1 Recall from last week

Total positive cells in $\mathcal{B}_{\geq 0}$:

- Have the product structure (By Cr map)
- Have the Marsh-Rietsch parametrization

! Remains to show $\mathcal{B}_{\geq 0}$ does not intersect with lower-dimensional Deohdar component. (For connected component argument for step 3)

2 Continuing from last week

Illustrate with example $G = SL_3$, $u = 1, \underline{w} = s_1 s_2 s_1$ fixed reduced expression.
We have positive subexpression $\underline{u}_+ = (1\ 1\ 1)$,
distinguished, non-positive subexpression $\underline{u} = (s_1\ 1\ s_1)$.

Want to show: $\mathcal{B}_{\geq 0} \cap \mathring{\mathcal{B}}_{\underline{u},\underline{w}}(\mathbb{R}) = \emptyset$.
It is known that $\mathcal{B}_{\geq 0} \cap \mathring{\mathcal{B}}_{u,w}(\mathbb{R}) =: \mathcal{B}_{u,w,>0}$ is a connected component of $\mathring{\mathcal{B}}_{u,w}(\mathbb{R})$.
So here we suppose otherwise that $\mathcal{B}_{u,w,>0} \cap \mathring{\mathcal{B}}_{\underline{u},\underline{w},>0}(\mathbb{R}) \neq \emptyset$,
then $\mathcal{B}_{u,w,>0}$ contains a connected component of $\mathring{\mathcal{B}}_{\underline{u},\underline{w},>0}(\mathbb{R})$.

Using Marsh-Rietsch parametrization, we get $\mathring{\mathcal{B}}_{\underline{u},\underline{w}}(\mathbb{R}) = \dot{s}_1 y_2(\neq 0) x_1(\mathbb{R}) \dot{s}_1^{-1} \cdot B^+$
which has connected components:

$$\dot{s}_1 y_2(>0) x_1(\mathbb{R}) \dot{s}_1^{-1} \cdot B^+, \qquad \dot{s}_1 y_2(\neq 0) x_1(<0) \dot{s}_1^{-1} \cdot B^+$$

In particular $\mathcal{B}_{\geq 0}$ contains at least one of the above component.

Since $\mathcal{B}_{\geq 0}$ is stable under the action of $G_{\geq 0}$, we have:

$$x_1(>0)\mathcal{B}_{\geq 0} \subseteq \mathcal{B}_{\geq 0} \qquad \Rightarrow \qquad x_1(>0)\dot{s}_1 y_2(\neq 0) x_1(\mathbb{R}) \dot{s}_1^{-1} \cdot B^+ \cap \mathcal{B}_{\geq 0} \neq \emptyset$$

By SL_2 calculation, we have:

$$x_1(>0)\dot{s}_1 y_2(\neq 0) x_1(\mathbb{R}) \dot{s}_1^{-1} \cdot B^+ = y_1(>0) x_1(<0) \alpha_1^\vee(>0) y_2(\neq 0) \dot{s}_1^{-1} \cdot B^+ \subseteq y_1(>0) y_2(\neq 0) y_1(<0) \cdot B^+$$

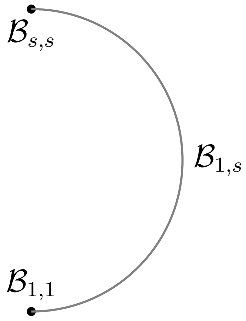
Since the rightmost set $\subseteq \mathring{\mathcal{B}}_{\underline{u},\underline{w}}(\mathbb{R})$, and $\mathcal{B}_{u,w,>0}$ is a connected component of $\mathring{\mathcal{B}}_{u,w}(\mathbb{R})$,
$\mathring{\mathcal{B}}_{u,w}(\mathbb{R}) \cap \mathring{\mathcal{B}}_{\underline{u},\underline{w}}(\mathbb{R})$ is a union of connected components,
which contains $\{y_1(a) y_2(b) y_1(c) \cdot B^+ : a > 0, c < 0, b \text{ with some fixed sign}\}$.
Then by taking limit $a \to 0, b \to 0$, we get $\{y_1(c) \cdot B^+; c < 0\} \subseteq \mathcal{B}_{\geq 0}$ which is a contradiction.

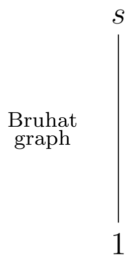
3 This week: Regularity theorem

Before going into the statement, first see 2 examples of $SL_2,\ SL_3$.

