Solution 3

Exercise 2.17

(i) Let V be a subset of X. A point $z \in V$ is said to be an interior point of V if there is $r > 0$ such that $B(z, r) \subseteq V$. If we put $\mathrm{int}(V)$ the set of all interior points of V, show that $\mathrm{int}(V)$ is an open subset of X.

(ii) A metric d on X is said to be non-archimedean if it satisfies the strong triangle inequality, that is, $d(x, y) \leq \max(d(x, z), d(z, y))$ for all x, y and $z \in X$ (see also Example 1.2(iv)). Show that if d is a non-archimedan metric on X, then for every closed ball $\overline{B}(a, r) := \{x \in X : d(a, x) \leq r\}$ is an open set in X.

Solution. (i) Let $x \in \mathrm{int}(V)$. By the definition of interior points, there is $r > 0$ such that $B(x, r) \subseteq V$. In particular, for any $y \in B(x, r/2)$, we have

$$B(y, r/2) \subseteq B(x, r) \subseteq V,$$

so that $y \in \mathrm{int}(V)$. Thus

$$B(x, r/2) \subseteq \mathrm{int}(V).$$

As $x \in \mathrm{int}(V)$ is arbitrary, we prove that $\mathrm{int}(V)$ is an open set.

(ii) Let $B = \overline{B}(a, r)$ be an arbitrary closed ball for some $r > 0$. Let $x \in B$. We will show that $B(x, r) \subseteq B$. Indeed, if $y \in B(x, r)$, then

$$d(a, y) \leq \max(d(a, x), d(x, y)) \leq \max(r, r) = r,$$

since d is a non-archimedean metric. Hence $y \in \overline{B}(a, r) = B$. Therefore $B(x, r) \subseteq B$ for any $x \in B$, whence B is open.

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