$ admits a Lagrangian section, a theorem of Duistermaat [45] will give us global *action-angle coordinates* so that symplectically we have

$$X = T^*B/\Lambda^\vee,$$

where the lattice $\Lambda^\vee \subset T^*B$ is locally generated by $dx_1, \ldots, dx_n$, and ω can be identified with the canonical symplectic form

$$\omega = \sum_{i=1}^{n} dx_i \wedge du_i$$

on $T^*B/\Lambda^\vee$. Here $u_1, \ldots, u_n$ are the fiber coordinates on T^*B.

Then by the SYZ conjecture, the mirror of X should be the fiberwise dual of μ, i.e.

$$\check{X} := TB/\Lambda,$$

where the lattice $\Lambda \subset TB$ is locally generated by $\partial/\partial x_1, \ldots, \partial/\partial x_n$. Note that $\check{X}$ is naturally a complex manifold with holomorphic coordinates given by $z_i := \exp(x_i + \mathbf{i} y_i)$, where $y_1, \ldots, y_n$ are fiber coordinates on TB dual to $u_1, \ldots, u_n$. X is equipped with the holomorphic volume form

$$\check{\Omega} := d \log z_1 \wedge \cdots \wedge d \log z_n.$$

There is an explicit fiberwise Fourier–type transform, called the *semi-flat SYZ transform* $\mathcal{F}^{\text{semi-flat}}$, that carries $\exp \mathbf{i}\omega$ to $\check{\Omega}$ (see e.g. [23, Section 2]). Many other details on semi-flat SYZ mirror symmetry were worked out by Leung in [120].

If we now switch to the complex affine structure on B, we obtain a symplectic structure on $\check{X}$ compatible with its complex structure so that the mirror $\check{X}$ becomes a Kähler manifold. Furthermore, if the function ϕ above satisfies the *real Monge-Ampère equation*

$$\det\left(\frac{\partial^2 \phi}{\partial x_i \partial x_j}\right) = \text{constant},$$

we get T^n-invariant Ricci-flat metrics on both X and $\check{X}$. The McLean metric on B is then called a *Monge-Ampère metric* and B is called a *Monge-Ampère manifold.* This links mirror symmetry to the study of real Monge-Ampère equations and affine Kähler geometry, where Cheng and Yau had made substantial contributions [35, 36, 37] way before mirror symmetry was discovered. It turned out that the construction of Monge-Ampère metrics on affine manifolds with singularities is a very difficult question. The highly nontrivial works of Loftin-Yau-Zaslow [131, 132] constructed such metrics near the "Y" vertex, a typical type of singularity in the 3-dimensional case. But besides this, not much is known.

4. Constructing SYZ fibrations

Right after the introduction of the SYZ conjecture in 1996, a great deal of effort was devoted to finding examples of special Lagrangian torus fibrations, or *SYZ fibrations*, on Calabi-Yau manifolds.[2] Zharkov [182] first constructed *topological* torus fibrations on Calabi-Yau hypersurfaces in a smooth projective toric variety $\mathbb{P}_\Delta$, which includes the important quintic 3-fold example. He obtained his fibrations by deforming the restriction of the moment map on $\mathbb{P}_\Delta$ to the boundary $\partial\Delta$ of the moment polytope to a nearby smooth Calabi-Yau hypersurface.

Applying similar ideas and a gradient-Hamiltonian flow, W.-D. Ruan found *Lagrangian* torus fibrations on quintic 3-folds in a series of papers [151, 152, 153]. He also carried out a nontrivial computation of the monodromy of the fibrations, which was later used by Gross [72] to work out a topological version of SYZ mirror symmetry for Calabi-Yau manifolds. On the other hand, Mikhalkin [137] produced smooth torus fibrations on hypersurfaces in toric varieties by using tools from tropical geometry.

In general, constructing special Lagrangian submanifolds is a very difficult problem. One promising approach is by *mean curvature flow*. Thomas [164] formulated a notion of stability for classes of Lagrangian submanifolds with Maslov index zero in a Calabi-Yau manifold, and it should be mirror to the stability of holomorphic vector bundles. Thomas-Yau [165] conjectured that there should exist a unique special Lagrangian representative in a Hamiltonian isotopy class if and only if the class is stable, and that such a representative should be obtained by mean curvature flow, the long time existence of which should hold. They further proposed a Jordan-Hölder–type decomposition for special Lagrangian submanifolds and related this to formation of singularities in mean curvature flow.

These proposals and conjectures have a big influence on the development of Calabi-Yau geometry and the SYZ conjecture, and a lot of advances in this area have been seen: [32, 33, 104, 118, 140, 141, 143, 158, 159, 160, 170, 171, 172, 173]; see [174] and [142] and references therein for more details. Unfortunately, despite so much effort, existence of special Lagrangian torus fibrations on the quintic 3-fold is still unknown.

