MMAT 5000 Analysis I, 2016/17, Test

Answer ALL Questions 29 Oct, 2016. 10:00-12:00

1. Let (X, d) be a metric space. Put $\rho(x, y) := \frac{d(x,y)}{1+d(x,y)}$, for all $x, y \in X$.

   (i) (5 *points*) Show that ρ is a metric on X.

   (ii) (8 *points*) Let(x_n) be a sequence in X. Show that (x_n) is convergent with respect to the metric d if and only if it is also convergent with respect to the metric ρ.

   (iii) (7 *points*) Let A be a subset of X. Show that A is open with respect to d if and only if it is open with respect to ρ.

2. (i) (4 *ponits*) Let $A = \{\sqrt{2n} : n = 1, 2, ...\}$. What is the closure of A in $\mathbb{R}$?

   (ii) (4 *points*) Let A be as in Part (i). What is the boundary set of A in $\mathbb{R}$?

   (iii) (4 *points*). For a pair of non-empty subsets A and B of $\mathbb{R}$, let $A + B$ be the set defined by $\{a + b : a \in A,\ b \in B\}$. Show that $\overline{A} + \overline{B} \subseteq \overline{A + B}$.

   (iv) (8 *points*) If A and B both are non-empty closed subsets of $\mathbb{R}$, does it imply that the set $A + B$ is also closed in $\mathbb{R}$?

3. Let $(X_i, d_i)_{i=1}^{\infty}$ be a sequence of metric spaces. Suppose that $d_i(x_i, y_i) \leq 1$ for all $x_i, y_i \in X_i$ and for all $i = 1, 2....$ Put $X := \{(x_i) : x_i \in X_i,\ i = 1, 2, ...\}$. Define

$$d((x_i), (y_i)) = \sum_{i=1}^{\infty} \frac{d_i(x_i, y_i)}{2^i}$$

   for (x_i) and (y_i) in X.

   (i) (5 *ponits*) Show that d is a metric on X.

   (ii) (7 *points*) Let W_i be a non-empty open subset of X_i for $i = 1, ..., N$. where $1 \leq N \leq \infty$. Let $Y = \{(x_i) \in X : x_i \in W_i,\ \ i = 1, 2, ..., N\}$. Show that if $N < \infty$, then Y is an open subset of X with respect to the metric d.

   (iii) (8 *points*) Give an example to show that the assertion in Part (ii) does not hold in general if $N = \infty$.

End

1 Answer

(1.i) See the homework.

(1.ii) Notice that since $d(x,y) \geq 0$ for all $x, y \in X$, we see that $\rho(x,y) < 1$ for all $x, y \in X$ and thus we have

$$d(x,y) = \frac{\rho(x,y)}{1-\rho(x,y)} \tag{1}$$

for all $x, y \in X$. Thus we have $d(x_n, x) \to 0$ if and only if $\rho(x_n, x) \to 0$. So Part (ii) follows.

(1.iii) Suppose that A is ρ-open. Then for any $z \in A$, there is $r > 0$ such that if $\rho(z,x) < r$ implies $x \in A$. On the other hand, notice that $\rho(x,y) \leq d(x,y)$ for all $x, y \in X$. So if $d(z,x) < r$, then $\rho(z,x) < r$ and hence $x \in A$. Thus A is d-open.
Conversely, assume that A is d-open. Let $z \in A$. So, there is $\delta > 0$ such that if $d(z,x) < \delta$, then $x \in A$. On the other hand, if $\rho(x,y) < 1/2$, then we have $d(x,y) \leq 2\rho(x,y)$ by the Eq 1 above. Now take $0 < r < \min(\delta/2, 1/2)$. So if $\rho(z,x) < r$, then we have $d(z,x) \leq 2\rho(z,x) < 2r < \delta$. It follows that $x \in A$ by the choice of δ. So A is ρ-open.

(2.i) One can directly check that $\overline{A} = A$ (need to check).

(2.ii) The boundary set of A is A itself (need to check).

(2.iii) Let $x \in \overline{A}$ and $b \in \overline{B}$. Then there are sequences (x_n) and (y_n) in A and B respectively such that $\lim x_n = a$ and $\lim y_n = b$. This implies that $\lim(x_n + y_n) = x + y$ and so $x + y \in \overline{A+B}$.

(2.iv) Let A be as in Part (i) and $B = \{-\sqrt{2n-1} : n = 1, 2, ...\}$. Then A and B both are closed subsets. Notice that $\lim(\sqrt{2n} - \sqrt{2n-1}) = 0$. So $0 \in \overline{A+B}$ but notice that $0 \notin A + B$. Therefore $A + B$ is not closed.

(3.i) Notice that since $d_i \leq 1$ on X_i. So the series d is convergent and thus d is well defined. Also one can directly check that d is a metric on X (check!).

(3.ii) Fix an element $a = (a_i) \in Y$. So $a_i \in W_i$ for all $i = 1, ..., N$. Since each W_i is open in X_i, so for each $i = 1, 2..., N$, we can find $r_i > 0$ such that $B(a_i, r_i) \subseteq W_i$. Let $0 < r < \min\{\frac{r_1}{2}, \cdots, \frac{r_N}{2^N}\}$. Now if for $x = (x_i) \in X$ with $d(a,x) < r$, then $\frac{d_i(a_i, x_i)}{2^i} < r$ for all $i = 1, 2....$ In particular, we have $d_i(a_i, x_i) < r_i$ for all $i = 1, ..., N$ by the choice of r and hence $x_i \in W_i$ for all $i = 1, .., N$. This gives $x \in Y$ if $x \in B(a, r)$. Therefore, a is an interior point of Y for all $a \in Y$.

(3.iii) Consider $X_1 = X_2 = \cdots = \{0,1\}$ and each d_i is the discrete metric on X_i. Put $W_1 = W_2 \cdots = \{0\}$. Then each W_i is open in X_i for all $i = 1, 2...$ and $Y = W_1 \times W_2 \times \cdots = \{(0, 0, ...)\}$. We are going to show that Y is not open. Let $a = (0, 0, ...)$. It needs to show that for each $r > 0$, there is $x = (x_i) \in X$ such that $d(a, x) < r$ but $x \neq a$. Notice that since $\lim_j \sum_{i \geq j} \frac{1}{2^i} = 0$, there is a positive integer N such that $\sum_{i \geq N} \frac{1}{2^i} < r$. So if we let $x_1 = 0$ for all $i = 1, .., N-1$ and $x_i = 1$ for $i \geq N$. This implies that

$$d(a, x) = \sum_i \frac{d_i(a_i, x_i)}{2^i} = \sum_{i \geq N} \frac{d_i(a_i, x_i)}{2^i} = \sum_{i \geq N} \frac{1}{2^i} < r.$$

So $x \in B(a, r)$ but $a \neq x$ as desired and hence Y is not open.