Date 9 · 2 2022

Recall Last time.

Bruhat decomposition $G = \coprod_{w\in W} C(w)$, where $C(w) = BwB$

We also have the multiplication formula for the Bruhat cells

For $s\in S$
$w\in W$
$$C(s)\,C(w) = \begin{cases} C(sw), & \text{if } sw > w \\ C(w) \amalg C(sw), & \text{if } sw < w. \end{cases}$$

The closure relation $\overline{C(w)} = \coprod_{\underset{\underset{\text{Bruhat order}}{\uparrow}}{w' \le w}} C(w')$

$$\overbrace{st\cdots}^{m_{st}\text{ terms}} = \overbrace{ts\cdots}^{m_{st}\text{ terms}}$$

$$W = \langle s\rangle_{s\in S} \,/\, \underset{\underset{\text{reflection}}{\uparrow}}{s^2 = 1},\ (st)^{\underset{\underset{\text{Coxeter relation}}{\uparrow}}{m_{st}}} = 1$$

Hecke algebra

Definition: Let (W,S) Coxeter gp

The (generic) Hecke alg. H associated to W is an algebra over $\mathbb{Z}[q]$, A
where q is an indeterminate generated by T_s for $s\in S$, subject to the relations

- Coxeter relation
$$\underbrace{T_s T_t \cdots}_{m_{st}\text{ terms}} = \underbrace{T_t T_s \cdots}_{m_{st}\text{ terms}}$$

e.g. Hec 1 = T_1.

- quadratic relation
$$(T_s - q)(T_s + 1) = 0 \quad (\text{i.e. } T_s^2 = (q-1)T_s + q)$$

Rmk ① We may specify the indeterminant q to some number.
If $q=1$, then $T_s^2 = 1$. i.e. T_s is of order 2.
In this case, $H_{q=1} = \mathbb{Z}[W]$, the gp. algebra.
If $q\ne 0$, then $T_s^{-1} = q^{-1}T_s - q^{-1}(q-1)$. So T_s is invertible in H_q.
If $q=0$, then $H_{q=0}$ is the 0-Hecke algebra.

② When q is a power of a prime number, then
H_q is the Hecke algebra of $G(\mathbb{F}_q)$ if W is the Weyl gp of G.
When $q=0$, then H_0 is closely related to the total positivity.

Prop. Let $w\in W$. Let $w = s_1\cdots s_n$ be a reduced expression. Then
$T_w := T_{s_1}T_{s_2}\cdots T_{s_n} \in H$ is independent of the choice of reduced expression.

Pf. This follows from the fact that any two reduced expressions of w are related by the Coxeter relations.

$$\begin{array}{ccc} \cdots\cdots(st\cdots\cdots)\cdots\cdots & & \cdots(T_sT_t\cdots)\cdots \\ \wr\wr & \Longrightarrow & \| \\ \cdots\cdots\underbrace{(ts\cdots\cdots)}_{m_{st}\text{ terms}}\cdots\cdots & & \cdots(T_tT_s\cdots)\cdots \end{array}$$

□

Prop. H is spanned by $\{T_w\}$ as $\mathbb{Z}[q]$-module.

Pf. Let $H' = \sum_{w\in W} \mathbb{Z}[q]\, T_w \subseteq H$.

It suffices to show that $T_s H' \subseteq H'$

$$T_sT_w = \begin{cases} T_{sw} & \text{if } sw>w \\ (q-1)T_w + qT_{sw}, & \text{if } sw<w. \end{cases}$$

Let $w = s_1\cdots s_n$ reduced exp.
Then
$sw = ss_1\cdots s_n$ is reduced

Let $sw = s_1\cdots s_l$ reduced
Then $w = ss_1\cdots s_l$ reduced

$$\begin{aligned} T_sT_w &= T_sT_sT_{sw} \\ &= ((q-1)T_s + q)T_{sw} \\ &= (q-1)T_sT_{sw} + qT_{sw} \\ &= (q-1)T_w + qT_{sw}. \end{aligned}$$

Then $T_sH' \subseteq H'$, Since H is generated by $\{T_s\}$ as an algebra, $H = H'$.