In contrast, noncompact examples of special Lagrangian fibrations are much easier to come by. Harvey and Lawson's famous paper on calibrated geometries [94] gave the simplest of such examples: the map defined by

$$\begin{gathered} f : \mathbb{C}^3 \to \mathbb{R}^3, \\ (z_1, z_2, z_3) \mapsto \left(\operatorname{Im}(z_1 z_2 z_3), |z_1|^2 - |z_2|^2, |z_1|^2 - |z_3|^2\right), \end{gathered}$$

is a special Lagrangian fibrations with fibers invariant under the diagonal T^2-action on $\mathbb{C}^3$. This was later largely generalized by independent works of Goldstein [65] and Gross [71], who constructed explicit special Lagrangian torus fibrations on any *toric* Calabi-Yau n-fold. The discriminant loci of these examples are of real codimension two and can be described explicitly.

[2] In [69, 70], instead of constructing such fibrations, Gross assumed their existence and deduced interesting consequences which were predicted by mirror symmetry.

Another set of noncompact examples, which has a historic impact on the development of SYZ mirror symmetry and special Lagrangian geometry, was discovered by Joyce [103]. It was once believed that special Lagrangian fibrations would always be smooth and hence have codimension two discriminant loci. But the examples of Joyce indicated that this is unlikely the case. What he constructed are explicit S^1-invariant special Lagrangian fibrations that are only *piecewise smooth* and have *real codimension one discriminant loci*. The set of singular points of such a fibration is a Riemann surface whose amoeba-shaped image gives the codimension one discriminant locus. Joyce argued that his examples exhibited the generic behavior of discriminant loci of special Lagrangian fibrations.

This important work of Joyce not only deepens our understanding of possible singularities of special Lagrangian fibrations, but also forces us to rethink about the SYZ conjecture. Originally, a mirror pair of Calabi-Yau manifolds X and $\check{X}$ are expected to have special Lagrangian torus fibrations to the same base B and share the same discriminant loci. Now Joyce's examples suggest that while the discriminant locus on one side may be of codimension two, that on the mirror side can well be of codimension one. The best one can hope for is that as one approaches the large complex structure limits, the discriminant loci on both sides converge to the same codimension two subset in B.

More precisely, let $\mathfrak{X} \to D$ and $\check{\mathfrak{X}} \to D$ be maximally unipotent degenerations of Calabi-Yau manifolds mirror to each other, where D is the unit disk and $0 \in D$ are the large complex structure limits on both sides. We choose a sequence $\{t_i\} \subset D$ converging to 0, and let $g_i, \check{g}_i$ be Ricci-flat metrics on $\mathfrak{X}_{t_i}, \check{\mathfrak{X}}_{t_i}$ respectively, normalized to have fixed diameters C. Then a limiting version of the SYZ conjecture can be expressed as:

(i) there are convergent subsequences of $(\mathfrak{X}_{t_i}, g_i)$ and $(\check{\mathfrak{X}}_{t_i}, \check{g}_i)$ converging (in the Gromov-Hausdorff sense) to metric spaces (B_∞, d_∞) and $(\check{B}_\infty, \check{d}_\infty)$ respectively;
(ii) the spaces B_∞ and $\check{B}_\infty$ are affine manifolds with singularities which are both homeomorphic to S^n;
(iii) outside a real codimension 2 locus $\Gamma \subset B_\infty$ (resp. $\check{\Gamma} \subset \check{B}_\infty$), d_∞ (resp. $\check{d}_\infty$) is induced by a Monge-Ampère metric; and
(iv) the Monge-Ampère manifolds $B_\infty \setminus \Gamma$ and $\check{B}_\infty \setminus \check{\Gamma}$ are Legendre dual to each other.

This was proposed independently by Gross-Wilson [92] and Kontsevich-Soibelman [114]. In fact, the general question of understanding the limiting behavior of Ricci-flat metrics was raised by Yau in his famous lists of open problems [179, 180]. Motivated by the SYZ picture of mirror symmetry, this question has been studied extensively in the last 15 years, and substantial progress has been made by Gross-Wilson [92], Tosatti [166, 167], Ruan-Zhang [154], Zhang [181], Rong-Zhang [149, 150] and more recently, Gross-Tosatti-Zhang [91, 90].

The metric spaces B_∞ and $\check{B}_\infty$ should be viewed as limits of bases of SYZ fibrations on the Calabi-Yau families. Applying the above picture, one may try to construct the mirror of a maximally unipotent degeneration of Calabi-Yau manifolds $\mathfrak{X} \to D$ by first identifying the Gromov-Hausdorff limit B_∞, taking the Legendre dual $\check{B}_0$ of $B_\infty \setminus \Gamma$, and then compactifying the quotient $\check{X}_0 := T\check{B}_0/\Lambda$.