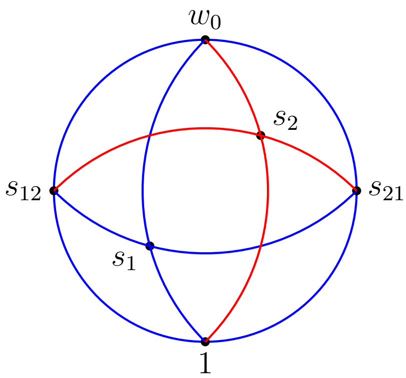
3.1 Example for regularity theorem

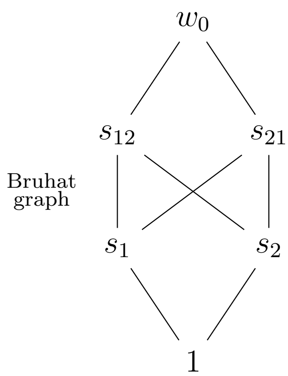
$G = SL_2$





$G = SL_3$
(In the figure, red segments are behind the sphere)
(Note: The TP cells involves calculation, not easy to see)





Then the TP cells are given as follows:
3-dim: $\mathcal{B}_{1,w_0,>0} \rightsquigarrow$ the open 3-ball.

2-dim:	$\mathcal{B}_{1,s_{12},>0}$	$\mathcal{B}_{1,s_{21},>0}$	$\mathcal{B}_{s_1,w_0,>0}$	$\mathcal{B}_{s_2,w_0,>0}$
	lower left sphere	lower right sphere	top front sphere	top back sphere

$\rightsquigarrow$ 4 open 2-balls (open discs).

1-dim: $\mathcal{B}_{1,s_1,>0},\ \mathcal{B}_{1,s_2,>0},\ \mathcal{B}_{s_1,s_{12},>0},\ \mathcal{B}_{s_1,s_{21},>0},\ \mathcal{B}_{s_2,s_{12},>0},\ \mathcal{B}_{s_2,s_{21},>0},\ \mathcal{B}_{s_{12},w_0,>0},\ \mathcal{B}_{s_{21},w_0,>0}$
$\rightsquigarrow$ 8 open 1-balls (half circles).

0-dim: $\mathcal{B}_{1,1,>0},\ \mathcal{B}_{s_1,s_1,>0},\ \mathcal{B}_{s_2,s_2,>0},\ \mathcal{B}_{s_{12},s_{12},>0},\ \mathcal{B}_{s_{21},s_{21},>0},\ \mathcal{B}_{w_0,w_0,>0}\ \rightsquigarrow$ 6 points.

The above leads to a triangulation of a closed 3-ball.

3.2 Statement of regularity theorem(s)

Theorem 1. *The closure of a TP cell is homeomorphic to a closed ball.*
More precisely, $\overline{\mathcal{B}_{v,w,>0}} = \bigsqcup_{v \leq v' \leq w' \leq w} \mathcal{B}_{v',w',>0}$ *is a regular CW complex.*

Note: regular CW complex means that:

- the closure of each stratum is a union of other strata
- the closure of each stratum is homeomorphic to a closed ball

References of proof:

- Hersh, Regular cell complexes in total positivity, 2014 (for the closure of a TP cell in the big Schubert cell)
- Galashin-Karp-Lam, The totally nonnegative Grassmannian is a ball, 2021 (for the closure of TP cell)
- Bao-He, Product structure and regularity theorem for totally nonnegative flag varieties, 2022 (for general Kac-Moody groups)

Remark: Both [GKL] and [BH] also study the partial flag variety (where it gets more complicated) and the regularity theorem holds in the general case (in addition to the case of $\mathcal{B} := G/B^+$).

3.3 Structure of the proof (by [BH])

The proof consists of inputs from 3 different areas:
Lie-theoretic input: Product structure
$\rightsquigarrow$ the closure of $\mathcal{B}_{v,w,>0}$ is a topological manifold with boundary $= \partial\overline{\mathcal{B}_{v,w,>0}} := \overline{\mathcal{B}_{v,w,>0}} - \mathcal{B}_{v,w,>0}$.

Remark: Note that in general we do not have

$$\text{boundary strata} \;=\; \text{boundary of manifold}$$

For example: $\mathbb{S}^1 = (\mathbb{S}^1 - \{p\}) \bigsqcup \{p\}$,
$\{p\}$ is a boundary strata, $\mathbb{S}^1$ is a manifold without boundary.

Topological input: Generalized Poincare conjecture:
Let X be a compact n-dimensional topological manifold with boundary such that
$\partial X \simeq \mathbb{S}^{n-1}, \ X - \partial X \simeq$ open ball of dimension n,
then $X \simeq$ closed ball of dimension n.
Remark: This theorem may not give an isomorphism between X and the n-ball.

Combinatorial input: Shellability of Coxeter groups
$\rightsquigarrow$ the boundary strata are glued together in the desired way (to use Poincare conjecture).

