MATH 4010 | 03/03/2016

Tutorial 6

Applications of Hahn-Banach Thm

<u>Recall</u>: (We focus on real-valued version)

① Hahn-Banach Thm on Vector spaces

<u>Thm</u>: Let X be a real vector space and p is a sublinear functional on X. If f is a linear functional on a subspace Z of X and satisfies

$$f(x) \leq p(x) \quad , \forall x \in Z$$

Then f has a linear extension $\tilde{f}$ defined on X such that

$$\tilde{f}(x) = f(x) \, , \ \forall x \in Z$$
$$\tilde{f}(x) \leq p(x) \, , \ \forall x \in X .$$

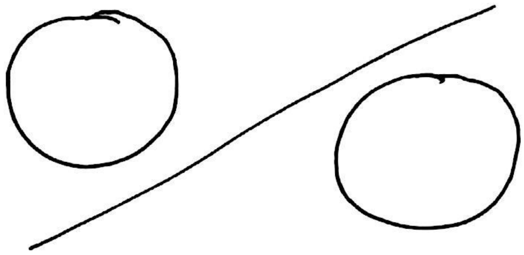
② Hahn-Banach Thm on normed spaces.

<u>Thm</u>: Let f be a bounded linear functional on a subspace Z of a normed space X. Then there exists a bounded linear functional $\tilde{f}$ on X such that

$$\tilde{f}(x) = f(x) \, , \ \forall x \in Z$$
$$\|\tilde{f}\|_X = \|f\|_Z .$$

<u>Remark</u>: Hahn-Banach Thm is one of fundemental thms in Functional Analysis and has a lot of applications. As we known, it gurantees that a normed space is richly supplied with bounded linear functionals, i.e. $\exists$ bdd linear fcnal f s.t. $f(x_1) \neq f(x_2)$, $\forall x_1 \neq x_2$. Based on Hahn-Banach Thm, we also obtain the theory of dual spaces and adjoint operators.

Eg 1: (Seperation of convex sets)



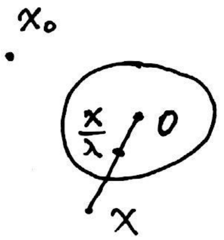
Let X be a real normed space and let A, B be two nonempty disjoint convex subsets of X

(i) If A is open, then $\exists$ a bounded linear functional f on X and $c \in \mathbb{R}$ s.t. $f(a) < c \leq f(b)$, $\forall\, a \in A$, $b \in B$.

(ii) If A is compact and B is closed, then $\exists$ a bounded linear functional f on X and $c_1, c_2 \in \mathbb{R}$ s.t. $f(a) \leq c_1 < c_2 \leq f(b)$, $\forall a \in A$ and $b \in B$.

Remark: The hyperplane $H_c = \{x \in X : f(x) = c\}$ seperates two disjoint convex sets A and B.

Pf: Step 1: (Reduce the problem to be seprating a point from a convex set.)



Choose $a_0 \in A$, $b_0 \in B$. Set $x_0 \in a_0 - b_0$

Consider $D = A - B + x_0 = \{a - a_0 + b_0 - b \mid a \in A, b \in B\}$

Since A, B are convex and A is open, it is clear that

D is an open convex neighborhood of 0.

Moreover, $x_0 \notin D$, otherwise, $x_0 = a - b + x_0$ i.e. $a - b = 0$ for some $a \in A, b \in B$

A contradiction to $A \cap B = \phi$

Step 2: (Construct sublinear functional)

Define $p(x) = \inf\{\lambda > 0 \mid x \in \lambda D\}$ (which is called Minkowski fcnal)

Since D is open and $0 \in D$, $B(0, \rho) \subset D$ for some $\rho > 0$.

Thus $p(x) \leq \frac{\|x\|}{\rho}$, since $x \in \frac{\|x\|}{\rho} B(0,\rho) \subset \frac{\|x\|}{\rho} D$, $\forall\, x \in X$.

Furthermore, p satisfy $p(\alpha x) = \alpha x$, $\forall \alpha \geq 0$ and $p(x+y) \leq p(x) + p(y)$.

Indeed, $\forall \varepsilon > 0$, let $\lambda_1 = p(x) + \frac{\varepsilon}{2}$ and $\lambda_2 = p(y) + \frac{\varepsilon}{2}$, then

$\frac{x}{\lambda_1} \in D$ and $\frac{y}{\lambda_2} \in D$

So, $\frac{x+y}{\lambda_1 + \lambda_2} = \frac{\lambda_1}{\lambda_1 + \lambda_2} \frac{x}{\lambda_1} + \frac{\lambda_2}{\lambda_1 + \lambda_2} \frac{y}{\lambda_2} \in D$ since D is convex.

i.e. $p(x+y) \leq \lambda_1 + \lambda_2 = p(x) + p(y) + \varepsilon \Rightarrow p(x+y) \leq p(x) + p(y)$.