Unfortunately this approach is *never* going to work because the semi-flat complex structure on $\check{X}_0$ is not globally defined because of nontrivial monodromy of the affine structure around the singularities in $\check{B}_\infty$. To get the correct mirror, we need to deform the complex structure of $\check{X}_0$ by quantum corrections from holomorphic disks, as suggested by SYZ.

5. SYZ for compact Calabi-Yau manifolds

It is believed that the symplectic structure that we get using semi-flat SYZ mirror symmetry can naturally be compactified to give a global symplectic structure on the mirror. Indeed, Castaño-Bernard and Matessi [15] have shown that the topological compactifications constructed by Gross in [72] can be endowed with symplectic structures, thus producing pairs of compact symplectic 6-folds which are homeomorphic to known mirror pairs of Calabi-Yau 3-folds, including the quintic 3-fold and its mirror, and equipped with Lagrangian torus fibrations whose bases are Legendre dual integral affine manifolds with singularities.

On the other hand, as we mentioned above, the original SYZ proposal [163] pointed out that Ricci-flat metrics on the mirror should differ from semi-flat Calabi-Yau metrics [68] by instanton corrections coming from holomorphic disks whose boundaries wrap non-trivially around fibers of an SYZ fibration. Since a metric on the mirror is determined uniquely by its symplectic and complex structures, it is natural to expect that the instanton corrections can be used to perturb the mirror complex structure. This is the key idea underlying the SYZ conjecture. As holomorphic disks can be glued to give holomorphic curves, this explains why mirror symmetry can be used to solve enumerative problems.

Given an affine manifold with singularities B, let $\Gamma \subset B$ and $B_0 = B \setminus \Gamma$ be its singular and smooth loci respectively. What we want is the construction of a complex manifold X as a (partial) compactification of a small deformation of $X_0 := TB_0/\Lambda$. This is called the *reconstruction problem* in mirror symmetry.

The problem was first studied by Fukaya in [53] where he attempted to find suitable perturbations by directly solving the Maurer-Cartan equation that governs the deformations of complex structures on X_0. In the two-dimensional case, his heuristic arguments showed that the desired perturbations should come from gradient flow trees in B emanating from Γ. The latter should come from limits of holomorphic disks bounding Lagrangian torus fibers of an SYZ fibration when one approaches a large complex structure limit. Fukaya made a series of intriguing conjectures explaining how quantum corrections are modifying the mirror complex structure. Nevertheless the analysis required to make his intuitively clear picture rigorous seemed out of reach.[3]

Kontsevich-Soibelman [115] got around the analytic difficulties in Fukaya's arguments by working with rigid analytic spaces. They started with an integral affine structure on S^2 with 24 singular points such that the monodromy of the affine structure around each singular point is $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, and managed to reconstruct a non-Archimedean analytic $K3$ surface. The basic idea is to attach an automorphism to each gradient flow line in Fukaya's construction, and use them to modify the gluing between charts on the mirror, thereby canceling the effect of the non-trivial monodromy of the affine structure around the discriminant locus.

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[3] In a very recent work [25], Fukaya's program has been realized by making use of the relation between Witten-Morse theory and de Rham theory developed in [31]. In particular, this gives a new geometric interpretation of scattering diagrams.

A crucial step in their argument is a key lemma showing that when two gradient flow lines intersect, new lines attached with automorphisms can always be added so that the composition around each intersection point is the identity. This is called a *scattering diagram*, which ensures that the composition of automorphisms attached to lines crossed by a path is independent of the path chosen, so that the modified gluings are consistent.

At around the same time, Gross and Siebert launched their spectacular program [84, 85, 86, 73, 87] aiming at an algebraic-geometric approach to the SYZ conjecture. Motivated by the limiting version of the SYZ conjecture we discussed in the previous section and the observation by Kontsevich that the Gromov-Hausdorff limit will be roughly the dual intersection complex of the degeneration, they formulated an *algebraic-geometric SYZ procedure* to construct the mirror.

Starting with a toric degeneration of Calabi-Yau manifolds, the first step is to construct the dual intersection complex. Then one takes its (discrete) Legendre transform, and tries to reconstruct the mirror toric degeneration of Calabi-Yau manifolds from this Legendre dual. In this way, they can completely forget about SYZ fibrations. The claim is that all relevant information is encoded in the tropical geometry of the dual intersection complex, which is an integral affine manifold with singularities and plays the role of the base of an SYZ fibration.

Using the above key lemma of Kontsevich and Soibelman, together with many new ideas such as the use of log structures and techniques from tropical geometry, Gross and Siebert eventually succeeded in giving a solution to the reconstruction problem in *any* dimension [87]. Given an integral affine manifold with singularities satisfying certain assumptions and equipped with some additional structures like a polyhedral decomposition, they constructed a toric degeneration of Calabi-Yau manifolds which can be described explicitly and canonically via tropical trees in B. An important feature of their construction is that the Calabi-Yau manifolds constructed are defined over $\mathbb{C}$, instead of being rigid analytic spaces.

