Pf: Suppose not, then $\pi_1(M) \neq 1$.

By lemma 8, $\exists$ a closed curve $\gamma : [0,b] \to M$ such that $L(\gamma) \leq L(\alpha)$, $\forall \alpha \neq 1$.

Then γ has to be a geodesic and hence

$$\gamma'(0) = \gamma'(b).$$

We may also assume $|\gamma'(t)| \equiv 1$.

Let $x = \gamma(0) = \gamma(b)$. Then parallel transport along γ: $P^\gamma : T_xM \to T_xM$ has $\det P^\gamma = +1$ (Lemma 7)

Note that eigenvalues of P^γ are of the form ± 1, $e^{i\theta}$ ($\neq \pm 1$), and if $e^{i\theta}$ is an eigenvalue, then $e^{-i\theta}$ is also an eigenvalue.

Since $\det P^\gamma = +1$, then $\dim\{-1\text{-eigenspace}\}$ is even.

Hence $\underline{\dim M = \text{even}} \Rightarrow \dim\{+1\text{-eigenspace}\}$ is also even

Note that γ is a closed geodesics, $\gamma'(0) = \gamma'(b)$

& $P^{\gamma}(\gamma'(0)) = \gamma'(b) = \gamma'(0)$

$\Rightarrow$ dim$\{+1\text{-eigenspace}\} > 0$, hence ≥ 2.

Therefore, $\exists\, e \in T_xM$ s.t. $\begin{cases} P^{\gamma}(e) = e \text{ and} \\ \langle e, \gamma'(0)\rangle = 0. \end{cases}$

Now, let V be the parallel vector field along γ such that $V(0) = e$

Then $V(b) = P^{\gamma}(V(0)) = P^{\gamma}(e) = e$

$\Rightarrow$ V is well-defined vector field on the closed curve γ.

$\Rightarrow$ $\exists$ a 1-parameter family of closed geodesics $\{\gamma_u\}$ s.t. $\gamma_0 = \gamma$ & V = transversal vector field of $\{\gamma_u\}$ ($\gamma_u(t) = \exp_{\gamma(t)}(uV(t))$, $|u| \ll 1$)

Then 2^{nd} variation formula $\Rightarrow$

$$\frac{d^2L}{du^2}(0) = \int_0^b \left[|D_{\gamma'}V^{\perp}|^2 - \langle R_{\gamma' V}\gamma', V^{\perp}\rangle \right] dt$$

(γ_u closed $\Rightarrow$ bdy term $= 0$)

Since $\langle V(0), \gamma'(0)\rangle = \langle e, \gamma'(0)\rangle = 0$, we have

$$V^\perp \equiv V, \quad \forall t \in [0,b].$$

$$\Rightarrow \quad D_{\gamma'} V^\perp = D_{\gamma'} V = 0 \quad (\text{since } V \text{ parallel})$$

$$\therefore \quad \frac{d^2L}{du^2}(0) = -\int_0^b \langle R_{\gamma' V^\perp}\gamma', V^\perp \rangle \, dt$$

$$< 0 \quad (\text{since sectional curv} > 0)$$

Contradicting that γ is length minimizing.

$$\Rightarrow \quad \pi_1(M) = 1.$$

$\divideontimes$

Ch8 Morse index form and Bonnet-Myers Theorem

Let γ = normalized geodesic defined on $[a,b]$

$\mathcal{V} = \mathcal{V}(a,b) = \{ X = \text{piecewise } C^\infty \text{ vector field along } \gamma$
s.t. $\langle X, \gamma' \rangle = 0 \}$

$\mathcal{V}_0 = \mathcal{V}_0(a,b) = \{ X \in \mathcal{V} : X(a) = X(b) = 0 \}$

= the space of transversal vector fields of normal variations of γ.

Def: (1) $I(X,X) = I_a^b(X,X)$

$$= \int_a^b \left[|X'(t)|^2 - \langle R_{\gamma' X}\gamma', X \rangle \right] dt$$

$\forall X \in \mathcal{V}$.

(Note: $\int_a^b |X'(t)|^2 \underline{\underline{\text{means}}} \sum_{i=0}^{k-1} \int_{a_i}^{a_{i+1}} |X'(t)|^2 dt$)

where $a = a_0 < a_1 < \cdots < a_k = b$ s.t. $X|_{[a_i, a_{i+1}]} \in C^\infty$

(2) $I(X,Y) \overset{\text{def}}{=} \frac{1}{2}\left[I(X+Y, X+Y) - I(X,X) - I(Y,Y) \right]$

$\qquad\qquad\qquad \forall X, Y \in \mathcal{V}$

$\| \quad I_a^b(X,Y)$ is called the <u>index form</u> of γ.