Rmk. Infact, $\{T_w\}$ is a basis of H.
We will prove it when W is a finite Weyl gp and q is a power of prime.
The general case is proved by constructing a faithful repr. of H for $q\neq 0$.
(see [Lusztig, Hecke algebra with unequal parameters, §3])
For $q=0$, we will discuss later.

Now we assume that G is a conn reductive gp. W its Weyl gp.
(eg. $G = GL_n$)

$B \subseteq G$ Borel subgp, $B = TU$.

Let Φ be the root system. $\Phi = \Phi^+ \sqcup \Phi^-$

For any $\alpha\in\Phi^+$, $U_\alpha \subseteq U$ root subgp.

Eg. $G = GL_n$
$\alpha = e_i - e_j$

$$U_\alpha = \left\{ \begin{pmatrix} 1 & & * \\ & \ddots & \\ & & 1 \end{pmatrix} \leftarrow i\text{-th row} \right\}$$
j-th column

Fact: $U = \prod_{\alpha\in\Phi^+} U_\alpha$ for any order on Φ^+

Lie algebra version
$\mathfrak{u} = \mathrm{Lie}(U)$, $\mathfrak{u}_\alpha = \mathrm{Lie}(U_\alpha)$
$\mathfrak{u} = \bigoplus_{\alpha\in\Phi^+} \mathfrak{u}_\alpha$.

The idea is that for $\alpha, \beta \in \Phi^+$

$$U_\alpha U_\beta \subseteq U_\beta U_\alpha \prod_{\substack{\gamma\in a\alpha+b\beta \\ a,b\geq 1}} U_\gamma \leftarrow \text{in any order}$$

$B \subseteq G$ Borel $B = TU$

$B^- \subseteq G$ opposite Borel, $B^- = TU^-$

Eg $G = GL_n$, B upper triangular, B^- lower triangular.

Prop. Let $w \in W$. Then the multiplication map gives an isomorphism.

$$m : (U \cap {}^wU^-) \times (U \cap {}^wU) \xrightarrow{\sim} U \quad \text{Here } {}^gH = gHg^{-1}$$

Pf. $\Phi^+ = (\Phi^+ \cap w(\Phi^-)) \amalg (\Phi^+ \cap w(\Phi^+))$

$$\prod_{\alpha \in \Phi^+ \cap w(\Phi^-)} U_\alpha \subset U \cap {}^wU^-, \quad \prod_{\alpha \in \Phi^+ \cap w(\Phi^+)} \subset U \cap {}^wU.$$

But
$$U = \prod_{\alpha \in \Phi^+ \cap w(\Phi^-)} U_\alpha \prod_{\alpha \in \Phi^+ \cap w(\Phi^+)} U_\alpha$$
$$\subseteq (U \cap {}^wU^-) \cdot (U \cap {}^wU)$$

But $(U \cap {}^wU^-) \cdot (U \cap {}^wU) \subseteq U$

So all the inclusions above are equalities.

In particular, $U \simeq (U \cap {}^wU^-) \times (U \cap {}^wU)$ □.

Cor. We have an isomorphism

$$(U \cap {}^wU^-) \times B \xrightarrow{\sim} B\dot{w}B$$
$$(u, b) \longmapsto u\dot{w}b$$

$({}^wU)\dot{w}B = (\dot{w}U\dot{w}^{-1})\dot{w}B = \dot{w}B$

Pf. $B\dot{w}B = UT\dot{w}B = U\dot{w}B = (U \cap {}^wU^-)(U \cap {}^wU)\dot{w}B$

$$= \underbrace{(U \cap {}^wU^-)\dot{w}B}_{\text{surjectivity}} \subseteq \underbrace{\dot{w}U^-B \simeq U^- \times B}_{\text{injectivity.}} \ \square$$

$\mathfrak{g} = \mathfrak{u}^- \oplus \mathfrak{b}$ (Lie alg.)