Now the goal is to acquire a more conceptual understanding of mirror symmetry by passing through the tropical world. On the B-side, Gross and Siebert conjectured that the deformation parameter in their construction is a canonical coordinate[4] and period integrals of the family of Calabi-Yau manifolds can be expressed in terms of tropical disks in B. Evidences were first provided in the local cases, e.g. the local $\mathbb{P}^2$ example in [87, Remark 5.1].

On the A-side, one would like to understand the Gromov-Witten theory of a smooth fiber in terms of that of the central fiber of a toric degeneration. The recent independent works of Abramovich-Chen [34, 4] and Gross-Siebert [89] developed the theory of *log Gromov-Witten invariants*, generalizing previous works of Li-Ruan [123], Ionel-Parker [101, 102], and Jun Li [124] on relative Gromov-Witten theory. This constitutes a significant step towards the ultimate goal. If one can further prove a general correspondence theorem between tropical and holomorphic curves/disks, in the same vein as works of Mikhalkin [138, 139], Nishinou-Siebert [147] and Nishinou [146, 145], then we can connect the A-side (i.e. Gromov-Witten theory) of a Calabi-Yau manifold to the tropical world.

Albeit much work needs to be done, this lays out a promising and beautiful picture to explain the geometry of mirror symmetry via tropical geometry. We

[4]This was recently confirmed by Ruddat-Siebert [155]

refer the reader to the nice survey articles of the inventors [88, 75] for an overview of the Gross-Siebert program.

6. SYZ for noncompact Calabi-Yau manifolds

The lack of examples of special Lagrangian torus fibrations is one main obstacle in implementing the original SYZ proposal for compact Calabi-Yau manifolds (and perhaps one of the main reasons why Gross and Siebert attempted to develop an algebraic-geometric version). But there are plenty of noncompact examples where one can find explicit special Lagrangian torus fibrations, such as those constructed by Goldstein [65] and Gross [71] in the case of toric Calabi-Yau manifolds.

Moreover, open Gromov-Witten invariants which count stable maps from open Riemann surfaces to the manifold are well-defined in the toric case by works of Fukaya-Oh-Ohta-Ono [57, 58, 59] (and more generally in the S^1-equivariant case by the work of Liu [130]). So it makes perfect sense to carry out the SYZ proposal directly for toric Calabi-Yau manifolds, without retreating to the tropical world and an algebraic-geometric version.

This brings us to the realm of *local mirror symmetry*, which is derived from mirror symmetry for compact Calabi-Yau manifolds by taking certain limits in the Kähler and complex moduli spaces [107]. Since this mirror symmetry provides many interesting examples and has numerous applications, it has attracted a lot of attention from both physicists and mathematicians [121, 38, 97, 71, 72, 109, 66, 99, 100, 50, 51, 110, 156].

Let X be an n-dimensional toric Calabi-Yau manifold (which is necessarily *noncompact*). To carry out the SYZ construction, we use a special Lagrangian torus fibration $\mu : X \to B$ constructed by Goldstein and Gross; such a fibration is *nontoric*, meaning that it is not the usual moment map associated to the Hamiltonian T^n-action on X. The discriminant locus of this SYZ fibration has been analyzed in details by Gross and can be described explicitly.

Topologically, the base B is simply an upper half-space in $\mathbb{R}^n$, and it is an integral affine manifold with both singularities and boundary. The pre-image of the boundary $\partial B \subset B$ is a non-toric smooth hypersurface $D \subset X$ (a smoothing of the union of toric prime divisors). The discriminant locus $\Gamma \subset B$ is a real codimension two tropical subvariety contained in a hyperplane H which we call the *wall* in B. By definition, the wall(s) inside the base of an SYZ fibration is the loci of Lagrangian torus fibers which bound Maslov index 0 holomorphic disks in X. It divides the base into different chambers over which the Lagrangian torus fibers behave differently in a Floer-theoretic sense. In the case of the Gross fibration, the wall $H \subset B$, which is parallel to the boundary hyperplane ∂B, divides the base into two chambers B_+ and B_-.

We consider (virtual) counts of Maslov index 2 holomorphic disks in X bounded by fibers of μ over B_+ and B_-; these are disks which intersect with the hypersurface D at one point with multiplicity one. As a point moves from B_- to B_+ by crossing the wall, the virtual number of Maslov index 2 disks bounded by the corresponding Lagrangian torus fiber, or more precisely, *genus 0 open Gromov-Witten invariants*, jumps, exhibiting a *wall-crossing phenomenon*.