Notes: (i) $I(X,Y) = \int_a^b [\langle X', Y' \rangle - \langle R_{\gamma' X}\gamma', Y \rangle] dt$ (Ex!)

(ii) $I(X,Y)$ is bilinear (wrt scalar multiplication) and symmetric.

(iii) If V = transversal vector field of a normal variation $\{\gamma_u\}$ of the normalized geodesic γ, then $V \in \mathcal{V}_0 \subset \mathcal{V}$ and the 2^{nd} variation

$$L''(0) = I(V,V) \quad \left(\text{by } 2^{nd} \text{ variation formula}\right)$$

<u>Lemma 1</u>: Let • $\gamma : [a,b] \to M$ normalized geodesic

• $\gamma(b)$ conjugate to $\gamma(a)$

Then $\forall$ normal Jacobi field V with $V(a)=V(b)=0$ satisfies

$$I(V,V)=0.$$

<u>Pf</u>:
$$\begin{aligned}
I(V,V) &= \int_a^b |V'|^2 - \langle R_{\gamma' V}\gamma', V\rangle \\
&= \int_a^b |V'|^2 + \langle V'', V\rangle \quad (\text{Jacobi eqt.}) \\
&= \int_a^b |V'|^2 + \langle V', V\rangle' - |V'|^2 \\
&= \langle V', V\rangle\Big|_a^b = 0 \quad \divideontimes
\end{aligned}$$

<u>Note</u>: Therefore, if $\gamma(b)$ conjugate to $\gamma(a)$, then the index form of γ is <u>degenerate</u>.

Terminology: A geodesic $\gamma:[a,b]\to M$ is said to contain no conjugate point if $\gamma(a)$ has no conjugate point along γ.

Lemma 2: Let
- $\gamma:[a,b]\to M$ normalized geodesic
- γ has no conjugate point

Then $I(X,Y)$ is positive definite on $\mathcal{V}_0(a,b)$.

Lemma 3 Let
- $\gamma:[a,b]\to M$ normalized geodesic
- $\gamma(b)$ conjugate to $\gamma(a)$
- $\gamma(c)$ is not conjugate to $\gamma(a)$, $\forall\, c\in(a,b)$.

Then $I(X,Y)$ is semi-positive definite on $\mathcal{V}_0(a,b)$, but not positive definite.

Lemma 4 Let
- $\gamma:[a,b]\to M$ normalized geodesic

Then $\exists\, c\in(a,b)$ s.t. $\gamma(c)$ is conjugate to $\gamma(a)$

$\iff \exists\, X\in\mathcal{V}_0(a,b)$ s.t. $I(X,X)<0$.

Cor: If $\gamma:[a,b]\to M$ is a normalized geodesic which contains no conjugate point, then $\forall\,[\alpha,\beta]\subset[a,b]$, $\gamma|_{[\alpha,\beta]}$ has no conjugate point.

Pf: Suppose not, then $\exists\ [\alpha,\beta]$ s.t. $\gamma(\beta)$ conjugate to $\gamma(\alpha)$. Then by Lemma 3, $\exists\ J \neq 0 \in \mathcal{V}_0(\alpha,\beta)$ s.t. $I_\alpha^\beta(J,J) = 0 \quad (J(\alpha)=J(\beta)=0)$

Define a piecewise C^∞ vector field X along $\gamma:[a,b]\to M$ by

$$X = \begin{cases} J, & t\in[\alpha,\beta] \\ 0, & \text{otherwise.} \end{cases}$$

Then X is well-defined and belongs to $\mathcal{V}_0(a,b)$

$$I_a^b(X,X) = \int_a^b |X'|^2 - \langle R_{\gamma' X}\gamma', X\rangle$$

$$= \int_\alpha^\beta |J'|^2 - \langle R_{\gamma' J}\gamma', J\rangle$$

$$= I_\alpha^\beta(J,J) = 0.$$

Hence Lemma 2 $\Rightarrow$ $\gamma:[a,b]\to M$ contains conjugate point.