Cor. Let $G = \mathbb{G}(\mathbb{F}_q)$ ← $\mathbb{F}_q$-rational pts of a conn. reductive gp split over $\mathbb{F}_q$.

Then $\#(BwB/B) = q^{l(w)}$. eg. $G = GL_n(\mathbb{F}_q)$.

Pf. $BwB/B \simeq U \cap {}^wU^- = \prod_{\alpha \in \Phi^+ \cap w(\Phi^-)} U_\alpha^{\simeq G_a}$

Over $\mathbb{F}_q$, $\#(U_\alpha) = q$ So $\# BwB/B = q^{\#(\Phi^+ \cap w(\Phi^-))} = q^{l(w)}$ □.

Eg $G = GL_n$, $U_\alpha = \begin{pmatrix} 1 & & * \\ & \ddots & \\ & & 1 \end{pmatrix}$, $* \in \mathbb{F}_q$.

i.e. constant on each $B \times B$-orbit (BwB).

Definition. Let $G = \mathbb{G}(\mathbb{F}_q)$. The Hecke algebra $H(G)$ of G is the set of B-biinvariant functions on G (over $\mathbb{C}$), with the convolution product $*$, defined by $(f_1 * f_2)(g) = \frac{1}{\#(B)} \sum_{x \in G} f_1(x) f_2(x^{-1}g) = \frac{1}{\#(B)} \sum_{x \in G} f_1(gx) f_2(x^{-1})$.

This comes from the multiplication $m : G \times^B G \to G$, $m(g_1, g_2) = g_1g_2$

B action on $G \times G$ by $b \cdot (g_1, g_2) = (g_1 b^{-1}, bg_2)$

Prop. $H(G) = H_q$

Pf. (1) Let $\phi_w = 1_{BwB} \in H(G)$.

Then $\{\phi_w\}$ is a basis of $H(G)$ ← This follows from $G = \amalg BwB$.

(2) $\phi_s\phi_w = \begin{cases} \phi_{sw} & \text{if } sw > w \\ (q-1)\phi_w + q\phi_{sw}, & \text{if } sw < w. \end{cases}$

If $sw > w$, then ~~BsB~~ $C(s)C(w) = C(sw)$

Thus $\phi_s\phi_w$ is a function supported in $C(sw)$.

So $\phi_s\phi_w = c\phi_{sw}$. for some constant c.

Consider the function $\varepsilon: H(G) \to \mathbb{C}.\ f \mapsto \frac{1}{\#(B)}\sum_{x\in G} f(x)$.

Then ε is an alg hom and $\varepsilon(\phi_w) = q^{l(w)}$.

Now
$$\begin{array}{ccc} \varepsilon(\phi_s\phi_w) & = & \varepsilon(c\phi_{sw}) \\ \| & & \| \\ \varepsilon(\phi_s)\varepsilon(\phi_w) & & c\,\varepsilon(\phi_{sw}) \\ \| & & \| \\ q\cdot q^{l(w)} = q^{l(w)+1} & & c\,q^{l(sw)} = c\,q^{l(w)+1} \end{array}$$

So $c = 1$.

$\phi_s * \phi_s = ?$ $(q-1)\phi_s + q\phi_1$

$C(s)C(s) = C(1) \amalg C(s)$.

