Suggested Solution to Homework 4

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P126, 13.(Annihilator) Let $M \neq \emptyset$ be any subset of a normed space X. The annihilator M^a of M is defined to be the set of all bounded linear functionals on X which are zero everywhere on M. Thus M^a is a subset of the dual space X' of X. Show that M^a is a vector subspace of X' and is closed. What are X^a and $\{0\}^a$.

Proof. For any scalar α, β and $f, g \in M^a$,

$$(\alpha f + \beta g)(x) = \alpha f(x) + \beta g(x) = \alpha \cdot 0 + \beta \cdot 0 = 0, \forall x \in M.$$

Thus, $\alpha f + \beta g \in M^a$. So, M^a is a vector subspace of X'.
Let $f \in \overline{M^a}$. Then, there exists a sequence of bounded linear functionals $\{f_n\} \subset M^a$ such that $\lim_{n\to+\infty} f_n = f$. Therefore, for any $x \in M$,

$$f(x) = \lim_{n\to+\infty} f_n(x) = \lim_{n\to+\infty} 0 = 0.$$

which yields that $f \in M^a$. So, M^a is closed.
By the definition of annihilator,

$$X^a = \{f \in X' | f(x) = 0, \forall x \in X\} = \{\theta\}, \quad \text{where } \theta \text{ is the zero functional on } X.$$

$$\{0\}^a = \{f \in X' | f(0) = 0\} = X', \quad \text{since every bounded linear functional maps } 0 \in X \text{ to be } 0.$$

□

P238, 9.(Annihilator) Let X and Y be normed spaces, $T : X \to Y$ a bounded linear operator and $M = \overline{\mathscr{R}(T)}$, the closure of the range of T. Show that

$$M^a = \mathscr{N}(T^\times).$$

Proof. On the one hand, let $f \in M^a \subset Y'$, then

$$(T^\times f)(x) = f(Tx) = 0, \quad x \in X \text{ such that } Tx \in \mathscr{R}(T) \subseteq M.$$

So, $f \in \mathscr{N}(T^\times)$ which yields that $M^a \subseteq \mathscr{N}(T^\times)$. On the other hand, let $g \in \mathscr{N}(T^\times)$, then, for any $y \in M$, there exists a sequence of $\{x_n\} \in X$ such that $y = \lim_{n\to+\infty} Tx_n$. Since $g \in \mathscr{N}(T^\times)$ is continuous, we have

$$g(y) = g(\lim_{n\to+\infty} Tx_n) = \lim_{n\to+\infty} g(Tx_n) = \lim_{n\to+\infty} (T^\times g)(x_n) = 0.$$

So, $g \in M^a$ which yields that $\mathscr{N}(T^\times) \subseteq M^a$.
Therefore, $M^a = \mathscr{N}(T^\times)$. □

P239, 10. Let B be a subset of the dual space X' of a normed space X. The annihilator ${}^aB$ of B is defined to be

$${}^aB = \{x \in X | f(x) = 0 \text{ for all } f \in B\}.$$

Show that, in the above problem,

$$\mathscr{R}(T) \subset {}^a\mathscr{N}(T^\times).$$

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What does this mean with respect to the task of solving an equation $Tx = y$?

Proof. Let $y = Tx \in \mathscr{R}(T)$. Then, for any $f \in \mathscr{N}(T^{\times})$, since $T^{\times} f = 0$, we have

$$f(y) = f(Tx) = (T^{\times} f)(x) = 0.$$

which yields that $y \in^{a} \mathscr{N}(T^{\times})$. So, $\mathscr{R}(T) \subset^{a} \mathscr{N}(T^{\times})$.

This means that a necessary condition for the existence of solution to $Tx = y$ is that $f(y) = 0, \forall f \in \mathscr{N}(T^{\times})$. $\square$