This has been analyzed by Auroux [8, 9] and Chan-Lau-Leung [20] by applying the sophisticated machinery developed by Fukaya-Oh-Ohta-Ono [56]. Note that there is no scattering phenomenon in this case because there is only one wall. By taking fiberwise dual over each chamber in the base, and then gluing the resulting pieces together according to the wall-crossing formulas, we obtain the *instanton-corrected* or *SYZ mirror* $\check{X}$, which is a family of affine hypersurfaces (which are also noncompact Calabi-Yau manifolds) over the (complexified) Kähler moduli space of X [20, 3], and this agrees with predictions by physicists [38, 97].

This SYZ construction is very precise in the sense that it tells us exactly which complex structure on the mirror $\check{X}$ is corresponding to any given symplectic structure on X – the defining equation of the mirror $\check{X}$ is an explicit expression written entirely in terms of the Kähler parameters and disk counting invariants of X. For example, the SYZ mirror of $X = K_{\mathbb{P}^2}$ is given by[5]

$$\check{X} = \left\{ (u, v, z_1, z_2) \in \mathbb{C}^2 \times (\mathbb{C}^\times)^2 \mid uv = \delta(q) + z_1 + z_2 + \frac{q}{z_1 z_2} \right\}, \tag{6.1}$$

where q is the Kähler parameter which measures the symplectic area of a projective line inside the zero section of $K_{\mathbb{P}^2}$ over $\mathbb{P}^2$, and

$$\delta(q) = \sum_{k=0}^{\infty} n_k q^k \tag{6.2}$$

is a generating series of genus 0 open Gromov-Witten invariants.

Furthermore, the SYZ construction naturally defines the *SYZ map*, which is a map from the Kähler moduli space of X to the complex moduli space of $\check{X}$. As conjectured by Gross and Siebert [87, Conjecture 0.2 and Remark 5.1], the SYZ mirror family should be written in canonical coordinates. In the toric Calabi-Yau case, this is equivalent to saying that the SYZ map is inverse to a mirror map.

Evidences for this conjecture for toric Calabi-Yau surfaces and 3-folds were given in [20, 117], and Chan-Lau-Tseng [21] proved the conjecture in the case when X is the total space of the canonical line bundle over a compact toric Fano manifold. Recently, by applying orbifold techniques, the conjecture was proved for *all* toric Calabi-Yau manifolds (and orbifolds) in [19]. We call this an *open mirror theorem* because it provides an *enumerative meaning* to (inverses of) mirror maps, and gives an effective way to compute *all* the genus 0 open Gromov-Witten invariants.[6]

The main challenge in proving these results is the computation of the genus 0 open Gromov-Witten invariants defined by Fukaya-Oh-Ohta-Ono [57]. Since the moduli spaces of holomorphic disks are usually highly obstructed, these invariants are in general very difficult to compute. Currently, there are only very few techniques available (such as open/closed equalities, toric mirror theorems, degeneration techniques, etc). For example, the invariants in (6.2) can be computed as:

$$n_k = 1, -2, 5, -32, 286, -3038, 35870, \ldots$$

for $k = 0, 1, 2, 3, 4, 5, 6, \ldots$, which agrees with period computations in [66].

We should mention that the SYZ construction can be carried out also in the reverse direction [3] (see also [18, Section 5]). For example, starting with the conic

[5]More precisely, the SYZ mirror of $K_{\mathbb{P}^2}$ is given by the Landau-Ginzburg model $(\check{X}, W = u)$; see the next section.

[6]In their ICM lecture [82], Gross and Siebert sketched an alternative proof of this conjecture, using logarithmic Gromov-Witten theory [34, 4, 89] and with holomorphic disks replaced by tropical disks. Very recently, by applying the results in [19], Lau [116] showed that the generating functions of open Gromov-Witten invariants (such as $\delta(q)$ in the case of $X = K_{\mathbb{P}^2}$) are slab functions in the sense of Gross and Siebert.

bundle (6.1), one can construct an SYZ fibration using similar techniques as in [65, 71]. Although the discriminant locus is of *real codimension one* in this case, the SYZ mirror construction can still be carried out which gives us back the toric Calabi-Yau 3-fold $K_{\mathbb{P}^2}$, as expected.[7]

Nevertheless, outside the toric realm, it is not clear how SYZ constructions can be performed in an explicit way. One major problem is the well-definedness of open Gromov-Witten invariants. Only in a couple of non-toric cases (see Liu [130] and Solomon [161]) do we have a well-defined theory of open Gromov-Witten invariants.[8]

7. SYZ in the non–Calabi-Yau setting

Not long after its discovery, mirror symmetry has been extended to the non–Calabi-Yau setting, notably to Fano manifolds, through the works of Batyrev [10], Givental [62, 63, 64], Kontsevich [112], Hori-Vafa [98] and many others. Unlike the Calabi-Yau case, the mirror is no longer given by a manifold; instead, it is predicted to be a pair $(\check{X}, W)$, where $\check{X}$ is a non-compact Kähler manifold and $W : \check{X} \to \mathbb{C}$ is a holomorphic function. In the physics literature, such a pair $(\check{X}, W)$ is called a *Landau-Ginzburg model*, and W is called the *superpotential* of the model [168, 176].