We will focus on the combinatorial part more.

3.4 Partial order set

Definition: For a partial order set (poset) P:

- P is **bounded** if it has a least element and a greatest element.
- P is **pure** if all the maximal chain have the same length.
- P is **graded** if it is finite, bounded and pure.

Remark: In the usual sense, graded does not usually require finiteness. This definition above is used by combinatorialists.
Example: Let W be a Coxeter group, let $v, w \in W$ with $v \leq w$.
Set $[v, w] := \{u \in W : v \leq u \leq w\}$.
Then $([v, w], \leq)$ is graded.
Remark: We never really use poset of the whole infinite Coxeter groups.

3.5 Simplicial complex

Definition: Let Δ be a simplicial complex.

- A **facet** of Δ is a max. dimensional cell (i.e. not contained in other cells)
- Δ is **pure** if every facet is of the same dimension.
- Δ is **shellable** if it is pure, and facets can be given a linear order $F_1, F_2, \ldots, F_n$ such that $F_k \cap \left(\bigcup_{i=1}^{k-1} F_i \right)$ is a non-empty union of codimensional 1 facets.

For any poset P, we have the following construction of simplicial complex.
Definition: Let P be a poset. The **order complex** $\Delta(P)$ is the simplicial complex with vertices in P, and faces are the chains in P:
Each chain of length k corresponds to $(k-1)$-dimensional cell,
each subchain corresponds to a boundary cell.

Example: Let $P = \{x, y, z\}$ with $x \leq y, y \leq z, x \leq z$.
Then $x \leq y \leq z$ is a 2-dimensional cell (triangle), with:
1-dimensional cell boundaries (lines): $x \leq y,\ x \leq z,\ y \leq z$,
0-dimensional cell boundaries (points): $x,\ y,\ z$.

Definition: Let P be a graded poset.
P is called **EL-shellable** (edge-labelling-shellable)
if we may give each covering relation $x \lessdot y$ (i.e. if $x \leq z \leq y$ then $z = x$ or $z = y$) a label
such that for any $x < y$ in P, exists a unique increasing maximal chain
$c_0 : x \lessdot x_1 \lessdot x_2 \lessdot \cdots \lessdot x_n = y$, and for any other maximal chain from x to y,
labelling of $c_0 <$ labelling of c (in lexicalgraphical order).

We have the following theorem:

Theorem 2. *If P is EL-shellable, then $\Delta(P)$ is shellable.*

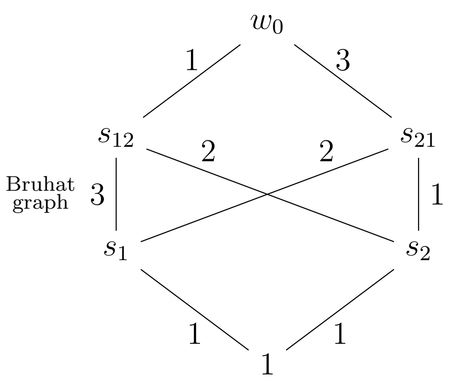
We also have:

Theorem 3 (Dyer)**.** *Reflection order gives shellability of $[v, w]$.*

(Note: Reflection order means a total order on the set of reflections.)

3.6 Example for shellability and EL-shellability

S_3 example: Let $s_1 \to 1,\ s_1s_2s_1 = s_{\alpha_1+\alpha_2} \to 2,\ s_2 \to 3$, i.e. $s_1 < s_1s_2s_1 < s_3$.



EL-shellability:
From 1 to w_0, there are 4 maximal chains:

$1 \lessdot s_1 \lessdot s_{12} \lessdot w_0$ (1,3,1)

$1 \lessdot s_1 \lessdot s_{21} \lessdot w_0$ (1,2,3) (This is c_0 from 1 to w_0)

$1 \lessdot s_2 \lessdot s_{12} \lessdot w_0$ (3,2,1)

$1 \lessdot s_2 \lessdot s_{21} \lessdot w_0$ (3,1,3)

Shellability:
Using the lexicographic order above, let
$F_1 = (1 \lessdot s_1 \lessdot s_{21} \lessdot w_0),\ F_2 = (1 \lessdot s_1 \lessdot s_{12} \lessdot w_0),\ F_3 = (1 \lessdot s_2 \lessdot s_{21} \lessdot w_0),\ F_4 = (1 \lessdot s_2 \lessdot s_{12} \lessdot w_0)$.
Then $F_2 \cap F_1 = (1 < s_1 < w_0)$,
$F_3 \cap (F_1 \cup F_2) = (1 < s_{21} < w_0) \cup (1 < w_0) = (1 < s_{21} < w_0)$.
$F_4 \cap (F_1 \cup F_2 \cup F_3) = (1 < w_0) \cup (1 < s_{12} < w_1) \cup (1 < s_2 < w_0) = (1 < s_{12} < w_1) \cup (1 < s_2 < w_0)$.
All of the above are non-empty unions of codimension-1 facets, so $\Delta(P)$ is shellable.

