MAT3280 Introductory to Probability, 2014-15 1^{st} term

Text: S. Ghahramani, Fundamentals of probability, 3^{rd} ed. Prentice Hall.

H.W: 10%
Test: 30% : 27/10/2014 (Mon)
Exam: 60% :

Ch 1: Axioms of Probability.

Notation:

- Ω: a set; an element x in Ω, write $x \in A$ (x belongs to Ω)
- subset: $A \subseteq \Omega$ ie: $\{x \in \Omega \mid x \in A\}$, $A, B, C \ldots$ are subsets of Ω
- Two sets A and B are said to be equal if $A \subseteq B$ and $B \subseteq A$
- intersection: $A \cap B = \{x \mid x \in A \text{ and } x \in B\}$ (or AB)
- union: $A \cup B = \{x \mid x \in A \text{ or } x \in B\}$
-
- complement: $A^c \equiv \{x \in \Omega \mid x \in A\}$
- difference: $A - B \equiv \{x \in \Omega \mid x \in A \text{ and } x \notin B\}$
- ϕ: empty set: A set ϕ does not contain any element.
- $A \times B \equiv \{(x,y) \mid x \in A, y \in B\}$, $\prod_{i=1}^{\infty} A_i \equiv \{(x_1, x_2, \ldots) \mid x_i \in A_i\}$

Remark: $\phi \subseteq \Omega$

- Two sets A and B are said to be mutually exclusive (or disjoint) if $AB = \phi$
- power set of Ω, $\mathcal{P}(\Omega) \equiv \{A \mid A \subseteq \Omega\}$

Basic Properties:

①: $\phi \subseteq \cancel{A}\,\Omega$, ;

②: $A \cap (B \cup C) = (A \cap B) \cup (A \cup C)$

$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

③: $\left(\bigcup_i A_i\right)^c = \bigcap_i A_i^c$

$\left(\bigcap_i A_i\right)^c = \bigcup_i A_i^c$

$\phi^c = \Omega$, $\Omega^c = \phi$

$\square$

From now on, a sample space, write Ω, means a non-empty set.

Def: A collection of subsets, write $\mathcal{E}$, of Ω is said to be a collection of events if it satisfies the following circls:

①: $\Omega \in \mathcal{E}$

②: if $E \in \mathcal{E} \Rightarrow E^c \in \mathcal{E}$

③: If $E_1, E_2 \cdots \in \mathcal{E}$, then $\bigcup_{n=1}^{\infty} E_n \in \mathcal{E}$

An element $E \in \mathcal{E}$ is called an event.

e.g.: Let $\Omega \equiv \{a, b, c, d\}$

- Put $\mathcal{E}_0 \equiv \mathcal{P}(\Omega)$ the collection of ~~all~~ all events. ie: ~~any~~ In this case, any subset of Ω is an event.
- Put $\mathcal{E}_{00} \equiv \{\phi, \Omega\}$ is a collection of events.

e.g.: Let $\Omega \equiv \{a, b, c, d\}$

Put $\mathcal{E}_1 \equiv \{\phi, \Omega, \{a\}, \{b, c\}\}$

Then $\mathcal{E}_1$ is a collection of events.

N.B.: $\{a\}$ is an event, $\{b\}$ is not an event.

e.g. Let $\Omega \equiv \{a, b, c, d\}$

Put $\mathcal{E}_2 \equiv \{\phi, \Omega, \{a\}, \{b\}, \{b, c\}\}$ is not a collection of events.

Remark: The notation of events depends on the choice of $\mathcal{E}$.

From now on, let $\mathcal{E}$ be a collection of events on Ω.

Prop: With the notation as above, we have

(i): $\phi \in \mathcal{E}$

(ii) If $A, B \in \mathcal{E}$ then

$AB \in \mathcal{E}$, $A \cup B \in \mathcal{E}$, $A \setminus B \in \mathcal{E}$.