It is natural to ask whether the SYZ proposal can be extended to this setting as well. Auroux [8] was the first to consider this question and in fact he considered a much more general setting, namely, pairs (X, D) consisting of a compact Kähler manifold X together with an effective anticanonical divisor D. The defining section of D defines a meromorphic volume form on X with simple poles only along D (and nowhere vanishing on $X \setminus D$), thus making it possible to speak about special Lagrangian torus fibrations on the complement $X \setminus D$.

Suppose that we are given such a fibration $\mu : X \setminus D \to B$. Then we can try to run the SYZ construction to produce the SYZ mirror $\check{X}$, i.e. considering the moduli space of pairs (L, ∇), where L is a fiber of μ and ∇ is a flat $U(1)$-connection over L, and then modifying by instanton corrections. Moreover, the superpotential W will naturally appears as the object mirror to Fukaya-Oh-Ohta-Ono's obstruction chain $\mathfrak{m}_0$ for the Floer complexes of Lagrangian fibers of μ.

When X is a compact toric Kähler manifold, a canonical choice of D is the union of all toric prime divisors. The moment map then provides a convenient Lagrangian torus fibration on X, which has the nice property that it restricts to a torus bundle on the open dense torus orbit $X \setminus D$. In this case, the SYZ mirror manifold $\check{X}$ is simply given by the algebraic torus $(\mathbb{C}^\times)^n$, because we have a torus bundle and there are no instanton corrections in the construction of the mirror manifold.

All the essential information is encoded in the superpotential W. Prior to the work of Auroux, it was Cho and Oh [39, 41] who first noticed that W can be expressed in terms disk counting invariants (or genus 0 open Gromov-Witten invariants). By classifying all holomorphic disks in X bounded by moment map fibers, they got an explicit formula for W in the case when X is Fano, and this agrees with earlier predictions obtained using physical arguments by Hori-Vafa [98]. This was later vastly generalized by works of Fukaya-Oh-Ohta-Ono [57, 58, 59] to all compact toric manifolds.

[7]More precisely, the SYZ mirror of (6.1) is the complement of a smooth hypersurface in $K_{\mathbb{P}^2}$.

[8]There are, however, recent works of Fukaya [54, 55] on defining disk counting invariants for compact Calabi-Yau 3-folds.

In [22], mirror symmetry for toric Fano manifolds was used as a testing ground to see how useful Fourier–type transforms, or what we call *SYZ transforms*, could be in the study of the geometry of mirror symmetry. For a toric Fano manifold X, we consider the open dense torus orbit $U_0 := X \setminus D \subset X$, which is also the union of Lagrangian torus fibers of the moment map. Symplectically, we can write $U_0 = T^*B_0/\Lambda^\vee$, where B is the moment polytope and B_0 denotes its interior. Then the SYZ mirror is $\check{X} := TB_0/\Lambda$ which is a bounded domain in $(\mathbb{C}^\times)^n$. To obtain the superpotential W, we consider the space

$$\tilde{X} := U_0 \times \Lambda \subset \mathcal{L}X$$

of fiberwise geodesic/affine loops in X. On $\tilde{X}$, we have an *instanton-corrected* symplectic structure $\tilde{\omega} = \omega + \Phi$, where Φ is a generating function of genus 0 open Gromov-Witten invariants which count (virtually) holomorphic disks bounded by moment map fibers.

An explicit SYZ transform $\mathcal{F}$ was then constructed by combining the semi-flat SYZ transform $\mathcal{F}^{\text{semi-flat}}$ with fiberwise Fourier series, and it was shown that $\mathcal{F}$ transforms the corrected symplectic structure $\tilde{\omega}$ on X precisely to the holomorphic volume form $e^W\check{\Omega}$ of the mirror Landau-Ginzburg model $(\check{X}, W)$, where W was obtained by taking fiberwise Fourier transform of Φ.

Moreover, $\mathcal{F}$ induces an isomorphism between the (small) quantum cohomology ring $QH^*(X)$ of X and the Jacobian ring $Jac(W)$ of W. The proof was by passing to the tropical limit, and observing that a tropical curve whose holomorphic counterpart contributes to the quantum product can be obtained as a gluing of tropical disks; see [22] for details. This observation was later generalized and used by Gross [74] in his study of mirror symmetry for the big quantum cohomology of $\mathbb{P}^2$ via tropical geometry.