Contradiction. ※

To prove Lemmas 2–4, we need the following

<u>Claim</u>: For $X, Y \in C^\infty$

$(*)\quad I_a^b(X,Y) = \langle X',Y\rangle\Big|_a^b - \int_a^b \langle X'' + R_{\gamma' X}\gamma', Y\rangle\, dt$

Pf: $I(X,Y) = \int_a^b \langle X',Y'\rangle - \langle R_{\gamma' X}\gamma', Y\rangle$

$= \int_a^b \langle X',Y\rangle' - \langle X'',Y\rangle - \langle R_{\gamma' X}\gamma', Y\rangle$

$= \langle X',Y\rangle\Big|_a^b - \int_a^b \langle X'' + R_{\gamma' X}\gamma', Y\rangle\, dt$

#

Claim: For piecewise C^∞ X, Y, with

$X_i = X\big|_{[a_i, a_{i+1}]} \in C^\infty$ where $a = a_0 < a_1 < \cdots < a_k = b$.

$(*)\quad I(X,Y) = \sum_{i=0}^{k-1} \langle X_i', Y\rangle\Big|_{a_i}^{a_{i+1}} - \sum_{i=0}^{k-1}\int_{a_i}^{a_{i+1}} \langle X_i'' + R_{\gamma' X_i}\gamma', Y\rangle\, dt$

Lemma 5: Let
- $\gamma : [a,b] \to M$ normalized geodesic
- $U \in \mathcal{V}(a,b)$

Then $I(U, \mathcal{V}_0) = 0 \iff U$ is a Jacobi field.

Pf: $(\Leftarrow)$ By $(*)$, $\forall\, Y \in \mathcal{V}_0$

$I(U,Y) = \sum_{i=0}^{k-1} \langle U', Y\rangle\Big|_{a_i}^{a_{i+1}} - \sum_{i=0}^{k-1}\int_{a_i}^{a_{i+1}} \langle U'' + R_{\gamma' U}\gamma', Y\rangle\, dt$

(Jacobi field $U \in C^\infty$, $Y(a) = Y(b) = 0$)

$= 0 - 0$. (by Jacobi eqt.)

$(\Rightarrow)$ Suppose $I(V, \mathcal{V}_0) = 0$.

Since V is piecewise C^∞, $\exists\, a = a_0 < a_1 < \cdots < a_k = b$

s.t. $V_i = V|_{[a_i, a_{i+1}]} \in C^\infty$, $i = 0, \cdots, k-1$.

Take a C^∞ function f on $[a, b]$ s.t.

$$\begin{cases} f(a_i) = 0, & \forall\, i = 0, \cdots, k-1 \\ f > 0, & \text{otherwise} \end{cases}$$

Let $X = V$, $Y = f(V'' + R_{\gamma' V}\gamma')$

Then Y is well-defined & $\in \mathcal{V}_0$

Hence $(*) \Rightarrow$

$$0 = I(V, Y) = \sum_{i=0}^{k-1} \langle V_i', Y \rangle \Big|_{a_i}^{a_{i+1}} - \sum_{i=1}^{k-1} \int_{a_i}^{a_{i+1}} \langle V'' + R_{\gamma' V}\gamma', f(V'' + R_{\gamma' V}\gamma') \rangle$$

$$= - \sum_{i=1}^{k-1} \int_{a_i}^{a_{i+1}} f |V'' + R_{\gamma' V}\gamma'|^2 \quad \left(\begin{matrix} \text{since } Y(a_i) = 0 \\ \forall\, i. \end{matrix} \right)$$

$\Rightarrow V'' + R_{\gamma' V}\gamma' = 0$ on $[a_i, a_{i+1}]$, $\forall\, i = 0, \cdots, k-1$.

Putting it back to the formula $(*)$, one has

$$0 = I(U,\tilde{Y}) = \sum_{i=0}^{k-1} \langle U', \tilde{Y} \rangle \Big|_{a_i}^{a_{i+1}}, \quad \forall \tilde{Y} \in \mathcal{V}_0 .$$

For a fixed $i_0 \in \{1, \cdots, k-1\}$, take $\tilde{Y}_{i_0} \in \mathcal{V}_0$

s.t.
$$\begin{cases} \tilde{Y}_{i_0}(a_i) = 0 & \text{if } i \neq i_0 \\ \tilde{Y}_{i_0}(a_{i_0}) = U'_{i_0+1}(a_{i_0}) - U'_{i_0}(a_{i_0}) . \end{cases}$$

Then
$$0 = I(U,\tilde{Y}) = -\langle U'_{i_0+1}(a_{i_0}), \tilde{Y}_{i_0}(a_{i_0}) \rangle + \langle U'_{i_0}(a_{i_0}), \tilde{Y}_{i_0}(a_{i_0}) \rangle$$
$$= -|\tilde{Y}_{i_0}(a_{i_0})|^2$$

$$\Rightarrow \quad U'_{i_0+1}(a_{i_0}) = U'_{i_0}(a_{i_0}) .$$

Since $i_0 \in \{1, \cdots, k-1\}$ is arbitrary, U is in fact C^1.