(iii) If $E_1, E_2 \dots$ is a seq of events, then

$\bigcap E_n \in \mathcal{E}$

Prop: Let ξ_0 be a collection of subsets of Ω. Then $\exists!$ a collection of events ξ in Ω st

(i) $\xi_0 \subseteq \xi$

and (ii) if ξ_1 is another collection of events st $\xi_0 \subseteq \xi_1$, then $\xi \subseteq \xi_1$

ie: ξ is the smallest ~~of~~ of events ~~that~~ which contains ξ_0

In this case, ξ is said to be the collection of events generated by ξ_0

eg: Let $\Omega \equiv \prod_{i=1}^{\infty}\{0,1\} \equiv \{(x_1, x_2 \ldots) \mid x_i = 0 \text{ or } 1\}$

For each $C_\ell \equiv \{(x_1, \ldots) \in \Omega \mid x_\ell = 1\}$

$\mathcal{E}_0 \equiv \{C_\ell \mid \ell = 1, 2 \ldots\}$

$\mathcal{E}_1 \equiv$ the σ-alg generated by $\mathcal{E}_0$.

Axioms of probability: Let Ω be a sample space and $\mathcal{E}$ be a ~~col~~ σ-algebra on Ω.

A function $P : \mathcal{E} \to \mathbb{R}$ is said to be a probability on Ω if it satisfies the following conditions:

① : $P(E) \geq 0$. ~~f~~ $\forall E \in \mathcal{E}$

② : $P(\Omega) = 1$

③ : if $E_1, E_2 \ldots$ are pairwise disjoint events, i.e. $E_i E_j = \phi$, $i \neq j$, then

$$P\left(\bigcup_{n=1}^{\infty} E_n\right) = \sum_{n=1}^{\infty} P(E_n).$$

In this case $P(E)$ is called the probability of E.

And $(\Omega, \mathcal{E}, P)$ is called a prob space e.g.: ~~Let~~ Let Ω be a finite sample space and $\xi \equiv \wp(\Omega)$.

For each $E \in \xi$, put $P_0(E) \equiv \frac{|E|}{|\Omega|}$

Then P_0 is a probability

e.g.: Let $\Omega = \{a, b\}$

$\cancel{\xi = \{a} \quad \xi \equiv \wp(\Omega) = \{\{a\},\{b\}, \{a,b\}, \phi\}$

Define $P\{a\} = \frac{1}{3}$, $P(\{b\}) = \frac{2}{3}$, $P(\Omega) = 1$,

$P(\phi) = 0$

Then P is a probability.

Remark: ~~If~~ Given (Ω, ξ), we can have different probality defined on Ω.

Prop. Let (Ω, ξ, P) be a prob space. Then we have

①: $P(\phi) = 0$

②: if $A, B \in \xi$ with $AB = \phi$, then
$P(A \cup B) = P(A) + P(B)$

③: if $A, B \in \xi$ then
$P(A \cup B) = P(A) + P(B) - P(AB)$

④. if $A \in \xi$, then $P(A^c) = 1 - P(A)$

⑤: if $A \subseteq B$, then $P(A) \leq P(B)$
Hence $P(A) \leq 1$. $\forall A \in \xi$.

Remk: if $P(A) = 0 \not\Rightarrow A = \phi$

Notation: A seq of events (E_n) is said to be increasing (resp. decreasing): if

$E_1 \subseteq E_2 \subseteq \ldots$ (resp $E_1 \supseteq E_2 \supseteq \ldots$)

Prop ① if (E_n) be an increasing seq of events then $P(\bigcup_{n=1}^{\infty} E_n) = \lim_n P(E_n)$

② if $E_n \downarrow$ then $P(\bigcap E_n) = \lim_n P(E_n)$

Prop: ~~let~~

(i): If $E_n \uparrow \Rightarrow P(\bigcup_{n=1}^{\infty} E_n) = \lim_n P(E_n)$

(ii): If $E_n \downarrow \Rightarrow P(\bigcap E_n) = \lim_n P(E_n)$

pf: (i): Let $F_1 \equiv E_1$, $F_2 \equiv E_2 - E_1$, ...

$F_n \equiv E_n - E_{n-1}$, ...

