Solution 4

Exercise 5.12 Let A be a subset of X.

(i) Show that if X is complete, then A is complete if and only if A is closed in X.

(ii) Show that if A is complete, then A is closed in X.

Solution. (i) Let A be a closed subset of a complete metric space X. Let (x_n) be a Cauchy sequence in A. Then (x_n) is also a Cauchy sequence in X. Since X is complete, (x_n) is convergent. Let $L := \lim_n x_n$. As (x_n) is a sequence in the closed set A, its limit L is also in A. Therefore A is complete.

On the other hand, if A is complete, then A is closed in X by (ii).

(ii) Let A be a complete subset of X. Here X may or may not be complete. Suppose (x_n) is a sequence in A converging to a limit L in X. We need to show that $L \in A$. As (x_n) is a convergent sequence, it follows from Proposition 5.3 that (x_n) is also a Cauchy sequence. Since A is complete, (x_n) converges to a limit L' in A. Now the uniqueness of limit implies that $L = L' \in A$. Therefore A is closed in X.

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