Then existence & uniqueness thm of ODE

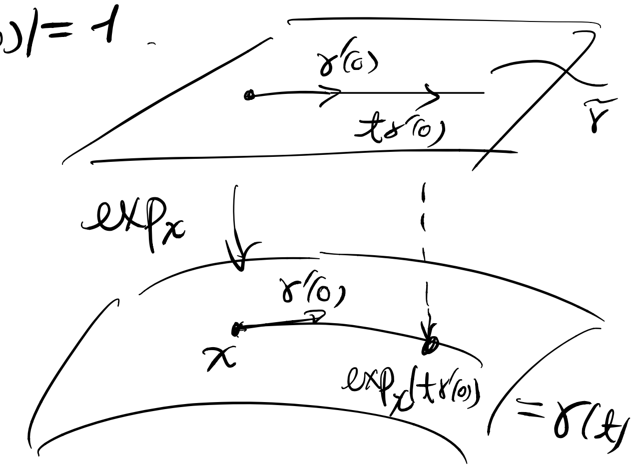
$\Rightarrow$ U is Jacobi. ※

Proof of Lemma 2

We may assume $a=0$, i.e. $\gamma : [0,b] \to M$

Define $\tilde{\gamma} = [a,b] \to T_xM : t \mapsto t\gamma'(0)$

where $x=\gamma(0)$, $|\gamma'(0)|=1$.



By assumption, γ has no conjugate point, hence $d\exp_x$ has no singular point along $\tilde{\gamma}$ $\Rightarrow$ $\exists$ nbd. $\mathcal{U}$ of $\tilde{\gamma}([0,b])$ in T_xM s.t.

$$\exp_x : \mathcal{U} \to M \text{ is an immersion.}$$

Then same proof as in Thm 2 of Ch 4, we can show that

(**) For any piecewise C^∞ curve $\sigma:[0,b]\to \exp_x\mathcal{U}$ connecting x to $\gamma(b)$, $L(\sigma)\geq L(\gamma)$. And equality holds $\Leftrightarrow$ σ = monotonic reparametrization of γ. (Ex!)

Now for any normal variation $\{\gamma_u\}$, $u\in(-\varepsilon,\varepsilon)$ with $\varepsilon>0$ small enough, we may assume $\gamma_u\subset \exp_x\mathcal{U}$. Then by (**)

$$L(u)\geq L(0), \quad \forall\, u\in(-\varepsilon,\varepsilon)$$

Since $L(u)$ is C^∞,

$$L''(0) = \lim_{s\to 0} \frac{L(-s)+L(s)-2L(0)}{s^2} \geq 0$$

Noting that any $X \in \mathcal{V}_0$ is a transversal vector field of a normal variation of γ, therefore

$$I(X,X) = L''(0) \geq 0 \quad , \quad \forall X \in \mathcal{V}_0$$

Suppose that $I(X,X)=0$, we have $\forall \varepsilon>0, Y \in \mathcal{V}_0$

$$0 \leq I(X \pm \varepsilon Y, X \pm \varepsilon Y) = I(X,X) \pm 2\varepsilon I(X,Y) + \varepsilon^2 I(Y,Y)$$

$$= \pm 2\varepsilon I(X,Y) + \varepsilon^2 I(Y,Y).$$

$$\Rightarrow \quad -\varepsilon I(Y,Y) \leq 2I(X,Y) \leq \varepsilon I(Y,Y)$$

$$\forall \varepsilon>0, \ Y \in \mathcal{V}_0.$$

Letting $\varepsilon \to 0$, we have $I(X,Y)=0, \ \forall Y \in \mathcal{V}_0$

Lemma 5 $\Rightarrow$ X = Jacobi

But $X(0)=X(b)=0$ and $\gamma(b)$ is not conjugate to $\gamma(0)$.

$\Rightarrow \quad X \equiv 0$

$\therefore \quad I$ is positive definite. ※

Lemma 6 (Cor. to lemma 2) (Minimality of Jacobi field)

Suppose
- $\gamma : [a,b] \to M$ normalized geodesic
- γ has no conjugate point
- U = Jacobi field along γ

Then $\forall\, X \in \mathcal{V}(a,b)$ with $X(a) = U(a)$ & $X(b) = U(b)$

$$I(U,U) \leq I(X,X).$$

Equality holds $\Leftrightarrow$ $X = U$.