$\Rightarrow \bigcup F_n = \bigcup E_n$ and $F_i F_j = \phi \quad \forall i \neq j$

$\Rightarrow P(\bigcup E_n) = P(\bigcup F_n) = \sum_{n=1}^{\infty} P(F_n) = P(F_1) + \sum_{n=2}^{\infty} P(F_n)$

$= P(E_1) + \lim_N \sum_{n=2}^{N} [P(E_n) - P(E_{n-1})] = \lim_N P(E_N)$.

(ii). If $E_n \downarrow \Rightarrow E_n^c \uparrow$

$\Rightarrow$ Since $\bigcap E_n = (\bigcup E_n^c)^c$

$\Rightarrow P(\bigcap E_n) = P(\bigcup E_n^c)^c = 1 - P(\bigcup E_n^c)$

$= 1 - (1 - \lim_n P(E_n)) = \lim_n P(E_n)$ $\square$

Cor: Let (E_n) be a seq of events. Then

$$P(\bigcup E_n) \le \sum_{n=1}^{\infty} P(E_n).$$

pf: N.B: By induction, we have

$$P(\bigcup_{n=1}^{N} E_n) \overset{(*)}{\le} \sum_{n=1}^{N} P(E_n), \quad \forall 1 \le N < \infty.$$

Let $\quad F_N \equiv \bigcup_{n=1}^{N} E_n$

$\Rightarrow \quad F_N \uparrow \quad$ and $\quad \bigcup F_N = \bigcup E_n$

Then $\quad P(\bigcup E_n) = P(\bigcup F_N) = \lim_N P(F_N)$

$\overset{\text{by } (*)}{\le} \lim_N \sum_{n=1}^{N} P(E_n).$ $\quad \square$

e.g.: Let $\Omega \equiv [0,1]$. Let $\mathcal{E}$ be the σ-algebra generated by $\{[a,b) \mid 0 \le a < b \le 1\}$

For each $[a,b)$, define
$$P_0([a,b)) = b-a$$

Then $\exists!$ P_0 extends to a probability P on $\mathcal{E}$

~~Hence $P([a,b)) = b-a$~~

Then $\quad P(\{\tfrac{1}{2}\}) = 0. \quad$ (N.B: $\{\tfrac{1}{2}\} \in \mathcal{E}$)

since $\{\tfrac{1}{2}\} = \bigcap_{n=3}^{\infty} [\tfrac{1}{2}, \tfrac{1}{2}+\tfrac{1}{n})$

~~N.B: $\{a\} \in \mathcal{E}, \forall a \in [0,1]$~~ $\quad \square$

From the eg above, we see that

$\{a\} \in \mathcal{E}, \quad \forall a \in [0,1]$

and $P\{a\} = 0$.

HW 1: P34: Ex 8, (18/09/2014)

P35: Ex. 12.

Deadline: 26/09 (Next Fri).

Ch 3. Conditional Probability

Throughout this chapter, (Ω, ξ, P) always denotes a prob space.

Let $A, B, C, \ldots \in \xi$.

~~Defn~~ Observation:

Let $\Omega = \{(1,0), (0,1), (0,0), (1,1)\}$

$\xi \equiv \wp(\Omega)$

$P(E) \equiv \dfrac{|E|}{4}$ for $E \in \xi$.

"1" represents successful

"0" " failure

Find the probability of both trials being successful given the first trial being failure

That is $A \equiv \{(1,1)\}$

let A $B \equiv \{(1,0), (1,1)\}$

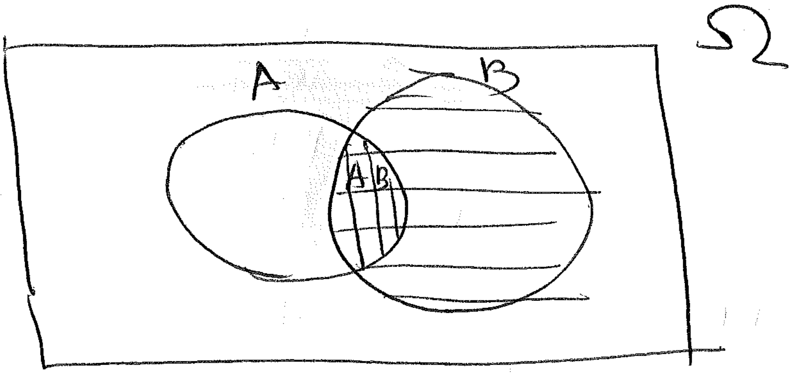
$$P(A|B) = \frac{|A|}{|B|} = \frac{1}{2}$$

