Ch5 Isometry, Space Forms

(M,g) = complete Riemannian manifold (connected)

Def: (M,g) with constant sectional curvature is called a space form.

Thm 1: $\forall\, c \in \mathbb{R}$ & $n \geq 2$, $\exists$ unique (up to isometry) simply-connected space form of dimension n and with constant sectional curvature c.

egs (Proof later)

- $c = 0$ $\quad(\mathbb{R}^n$, standard flat metric$)$
- $c = +1$ $\quad(\mathbb{S}^n$, standard metric$)$
- $c = -1$ $\quad\left(\mathbb{B}^n, \dfrac{4}{\left[1 - \sum_{i=1}^{n}(x^i)^2\right]^2}(dx^1)^2 \oplus \cdots \oplus (dx^n)^2\right)$

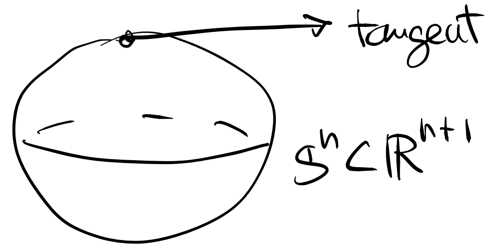
where $\mathbb{B}^n = \{(x^1, \cdots, x^n) : \sum_{i=1}^{n}(x^i)^2 < 1\}$.

(Hyperbolic n-space, unit ball model)

Def: Let M be a submanifold of $\overline{M}$ equipped with the induced metric. Then M is called a totally geodesic submanifold of $\overline{M}$ if a geodesic γ (of $\bar{M}$) tangents to M implies $\gamma \subset M$.

<u>Note</u>: Such a geodesic γ of $\bar{M}$ must be a geodesic of the submanifold M too.

<u>egs</u>:
- $\mathbb{R}^k \hookrightarrow \mathbb{R}^n : (x^1, \cdots, x^k) \mapsto (x^1, \cdots, x^k, 0, \cdots, 0)$ gives a totally geodesic submanifold of $\mathbb{R}^n$.
- $S^n \subset \mathbb{R}^{n+1}$ is <u>not</u> a totally geodesic submanifold of $\mathbb{R}^{n+1}$ as tangent lines to S^n don't stay on S^n



Let
- $M \subset \bar{M}$ be a submanifold
- M equipped with induced metric
- $D, \bar{D}$ = Levi-Civita connections of M & $\bar{M}$ respectively

$$\left(D_X Y = \left(\bar{D}_X Y\right)^{\text{tangent part}}, \ \forall X, Y \in \Gamma(TM) \subset \Gamma(T\bar{M}) \right)$$

Consider

$$S(X,Y) = D_X Y - \bar{D}_X Y \quad , \quad \forall X, Y \in \Gamma(TM)$$

(Note: S defines for vector fields on M, not $\bar{M}$)

Facts:
- $S(X_1 + X_2, Y) = S(X_1, Y) + S(X_2, Y)$
- $S(X,Y) = S(Y,X)$
- $\forall f \in C^\infty(M), \quad S(fX, Y) = S(X, fY) = fS(X,Y)$.

$\therefore$ S is "symmetric" tensor on M.

Pf of symmetric: $S(X,Y) - S(Y,X)$

$$= (D_X Y - \bar{D}_X Y) - (D_Y X - \bar{D}_Y X)$$

$$= (D_X Y - D_Y X) - (\bar{D}_X Y - \bar{D}_Y X)$$

$$= [X,Y] - [X,Y] = 0.$$ (both $D, \bar{D}$ are Levi-Civita) #

Therefore, we can define a symmetric bilinear form on T_xM, $\forall x \in M$:

$$\forall v, w \in T_xM, \quad S_x(v,w) = S(V,W)(x)$$

where V, W = any extensions of v, w respectively.

Def: This S is called the 2^{nd} fundamental form of M in $\bar{M}$.

Lemma 2 $M \subset \bar{M}$ totally geodesic

$\iff S \equiv 0$, where $S = 2^{nd}$ f.f. of M in $\bar{M}$

(i.e. $D_X Y = \bar{D}_X Y$, $\forall X, Y \in \Gamma(TM)$)

Pf: ($\Rightarrow$) Let $x \in M$ & $v \in T_x M \subset T_x \bar{M}$

Let γ = geodesic on $\bar{M}$ with

$\gamma(0) = x$, $\gamma'(0) = v$.