Pf: Note $U - X \in \mathcal{V}_0(a,b)$

Lemma 2 $\Rightarrow$ $0 \leq I(U-X, U-X)$
$$= I(U,U) - 2I(U,X) + I(X,X).$$

$$I(U,U) = \langle U', U\rangle\big|_a^b - \int_a^b \langle \cancel{U'' + R_{\gamma' U}\gamma'}, U\rangle$$
$$= \langle U', U\rangle\big|_a^b$$

$$I(U,X) = \langle U', X\rangle\big|_a^b - \int_a^b \langle \cancel{U'' + R_{\gamma' U}\gamma'}, X\rangle$$
$$= \langle U', U\rangle\big|_a^b = I(U,U) \quad \left(\text{Since } \begin{matrix} X(a) = U(a) \\ X(b) = U(b) \end{matrix}\right)$$

$\therefore$ $0 \leq I(U,U) - 2I(U,U) + I(X,X)$

$\Rightarrow$ $I(U,U) \leq I(X,X)$.

Equality $\Leftrightarrow$ $0 = I(U-X, U-X) \Leftrightarrow U-X$. #

Proof of Lemma 3

It is clear that $I(X,Y)$ is not positive definite (by lemma 1).

Take a parallel frame field $\{E_1(t), \cdots, E_n(t)\}$ along γ s.t. $E_1(t) = \gamma'(t)$.

Then $\forall\, X \in \mathcal{V}_0(0,b)$

$$X(t) = \sum_{i=2}^{n} f_i(t) E_i(t) \quad \text{with } f_i(0) = f_i(b) = 0.$$

$\forall\, \beta \in [0,b]$, define $\tau(X) \in \mathcal{V}_0(0,\beta)$ by

$$\tau(X)(t) = \sum_{i=2}^{n} f_i\left(\frac{b}{\beta}t\right) E_i\left(\frac{b}{\beta}t\right).$$

Then

$$I_0^\beta(\tau(X), \tau(X)) = \int_0^\beta \sum_{i=2}^{n} \left|\frac{d}{dt} f_i\left(\frac{b}{\beta}t\right)\right|^2$$
$$- \int_0^\beta \sum_{i,j=2}^{n} f_i\left(\frac{b}{\beta}t\right) f_j\left(\frac{b}{\beta}t\right) \left\langle R_{\gamma'(t) E_i(\frac{b}{\beta}t)} \gamma'(t), E_j\left(\frac{b}{\beta}t\right)\right\rangle$$

So $\lim\limits_{\beta \to b} I_0^\beta(\tau(X), \tau(X)) = I_0^b(X, X)$

Since $\gamma(b)$ is the only conjugate point, lemma 2

$$\Rightarrow \quad I_0^\beta(\tau(X), \tau(X)) \geq 0.$$

Hence $I_0^b(X, X) \geq 0$. $\therefore$ I_0^b is semi-positive definite. $\times$

To prove Lemma 4, we need

<u>Lemma 7</u> Let
- $\gamma : [0, b] \to M$ normalized geodesic
- $\gamma(b)$ is not conjugate to $\gamma(0)$

Then $\forall\, v \in T_{\gamma(b)}M$, $\exists!$ Jacobi field V along γ s.t. $V(0) = 0$ and $V(b) = v$.

(Pf: Ex!)

<u>Pf of Lemma 4</u>

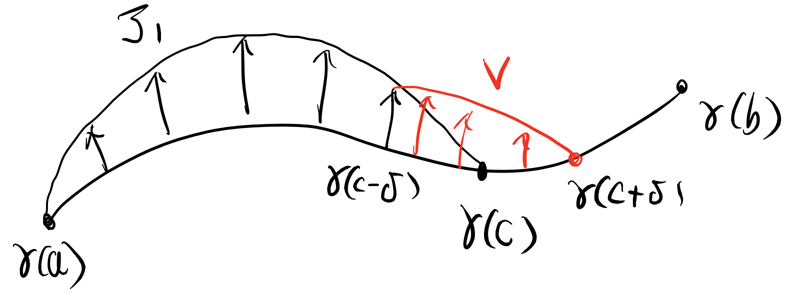
($\Rightarrow$) If $\exists\, c \in (a, b)$ s.t. $\gamma(c)$ conjugate to $\gamma(a)$.