$$= \frac{\frac{|A|}{4}}{\frac{|B|}{4}} = \frac{P(A \cap B)}{P(B)}$$

Def: Let $A, B \in \xi$. Assume that $P(B) > 0$.

The conditional probability of A given B, write $P(A|B)$, is defined by

$$P(A|B) = \frac{P(AB)}{P(B)}$$



e.g: Assume that A die with faces 1, 2, 3 colored ~~green~~ blue;

faces 4, 5, 6 colored red.

Suppose that $P(\{1\}) = P(\{3\}) = P(\{5\}) = \frac{1}{9}$

$P(\{2\}) = P(\{4\}) = P(\{6\}) = \frac{2}{9}$

~~solution~~ What is the prob that the outcome is even given the die lands blue face up?

Sol: Let $B = \{$the die lands blue face up$\}$

$A = \{$the die shows even number$\}$

Note: $\cancel{P(B)}\ B = \{1, 2, 3\}$

$$\therefore P(B) = P(1) + P(2) + P(3)$$
$$= \frac{1}{9} + \frac{2}{9} + \frac{1}{9} = \frac{4}{9}$$

and $AB = \{2\} = \frac{2}{9}$

$$\therefore P(A|B) = \frac{P(AB)}{P(B)} = \frac{\frac{2}{9}}{\frac{4}{9}} = \frac{2}{4} = \frac{1}{2}$$

$\square$

Prop: With the notation as above, fix $B \in \mathcal{E}$ with $P(B) > 0$, define

$Q: \mathcal{E} \to [0, \infty)$ by

$$Q(A) = P(A|B)$$

Then Q is a prob on Ω.

$\square$

~~Prop~~ 2

Notation: A sequence of events, $E_1, E_2, \ldots$ is said to be a partation on Ω if $E_iE_j = \phi$, $i \neq j$ and $\Omega = \bigcup_{i=1}^{\infty} E_i$ and $P(E_i) > 0$. for all i.

$1 \leq N \leq \infty$.

Prop: Let $\{E_1, E_2 \ldots E_N\}$ be a partation on Ω.

Let $A \in \mathcal{E}$. Then

$$P(A) = \sum_{i=1}^{\cancel{\infty} N} P(A|E_i)P(E_i)$$

pf: Let $A \in \mathcal{E}$.

Since $\Omega = \bigcup_{i=1}^{N} E_i$, $A = A\Omega = \bigcup_{i=1}^{N} (AE_i)$

On the other hand, since $E_iE_j = \phi$, $i \neq j$,

$AE_i \cap AE_j = \phi$, $i \neq j$.

Hence $P(A) = \sum_{i=1}^{n} P(AE_i)$

Note $P(AE_i) = P(A|E_i)P(E_i)$

$\square$

Prop (Bayes' Formula). With the notation as above, then

$$P(E_i|A) = \frac{P(A|E_i)P(E_i)}{\sum_{j=1}^{N} P(A|E_j)P(E_j)} \quad (\text{assume } P(A)>0),$$

for all $1 \le i \le N$ ~~$(1 \le N \le \infty)$~~ $(1 \le i \le N)$

pf: Fix $1 \le i \le N$. (For simply, assume $N<\infty$)

Note: $P(E_i|A) = \dfrac{P(E_iA)}{P(A)}$;

and $P(E_iA) = P(A|E_i)P(E_i)$.