As for manifolds of general type, there are currently two main approaches to their mirror symmetry along the SYZ perspective. One is by Abouzaid-Auroux-Katzarkov [3] in which they considered a hypersurface H in a toric variety V and constructed a Landau-Ginzburg model that is SYZ mirror to the blowup of $V \times \mathbb{C}$ along $H \times \{0\}$. In particular, when H is the zero set of a bidegree $(3, 2)$ polynomial in $V = \mathbb{P}^1 \times \mathbb{P}^1$, their construction produces a mirror of the genus 2 Riemann surface, which is in agreement with a previous proposal by Katzarkov [108, 106, 157].

Another approach, which is more in line with the Gross-Siebert program, is the work by Gross-Katzarkov-Ruddat [80], where they proposed that the mirror to a variety of general type is a reducible variety equipped with a certain sheaf of vanishing cycles. Presumably, the mirror produced in this approach should give the same data as the one produced by [3]. For example, the reducible variety should be the critical locus of the superpotential of the SYZ mirror Landau-Ginzburg model. But the precise relations between these two approaches are still under investigation.

8. Beyond SYZ

Besides providing a beautiful geometric explanation of mirror symmetry, the SYZ conjecture [163] has been exerting its profound influence on many related areas of mathematics as well. We are going to briefly describe several examples of applications in this regard.

HMS via SYZ. As we have seen, the SYZ conjecture is based upon the idea of D-branes in string theory. Recall that B-branes (i.e. D-branes in the B-model) are coherent sheaves over complex subvarieties while A-branes (i.e. D-branes in the A-model) are special Lagrangian submanifolds equipped with flat $U(1)$ connections. Therefore we may regard Kontsevich's Homological Mirror Symmetry (HMS) conjecture [111], which asserts that the (derived) Fukaya category of a Calabi-Yau manifold X is equivalent to the derived category of coherent sheaves on the mirror $\check{X}$, as a manifestation of the isomorphism between the A-model on X and the B-model on $\check{X}$. Hence, rather naturally, one may expect that the SYZ proposal, and in particular SYZ transforms, can be employed to construct geometric functors which realize the categorial equivalences asserted by the HMS conjecture.

For example, given a Lagrangian section of an SYZ fibration $\mu : X \to B$, its intersection point with a fiber L of μ determines a flat $U(1)$-connection on the dual torus $L^\vee$. Patching these flat $U(1)$-connections together give a holomorphic line bundle over the total space of the dual fibration, which is the mirror $\check{X}$ modulo quantum corrections. This simple idea, first envisioned by Gross [69, 70], was explored by Arinkin-Polishchuk [7] and Leung-Yau-Zaslow [122] to construct SYZ transforms which were then applied to prove and understand the HMS conjecture in the semi-flat Calabi-Yau case. Later, the same idea was also exploited to study the HMS conjecture for toric varieties [1, 2, 46, 48, 47, 16, 24, 40].

In some more recent works [18, 28, 27, 26, 81], SYZ transforms were used to construct geometric Fourier–type functors (on the objects level) which realize the HMS categorial equivalences for certain examples of toric Calabi-Yau manifolds such as resolutions of A_n-singularities and the smoothed conifold, where one encounters SYZ fibrations with singular fibers and hence nontrivial quantum corrections.

On the other hand, work in progress by K.-L. Chan, Leung and Ma [29, 30] have shown that SYZ transforms can also be applied to construct the HMS equivalences on the morphism level, at least in the semi-flat case. The ultimate goal is to construct a canonical geometric Fourier-type functor associated to any given SYZ fibration, which realizes the equivalences of categories asserted by the HMS conjecture, thereby enriching our understanding of the geometry of the HMS conjecture, and also mirror symmetry as a whole.

Ricci-flat metrics and disk counting. A remarkable observation in the SYZ paper [163] is that a Ricci-flat metric on the mirror can be decomposed as the sum of a semi-flat part (which was written down explicitly earlier in Greene-Shapere-Vafa-Yau [68]) and an instanton-corrected part which should come from contributions by holomorphic disks in the original Calabi-Yau manifold bounded by Lagrangian torus fibers of an SYZ fibration. This suggests a concrete and qualitative description of Ricci-flat metrics, which are almost never explicit.

In general, such a qualitative description is still extremely difficult to obtain because nontrivial examples of SYZ fibrations on compact Calabi-Yau manifolds are hard to find and open Gromov-Witten theory is not well-understood yet. However, recent works of Gaiotto-Moore-Neitzke [60, 61] have shed new light on the hyperkähler case. They proposed a new conjectural relation between hyperkähler metrics on the total spaces of complex integrable systems (the simplest example of which is the well-known Ooguri-Vafa metric [148]) and Kontsevich-Soibelman's wall-crossing formulas.