3.7 More facts about poset and complex

Definition:

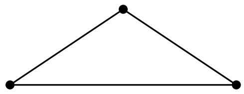
- A pure complex is called **thin** if every codim-1 face is contained in exactly 2 facets.
- It is called **subthin** if it is not thin and every codim-1 face is contained in at most 2 facets.

Facts: Let Δ be a finite shellable pure d-dimensional simplicial complex.

- If Δ is subthin, then it is homeomorphic to a closed ball.
- If Δ is thin, then it is homeomorphic to a sphere.

Example for above:

Subthin complex
(1-dim closed ball)



Thin complex
(1-dim sphere)

What we need is a complex $K(P)$ with cells indexed by a poset P, not by the chains in P. For such CW complex:

Proposition 4. *The complex $K(P)$ is shellable if $\hat{P} := P \sqcup \{\hat{1}\}$ is dual EL-shellable. Where $\hat{1}$ is the augmented greatest element. And dual of $(P, \leq)$ means $(P, \geq)$.*

Back to regularity theorem

Now we consider $\overline{\mathcal{B}_{v,w,>0}} = \bigsqcup_{v \leq v' \leq w' \leq w} \mathcal{B}_{u',w',>0}$.
$P = \{(v', w') : v \leq v' \leq w' \leq w\}$ and the order is given by $P \subseteq (W \times W, (\geq, \leq))$.
$\partial P = \{(v', w') : v \leq v' \leq w' \leq w, (v', w') \neq (v, w)\} = P - \{(v, w)\}$.
$\widehat{\partial P} = P$. (Note that (v, w) is the greatest element in P)
Note that P is dual EL-shellable and thin, then ∂P is homeomorphic to a sphere.

Combining Lie theoretic and combinatorial inputs, we see boundary of manifold $\overline{\mathcal{B}_{v,w,>0}}$ is homeomorphic to a sphere, and by induction hypothesis and generalized Poincare conjecture, we see $\overline{\mathcal{B}_{v,w,>0}}$ is homeomorphic to a closed ball.

4 Further problems

4.1 Arnold's problem & its generalizations

Question: Let $\mathcal{B}^* \subseteq \mathcal{B}$ be the big Richardson variety.
Then how many connected components of $\mathcal{B}^*(\mathbb{R})$ are there?
Answer[Shapiro-Shapiro-Veinstein, Rietsch]: Connected component of $\mathcal{B}^*(\mathbb{R}) \leftrightarrow$ connected components of a certain graph.
Number of connected components is known for type ADE and G_2.
The number for type F_4 is not in literature, but can be calculated by brute force.
The number for type BC remains an open problem.

Also one may ask for the number of connected components for arbitrary open Richardson variety $\mathring{\mathcal{B}}_{v,w}(\mathbb{R})$.

4.2 Another problem about Deodhar components

Question: Given connected component C of the top-dimensional Deodhar component, what is $\overline{C}$?
Answer: Recall that top-dimensional Deodhar component $\mathring{\mathcal{B}}_{\underline{v}_+,\underline{w}}(\mathbb{R}) \simeq (\mathbb{R}^\times)^{J^0_{\underline{v}_+}}$,
the connected component $\leftrightarrow$ $\{\pm 1\}^{J^0_{\underline{v}_+}}$.
What we have proved is that for the connected component corresponding to $\{+1\}^{J^0_{\underline{v}_+}}$ (TP cells), the closure only intersects the top-dimensional Deodhar components for smaller Richardson varieties, and the intersection is one connected component.

4.3 Symmetric spaces

Further reference: Lusztig, Total positivity in symmetric spaces.
Here as a special case of the symmetric space, we get the group G as $(G \times G)/K$, where $K = G_{\text{diag}}$.
$(G/K)_{\geq 0}$ is a disjoint union of the cells.
Question: Describe the topological/geometric structure of the closure of the cells.
Here G is a connected reductive group, σ is an involution on G.
$K = (G^\sigma)^\circ$ is the identity component of the fixed point G^σ.
G/K is symmtric space.

Note: Taking quotient somehow breaks total positivity.
Note 2: Symmetric space is useful on representation of real Lie group.

4.4 Related topics

- Postnikov, Positive Grassmannian, lectures by A. Postnikov (Combinatorial)
- Cluster algebra
- Amplituhedron (mathematical physics)