Step 3: (Construct linear funtional on subspace)

Set $Z = \{\alpha x_0\}$. Then Z is a subspace of X.

Define a functional g on Z as $g(\alpha x_0) = \alpha$.

Then $g(x_0) = 1$. Note that $x_0 \notin D$, one has $p(x_0) \geq 1$

So, $g(\alpha x_0) = \alpha \leq \alpha p(x_0) = p(\alpha x_0)$

By Hahn-Banach Theorem, $\exists$ a linear fcnal f on X s.t.

$$f(x) \leq p(x) \text{ and } f(x_0) = g(x_0) = 1$$

Since $p(x) \leq \frac{\|x\|}{\rho}$, f is bounded and $\|f\| \leq \frac{1}{\rho}$

Therefore, $\forall\ a \in A,\ b \in B$

$$f(a) - f(b) + 1 = f(a - b + x_0) \leq p(a - b + x_0) < 1$$ since $a - b + x_0 \in D$ and D is open.

i.e. $f(a) < f(b)$, $\forall\ a \in A, b \in B$.

The sets $f(A)$ and $f(B)$ are nonempty, disjoint convex sets and $f(A)$ is open. Taking $c = \sup_{a \in A} f(A)$, then (i) is proved.

(ii) Since A is compact and B is closed,

$$d(A, B) = \inf \{\|a - b\| \mid a \in A,\ b \in B\} > 0$$

Let $r = d(A, B)$. Then $A_r := \{x \in X \mid d(x, A) < r\}$ does not intersect with B. Then (i) yields that $\exists$ a bdd linear fcnal f on X and $c_2 \in \mathbb{R}$ s.t. $f(x) < c_2 \leq f(y)$, $\forall\ x \in A_r$, and $y \in B$

Since f is cts and A is compact, $f(A)$ is compact.

So $c_1 := \sup_{x \in A} f(x) < c_2$. This proves (ii)

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Eg 2. Show that a closed subspace of a reflexive space is reflexive.

<u>Recall</u>: $X = X^{**}$ — X is reflexive.

Let X be normed space, for any $x \in X$, we can define

$$g_x(f) = f(x), \quad \forall f \in X^*$$

Then $g_x \in X^{**}$ and $\|g_x\|_{X^{**}} = \|x\|$, i.e. $C: X \to X^{**}$ $x \mapsto g_x$ is an isomorphism

$X \subset X^{**}$. When C is surjective, i.e. $X = X^{**}$, X is reflexive.

Pf: Let X be a reflexive space and Z be a closed subspace of X.

It is clear that Z is a normed space so that $Z \subset Z^{**}$.

To show that Z is reflexive, it suffiries to prove $Z^{**} \subset Z$.

That is, $\forall z_0 \in Z^{**}$, $\exists x \in Z$ s.t. $z_0(f_0) = f_0(x)$, $\forall f_0 \in Z^*$.

Indeed, $\forall f \in X^*$, set $f_0 = f|_Z =: Tf$: Then $f_0 \in Z^*$ and $\|f_0\| \le \|f\|$.

So, $T: X^* \to Z^*$ is bounded. Let T^* be the adjoint operator of T

Then $T^*: Z^{**} \to X^{**}$. Set $z = T^* z_0$. Then $z \in X^{**}$.

Since X is reflexive, $\exists x \in X$ s.t. $z(f) = f(x)$, $\forall f \in X^*$.

Now, we claim that $x_0 \in Z$.

Assume that $x_0 \notin Z$. Since Z is closed, $d(x_0, Z) = \inf_{z \in Z} \|x - z\| > 0$.

Let $Y = Z \oplus \{x_0\} = \{y = z + \lambda x_0 \mid z \in Z, \lambda \in \mathbb{R}\}$.

Define a functional g on Y by $g(y) = \lambda d(x_0, Z)$

It is clear that g is linear and $g(z) = 0$, $\forall z \in Z$, $g(x_0) = d(x_0, Z)$

Moreover, $|g(y)| = |\lambda| d(x_0, Z) \le |\lambda| \left\| \frac{z'}{\lambda} + x_0 \right\| = \|z' + \lambda x_0\| = \|y\|$ since $\frac{z'}{\lambda} \in Z$.

So, $\|g\| \le 1$. By Hahn-Banach Thm, $\exists f \in X^*$ s.t. $f(z') = 0$, $f(x_0) = d(x_0, Z)$

and $\|f\| \le 1$. Thus $Tf = 0$.

However, $0 = z_0(f_0) = z_0(Tf) = (T^* z_0)(f) = z(f) = f(x) = d(x, Z)$

A contradiction. Therefore, $x \in Z$.

Now, we show that $z_0(f_0) = f_0(x)$, $\forall f_0 \in Z^*$. Indeed, by Hahn-Banach Thm, $\exists f \in X^*$ s.t. $f_0 = Tf$. Then

$$z_0(f_0) = z_0(Tf) = (T^* z_0)(f) = z(f).$$

and $f_0(x) = f(x)$, $\forall x \in Z$.

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