So $\phi_s * \phi_s = a\phi_s + b\phi_1$. for some $a, b \in \mathbb{C}$

Applying ε, $q^2 = aq + b$.

evaluate at $1 \in G$, $\frac{1}{\#B}\underset{q}{\underset{\|}{\#(BsB)}} = b$ So $b = q$, $a = (q-1)$

$\phi_s*\phi_s(1) = a\phi_s(1) + b\phi_1(1)$

So if $sw < w$, then
$$\begin{aligned} \phi_s * \phi_w &= \phi_s * (\phi_s * \phi_{sw}) \\ &= (\phi_s * \phi_s) * \phi_{sw} \\ &= ((q-1)\phi_s + q\phi_1) * \phi_{sw} \\ &= (q-1)\phi_w + q\,\phi_{sw} \end{aligned}$$

(3) By (2), we have an alg. hom.

$$H_q \to H(G), \quad T_w \mapsto \phi_w$$

This is surjective, Since H_q is spanned by $\{T_w\}$, $H(G)$ has a basis $\{\phi_w\}$.

We have $H_q \cong H(G)$ and $\{T_w\}$ is a basis of H_q.

Now we consider 0-Hecke alg.

Recall that

$$T_s T_w = \begin{cases} T_{sw}, & \text{if } sw > w \\ (q-1)T_w + q\,T_{sw}, & \text{if } sw < w. \end{cases}$$

If $q=1$, then we see the gp structure of W

If $q=0$, then we see the monoid structure of W.

Define the monoid action on W.

$$s * w = \max\{w, sw\}$$

Then in H_0, $T_s T_w = \pm T_{s*w}$. So $T_w * T_{w'} = \pm T_{w*w'}$.

in H_0, T_s is not invertible since $T_s^2 = -T_s$.

Geometric interpretation of $*$

$$\overline{BwB} = \coprod_{w' \leq w} Bw'B.$$

Fact $\overline{BwB} \cdot \overline{Bw'B} = \overline{B(w*w')B}$ (Exercise)

Pf. induction on $l(w)$.

Upshot: Monoid structure on $W \longleftrightarrow$ total positivity on G.

Construction of total positivity.

" Combinatorial datum on the root systems" $\rightsquigarrow$ " reductive gps/monoids".

We will use GL_n as the example

— Construct U^-.

Recall that the Lie alg. $u_- = \mathrm{Lie}(U^-)$ is the Lie alg. gen. by f_i for $i \in I$,

with the Serre relation $\quad \mathrm{ad}(f_i)^{-\langle \alpha_i, \alpha_j \rangle + 1}(f_j) = 0 \quad \forall i \neq j$.

For the gp U^-, there is a similar, but more complicated construction.

(see Springer, Algebraic groups, §10).

- Construct $U^-_{\geq 0}$

$$y_i(a) = \begin{pmatrix} 1 & \overset{i\text{-th column}}{\downarrow} & & \\ & \ddots & & \\ & a & \ddots & \\ & & & 1 \end{pmatrix} \rightarrow (i+1)\text{-th row}.$$

We start with $y_i(a)$, $i \in I$, $a > 0$.

Recall that for $G = GL_n$, $U^-_{\geq 0}$ is the submonoid gen. by $y_i(a)$.

This is our definition of $U^-_{\geq 0}$ in general.

Relations: — $y_i(a) y_i(b) = y_i(a+b) \quad \forall i \in I,\ a, b > 0$

— $y_i(a) y_j(b) = y_j(b) y_i(a)$ if $\underset{i}{\circ} \quad \underset{j}{\circ}$ in Dynkin diagram.

For GL_n, this means $|i-j| \geq 2$.

– $y_i(a)\,y_j(b)\,y_i(c) = y_j\left(\frac{bc}{a+c}\right) y_i(a+c)\, y_j\left(\frac{ab}{a+c}\right)$, if $\underset{i}{\circ}\!\!-\!\!\!-\!\!\underset{j}{\circ}$ (i.e. type A_2)

(This is basically a direct calculation in GL_3).

Rmk. ① This formula only works if $a+c \neq 0$. Thus one needs a more complicated formula for $U^-(\mathbb{R})$.

② Why no relation like $y_i(a)\,y_j(b) = ?$

Key observation $y_i(a) \in Bs_iB$ for $a \neq 0$. / Easy calculation in GL_2.

So $y_i(a)\,y_j(b) \in C(s_i)\,C(s_j) = C(s_is_j)$.

