Solution 1

Exercise 1.7

1. Let (X, d) be a metric space. Define

$$\rho(x, y) = \frac{d(x, y)}{1 + d(x, y)}$$

for $x, y \in X$. Show that ρ is also a metric on X.

Solution. Clearly $\rho : X \times X \to \mathbb{R}$ is a well-defined function. Now we check that ρ satisfies conditions (i)-(iv) in Definition 1.1: For any $x, y, z \in X$,

(i) $\rho(x, y) = \frac{d(x,y)}{1+d(x,y)} \geq 0$ (by condition (i) of d);

(ii) $\rho(x, y) = 0$ if and only if $d(x, y) = 0$ if and only if $x = y$ (by condition (ii) of d);

(iii) $\rho(x, y) = \frac{d(x,y)}{1+d(x,y)} = \frac{d(y,x)}{1+d(y,x)} = \rho(y, x)$ (by condition (iii) of d);

(iv) Note that $\phi(x) := \frac{x}{1+x} = 1 - \frac{1}{1+x}$ is an increasing function on $[0, \infty)$. Hence, by condition (iv) of d, we have

$$\begin{aligned}
\rho(x, y) &= \phi(d(x, y)) \leq \phi(d(x, z) + d(z, y)) \\
&= \frac{d(x, z)}{1 + d(x, z) + d(z, y))} + \frac{d(z, y)}{1 + d(x, z) + d(z, y)} \\
&\leq \frac{d(x, z)}{1 + d(x, z)} + \frac{d(z, y)}{1 + d(z, y)} \\
&= \rho(x, z) + \rho(z, y).
\end{aligned}$$

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2. Let (X, d_X), (Y, d_Y) be metric spaces. Define

$$\rho((x, y), (x', y')) = d_X(x, x') + d_Y(y, y')$$

for $x, x' \in X$ and $y, y' \in Y$. Show that ρ is a metric on the product space $X \times Y = \{(x, y) : x \in X; y \in Y\}$.

Solution. Clearly $\rho : X \times X \to \mathbb{R}$ is a well-defined function. Now we check that ρ satisfies conditions (i)-(iv) in Definition 1.1: For any $(x, y), (x', y'), (x'', y'') \in X \times Y$,

(i) $\rho((x, y), (x', y')) = d_X(x, x') + d_Y(y, y') \geq 0$;

(ii) $\rho((x, y), (x', y')) = 0$ if and only if $d_X(x, x') = 0$ and $d_Y(y, y') = 0$ if and only if $x = x'$ and $y = y'$ if and only if $(x, y) = (x', y')$;

(iii) $\rho((x, y), (x', y')) = d_X(x, x') + d_Y(y, y') = d_X(x', x) + d_Y(y', y) = \rho((x', y'), (x, y))$

(iv) By condition (iv) of d_X and d_Y, we have

$$\begin{aligned}\rho((x,y),(x',y')) &= d_X(x,x') + d_Y(y,y') \\ &\le d_X(x,x'') + d_X(x'',x') + d_Y(y,y'') + d_Y(y'',y') \\ &= \rho((x,y),(x'',y'')) + \rho((x'',y''),(x',y')).\end{aligned}$$

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3. Let (X,d) be a metric space and let A be a subset of X. We say that A is bounded if there is $M>0$ such that $d(a,a') \le M$ for all a,a' in A.

   Show that if $A_1,\ldots,A_N$ $(N<\infty)$ are all bounded subsets of X, $A_1 \cup \cdots \cup A_N$ is also a bounded subset of X.

   Solution. Without loss of generality, we assume that each A_k, $k=1,\ldots,N$ is non-empty. For each $k=1,\ldots,N$, pick $a_k \in A_k$. Set $D = \max\{d(a_j,a_k) : j,k = 1,\ldots N\}$.

   Since each A_k, $k=1,\ldots,N$, is bounded, there is $M_k>0$ such that $d(a,a') \le M_k$ for all $a,a' \in A_k$. Set $M = \max_{1\le k\le N} M_k$ and $C = D+2M$. Now for any $x,y \in A_1 \cup \cdots \cup A_N$, we have $x \in A_i, y \in A_j$ for some i,j, $1 \le i,j \le N$. Therefore

$$\begin{aligned}d(x,y) &\le d(x,a_i) + d(a_i,a_j) + d(a_j,y) \\ &\le M + D + M = C.\end{aligned}$$

   Hence $A_1 \cup \cdots \cup A_N$ is bounded. ◀