To describe their proposal in a bit more details, let us consider a complex integrable system $\psi : M \to B$, i.e. M is holomorphic symplectic and the fibers of ψ are complex Lagrangian submanifolds. More precisely, what Gaiotto, Moore and Neitzke were looking at in [60, 61] were all meromorphic Hitchin systems, in which case complete hyperkähler metrics were first constructed by Biquard-Boalch [11]; see e.g. [61, Section 4.1]. They made use of the fact that any hyperkähler metric is characterized by the associated twistor space, so they tried to construct a $\mathbb{C}^\times$-family of holomorphic Darboux charts on M which satisfy the hypotheses of the theorem of Hitchin et al. [96]. In particular, they required the coordinates to satisfy certain wall-crossing formulas which describe the discontinuity of the coordinates across the so-called *BPS rays*, where the (virtual) counts of BPS states jump.

These wall-crossing formulas turn out to be equivalent to those used by Kontsevich-Soibelman [115] and Gross-Siebert [87] in their constructions of toric degenerations of Calabi-Yau manifolds (and on the other hand they are the same as the wall-crossing formulas in motivic Donaldson-Thomas theory [105, 113]). In view of this and the SYZ conjecture, and also the fact that a vast family of examples of noncompact SYZ fibrations on meromorphic Hitchin systems, including many in complex dimension two (e.g. gravitational instantons, log-Calabi-Yau surfaces) have been constructed via hyperkähler rotation in [11], it is natural to expect that the hyperkähler metrics on those examples of complex integrable systems considered by Gaiotto, Moore and Neitzke can be expressed in terms of (virtual) counting of holomorphic disks.

This was done for the simplest example – the Ooguri-Vafa metric in [17]. More recent works of W. Lu [134, 135] have demonstrated that in general the twistor spaces and holomorphic Darboux coordinates on meromorphic Hitchin systems studied in [60, 61] produced the same data as those required to run the Gross-Siebert program [87], hence showing that there must be some (perhaps implicit) relations between the hyperkähler metrics and tropical disks counting. In his PhD thesis [129], Y.-S. Lin considered elliptic $K3$ surfaces and tropical disk counting invariants. He proved that his invariants satisfy the same wall-crossing formulas as those appeared in [60, 61]. Evidently, the hyperkähler metrics on those K3 surfaces are closely related to disk counting as well. More recent works of Stoppa and his collaborators [162, 49] have also shown the intimate relations between wall-crossing formulas in motivic Donaldson-Thomas theory and the constructions of Gaiotto-Moore-Neitzke.

Other applications of SYZ. Let us also mention two recent unexpected applications of SYZ constructions, without going into the details.

In their recent joint project [77], Gross-Hacking-Keel constructed SYZ mirror families to log Calabi-Yau surfaces, i.e. pairs (Y, D) where Y is a nonsingular projective rational surface and $D \in |-K_Y|$ is a cycle of rational curves, by extending the construction in [87] to allow integral affine manifolds with more general (i.e. worse) singularity types. Amazingly, their results could be applied to give a proof of a 30-year-old conjecture of Looijenga [133] concerning smoothability of cusp singularities.

In a more recent paper [78], they applied their construction to prove a Torelli theorem for log Calabi-Yau surfaces, which was originally conjectured in 1984 by Friedman [52]. On the other hand, their construction is also closely connected with the theory of cluster varieties, and they have suggested a vast generalization of the Fock-Goncharov dual bases [76, 79]. For a nice exposition of these exciting new results and developments, we refer the reader to the nice survey article by Gross and Siebert [83].

In another direction, the SYZ construction has been unexpectedly applied to construct new *knot invariants*. For a knot K in S^3, its conormal bundle N^*K is canonically a Lagrangian cycle in the cotangent bundle T^*S^3. In [43], Diaconescu-Shende-Vafa constructed a corresponding Lagrangian cycle L_K in the resolved conifold $X := \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$, which is roughly speaking done by lifting the conormal bundle N^*K off the zero section and letting T^*S^3 undergo the conifold transition. Their construction was motivated by a mysterious phenomenon called *large N duality* in physics.

In [6], Aganagic-Vafa defined a new knot invariant by a generalized SYZ construction applied to the pair (X, L_K). More precisely, their invariant is a generating function of genus 0 open Gromov-Witten invariants for the pair (X, L_K). It turned out that the resulting function is always a *polynomial* and they conjectured that it is nothing but a deformation of the classical *A-polynomial* in knot theory [42]. Furthermore, an interesting relation between their invariant and augmentations of the contact homology algebra of K [144] was suggested. Substantial evidences for this relation was obtained in a recent paper [5].

These two new applications of the SYZ conjecture, together with many more yet to come, open up new directions in mirror symmetry and many other branches of mathematics and physics,[9] and they are all pointing towards further beautiful and exciting research works in the future.

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