Since $\underset{i}{\circ}\!\!-\!\!\!-\!\!\underset{j}{\circ}$, s_is_j has only one reduced expression $s_i \cdot s_j$.

So no relation like $y_i(a)\,y_j(b) = ?$

We have $s_is_js_i = s_js_is_j$ Coxeter relation.

So relation $y_i(a)\,y_j(b)\,y_i(c) = y_j(?)\,y_i(?)\,y_j(?)$

Prop. (1) For any $w \in W$, $U^-_{w,>0} = \{y_1(a_1)\cdots y_n(a_n) : a_1, \cdots, a_n > 0\} \simeq \mathbb{R}^n_{>0}$ ← cell

is independent of the choice of reduced expressions $w = s_1 \cdots s_n$.

(2) $U^-_{\geq 0} = \coprod_{w \in W} U^-_{w,>0}$. cellular decomposition

(3) $U^-_{\geq 0}$ is the monoid gen. by $\{y_i(a)\}_{i \in I, a>0}$ subject to the relations (i) $y_i(a)\,y_i(b) = y_i(a+b)$

(ii) Coxeter relations for y_i & y_j.

Pf. (1)

$$\begin{array}{ccc} \cdots(s_is_j\cdots)\cdots & & \cdots(y_i(>0)\,y_j(>0)\cdots)\cdots \\ \wr & \Longrightarrow & \| \\ \cdots(s_js_i\cdots)\cdots & & \cdots(y_j(>0)\,y_i(>0)\cdots)\cdots \\ \text{reduced expressions} & & \text{in } U^- \end{array}$$

(2) Set $U'_{\geq 0} = \bigcup_{w \in W} U^-_{w,>0}$

Claim : $y_i(>0)\,U^-_{w,>0} = U^-_{s_i * w,>0}$, here $*$ is the monoid action.

If $s_iw > w$, then $w = s_1 \cdots s_n$ reduced expression

$\underline{sw = ss_1 \cdots s_n}$

So $y_i(>0)\,U^-_{w,>0} = U^-_{s_iw,>0}$

If $s_iw < w$, then $y_i(>0)\,U^-_{w,>0} = y_i(>0)\,y_i(>0)\,U^-_{sw,>0} = y_i(>0)\,U^-_{sw,0}$

$= U^-_{w,>0}$

Now $y_i(>0)\cdot U'_{\geq 0} \subseteq U'_{\geq 0}$

Since $U^-_{\geq 0}$ is gen. by $y_i(>0)$, $U^-_{\geq 0} = U'_{\geq 0}$.

Since $y_i(>0) \subseteq C(s_i)$, $U^-_{w,>0} \subseteq C(w)$

Thus $\bigcup_{w\in W} U^-_{w,>0}$ is a disjoint union since $\bigcup_{w\in W} C(w)$ is a disjoint union.

(3). Let $s_1\cdots s_n$ be any expression and $w = s_1 * \cdots * s_n$

Then $y_1(>0)\cdots y_n(>0) = U^-_{w,>0}$ by part (2).

So any relation comes from

$$s_1 * \cdots * s_n = t_1 * \cdots * t_m.$$

If non-reduced, then after Coxeter relations, exchange property.

we have $s_1 * \cdots * s_n = s_1 * \cdots * (s_i * s_{i+1}) * \cdots * s_n$ where $s_i = s_{i+1}$

use the relation $y_i(a)\, y_{i+1}(b) = y_i(a+b)$

replace $s_1 * \cdots * s_n$ by $s_1 * \cdots * \widehat{s_{j+1}} * \cdots * s_n$.

Now after this consideration, we may assume that $s_1\cdots s_n$, $t_1\cdots t_m$ are reduced. Then $n=m$, and $s_1\cdots s_n \underset{\text{Coxeter}}{\sim} t_1\cdots t_m$

So $y_{s_1}\cdots y_{s_n} = y_{t_1}\cdots y_{t_m}$ comes from Cox relation □