By above prop, $P(A) = \sum_{j=1}^{N} P(A|E_j)P(E_j)$

$\square$

e.g: There is a blood test for a disease E.
If a ~~people~~ person has disease E, then the test will show +ve A^+.

However every 100 health person, there are two ~~peop~~ person also show +ve A^+.

We know that every 10000 person, ~~has one~~ there ~~p~~ is 1 person has this disease

What is the prob of a ~~person~~ person really having ~~d~~ ~~with~~ this disease if ~~A~~ the test shows a +ve result?

sol: Aim ~~to~~ to find $P(E|A^+)$.

We know that

$$P(A^+|E) = 1 \quad , \quad P(E) = 0.0001$$

$$P(A^+|E^c) = 0.02$$

Then $$P(E|A^+) = \frac{P(A^+|E)P(E)}{P(A^+|E)P(E) + P(A^+|E^c)P(E^c)}$$

$$\Rightarrow P(E|A^+) = 0.00497$$

Remark: if $P(E) = 0.1 \Rightarrow P(E|A^+) = 0.847$.

$\square$

e.g: There are three cards. Suppose that the ~~b~~ both sides of the first card are in red color the both sides of the second card are ~~a~~ in black.
~~the third~~
~~at~~; one side of the third card is in red and another side is in black.

~~Three cards a~~ One card is randomly ~~the~~ selected in these three cards.
If the upper side of the chosen card is in red, what is the prob of another side is in black?

sol: Let RR, BB, RB be three cards.

~~[illegible]~~ Let R = the event that the upper side of chosen card is in red.

Aim to find $P(RB \mid R) = ?$
Not:
$$P(RR) = P(RB) = P(BB) = \frac{1}{3}.$$
and
$$P(RB \mid R) = \frac{P(R \mid RB)\,P(RB)}{P(R \mid RB)\,P(RB) + P(R \mid BB)\,P(BB) + P(R \mid RR)\,P(RR)}$$

Not: $P(R \mid RB) = \frac{1}{2}$
$P(R \mid BB) = 0$
$P(R \mid RR) = 1$

$$\Rightarrow P(RB \mid R) = \frac{1}{3}$$ $\square$

Def: Two events A and B are said to be independent (more precise P-independent) if $P(AB) = P(A)P(B)$.

e.g: Let $\Omega = \{(1,0), (0,1), (0,0), (1,1)\}$

$\mathcal{E} = \mathcal{P}(\Omega)$, $P(E) = \frac{|E|}{4}$.

Let $A = \{(1,0), (1,1)\}$ $\Rightarrow P(A) = P(B) = \frac{1}{2}$

$B = \{(0,1), (1,1)\}$

$\Rightarrow$ $A \cap B = \{(1,1)\}$ $\Rightarrow P(AB) = \frac{1}{4}$

$\therefore$ $P(AB) = \frac{1}{4} = P(A)P(B)$

Here A, B are indep.

e.g: ~~What~~ Let Ω and $\mathcal{E}$ be as above

Put $P_0(1,0) = P_0(0,1) = P_0(0,0) = \frac{1}{3}$

and $P_0(1,1) = 0$

Then $P_0(A) = \frac{1}{3} = P_0(B)$

$P_0(AB) = 0$

$\therefore$ A, B are not P_0- independent

Remark: The notition of ~~independents~~ independent events depends on the choice of P.

Remark: A, B ~~are~~ disjoint $\not\equiv$ A, B independent

e.g.: As above examples in 18.

- A, B are P-indep but A, B are not disjoint
- $\{(1,0)\}$ and $\{(0,1)\}$ are disjoint but not P_0-indep.

$\square$

Prop: Let A, B are independent events Then

①: A, B^c are indep

②: A^c, B^c are indep

③: $P(A|B) = P(A)$.

pf ①: $\vdash$: $P(AB^c) = P(A)\,P(B^c)$

Note: $A = AB \cup AB^c$ and $(AB) \cap (AB^c) = \phi$

Then $P(A) = P(AB) + P(AB^c)$

$\Rightarrow \quad P(AB^c) = P(A)(1 - P(B)) = P(A)\,P(B^c)$. $\square$