Then $\exists$ non-trivial normal Jacobi field J_1 along $\gamma|_{[a,c]}$ s.t. $J_1(a) = J_1(c) = 0$.

Define $J \in \mathcal{V}_0(a,b)$ by

$$J = \begin{cases} J_1 , & t \in [a,c] \\ 0 , & t \in [c,b] \end{cases}$$

Then $I_a^b(J,J) = I_a^c(J_1,J_1) + I_c^b(0,0) = 0$



Now take $\delta > 0$ small s.t.

$$\exp_{\gamma(c+\delta)} : T_{\gamma(c+\delta)}M \to M$$

is diffeo. on $B(3\delta) \subset T_{\gamma(c+\delta)}M$ (& $c+\delta < b$)

Since $d(\gamma(c-\delta), \gamma(c+\delta)) \leq 2\delta$, $\gamma(c-\delta)$ is not conjugate to $\gamma(c+\delta)$. Then lemma 7,

$\exists!$ Jacobi field V s.t.

$$V(c+\delta) = 0 \quad \text{and} \quad V(c-\delta) = J(c-\delta) = J_1(c-\delta).$$

Define $U = \begin{cases} J_1, & t \in [a, c-\delta] \\ V, & t \in [c-\delta, c+\delta] \\ 0, & t \in [c+\delta, b] \end{cases}$

Then $I_a^b(U,U) = I_a^{c-\delta}(J_1,J_1) + \underset{\left(I_{c-\delta}^{c+\delta}(J,J) \text{ by lemma } 6\right)}{\underset{\wedge}{I_{c-\delta}^{c+\delta}(V,V)}} + I_{c+\delta}^{b}(0,0)$

$$< I_a^{c-\delta}(J_1,J_1) + I_{c-\delta}^{c+\delta}(J,J) + I_{c+\delta}^{b}(0,0)$$

$$= I_a^b(J,J) = 0 \quad \#$$

($\Leftarrow$) If $\exists\, U \in \mathcal{V}_0(a,b)$ s.t. $I(U,U) < 0$, then

Lemma 2 & 3 $\Rightarrow$ $\exists$ conjugate point to $\gamma(a)$ in $\gamma([a,b])$ $\#$

<u>Fact</u> (Ex!) Applying lemma 4 to S^2, show that if $b > \pi$, then $\exists$ a piecewise smooth $f_0 : [0,b] \to \mathbb{R}$ such that

$$(***) \quad \begin{cases} f_0(0) = f_0(b) = 0 \\ \int_a^b [|f_0'|^2 - f_0^2] < 0. \end{cases}$$

Thm 8 (Bonnet-Myers)

Let • M = complete Riem mfd. ($n = \dim M$)

• $\mathrm{Ricci}_M \geq (n-1)C$, $C > 0$

Then M is compact and $\mathrm{diam}(M) \leq \frac{\pi}{\sqrt{C}}$.

Pf: Scaling $\Rightarrow$ we may assume $C = 1$.

Then we need to show that if $\gamma : [0,b] \to M$ normalized shortest geodesic connecting x to y, then $b \leq \pi$.

Take parallel frame field $\{E_1(t), \cdots, E_n(t)\}$ along γ such that $E_1(t) = \gamma'(t)$.

If $b > \pi$, define

$$X_i(t) = f_0(t) E_i(t), \quad i = 2, \cdots, n$$

where $f_0(t)$ is the function in $(\ast\ast\ast)$.

Then $X_i \in \mathcal{V}_0(0,b)$, $\forall\, i = 2, \cdots, n$, and

$$\sum_{i=2}^{n} I(X_i, X_i) = \sum_{i=2}^{n} \int_0^b \left[|X_i'|^2 - \langle R_{\gamma' X_i} \gamma', X_i \rangle \right] dt$$

$$= (n-1)\int_0^b (f_0')^2 - \int_0^b f_0^2 \sum_{i=2}^{n} \langle R_{E_1 E_i(t)} E_1, E_i(t) \rangle dt$$

$$\leq (n-1)\int_0^b \left[(f_0')^2 - f_0^2\right] dt \quad \left(\mathrm{Ricci}_M \geq n-1 \right)$$

$$< 0$$

$\Rightarrow \exists\, i_0$ s.t. $I(X_{i_0}, X_{i_0}) < 0$

$\Rightarrow \gamma$ is not minimizing (by lemma 4)

Contradiction.

$\therefore\ b \leq \pi$. #