$\Rightarrow \bar{D}_{\gamma'} \gamma' = 0$

By assumption, γ is also a geodesic of M

$\Rightarrow D_{\gamma'} \gamma' = 0$

Therefore $S(v,v) = S(\gamma'(0), \gamma'(0))$

$= D_{\gamma'}\gamma' - \bar{D}_{\gamma'}\gamma' = 0$

Symmetry of $S \Rightarrow S(v,w) = 0$, $\forall v, w \in T_x M$.

($\Leftarrow$) Suppose $S \equiv 0$

Let γ = geodesic of $\bar{M}$ such that

$\gamma(0) = x$ and $\gamma'(0) = v \in T_x M \subset T_x \bar{M}$

By Existence (& Uniqueness) of geodesic in M,

$\exists$ ζ = geodesic of M s.t.

$$\zeta(0) = x \,,\; \zeta'(0) = v \in T_x M$$

(and of course $\zeta \subset M$)

Then $S \equiv 0$

$$\Rightarrow \quad \bar{D}_{\zeta'(t)} \zeta'(t) = D_{\zeta'(t)} \zeta'(t) = 0$$

$\Rightarrow$ ζ is also a geodesic of $\bar{M}$

Uniqueness of geodesic on $\bar{M}$ $\Rightarrow$

$$\gamma = \zeta \subset M \,. \quad \#$$

<u>Lemma 3</u>: Let $M \subset \bar{M}$ be totally geodesic,

$K, \bar{K}$ = sectional curvatures of $M, \bar{M}$ respectively.

Then $\forall$ $x \in M$, $\forall$ 2-plane $\pi \subset T_x M \subset T_x \bar{M}$,

$$K(\pi) = \bar{K}(\pi) \,.$$

(Pf: Immediately from Lemma 2)

eg: Let $\gamma : (a,b) \to \bar{M}$ be a smooth curve parametrized by arc-length. Suppose $\exists$ isometry $\varphi : \bar{M} \to \bar{M}$ such that $\gamma((a,b)) = \{ y \in \bar{M} : \varphi(y) = y \}$.

Then γ is a normalized geodesic.

Pf: We 1st note that $\forall$ geodesic ζ in $\bar{M}$,

$\varphi \circ \zeta$ is also a geodesic in $\bar{M}$ (since φ = isom)

Now $\forall\, t_0 \in (a,b)$, take a geodesic

$$\zeta \in \bar{M} \text{ s.t. } \begin{cases} \zeta(0) = \gamma(t_0) \\ \zeta'(0) = \gamma'(t_0) \end{cases}$$

Since $\gamma((a,b))$ = fixed point set of φ,

$$d\varphi(\gamma'(t_0)) = \gamma'(t_0) \quad (\text{diff. } \varphi \circ \gamma = \gamma)$$

$$\Rightarrow \quad d\varphi(\zeta'(0)) = \zeta'(0)$$

Uniqueness of geodesic $\Rightarrow$ $\varphi \circ \zeta = \zeta$

$\Rightarrow \quad \zeta \subset \{ y \in \bar{M} : \varphi(y) = y \} = \gamma(a,b)$.

$\Rightarrow \quad \gamma$ is a normalized geodesic. #

Lemma 4: The set of fixed points of an isometry is a totally geodesic submanifold.
(not necessary connected)

Pf: Let $\varphi: \bar{M} \to \bar{M}$ be an isometry and

$M = \{y \in \bar{M} : \varphi(y) = y\}$ be the set of fixed points of φ

If M is a submanifold of $\bar{M}$, then the same argument as in the example implies M is totally geodesic. (Ex!)

So we only need to show the following

Claim: Let $x \in M$, $B(\delta) = \{v \in T_x\bar{M} : |v| < \delta\}$
$$B_\delta = \{y \in \bar{M} : d(x,y) < \delta\}$$
where $\delta > 0$ small enough s.t.
$$\exp_x : B(\delta) \to B_\delta \text{ is a diffeomorphism}$$
$(B_\delta = \exp_x(B(\delta)))$.

Let $\mathcal{F} \subset T_x\bar{M}$ be a linear subspace defined by
$$\mathcal{F} = \{v \in T_x\bar{M} : d\varphi(v) = v\}$$
Then $M \cap B_\delta = \exp_x(\mathcal{F} \cap B(\delta))$

(and hence M is a submanifold of $\overline{M}$)

<u>Pf of Claim</u>

(1) $M \cap B_\delta \subset \exp_x(\mathcal{F} \cap B(\delta))$

<u>Pf</u>: Let $y \in M \cap B_\delta \subset B_\delta$

$\Rightarrow \exists\, v \in B(\delta)$ s.t. $\exp_x v = y$.

Let $\gamma(t) = \exp_x(tv) : [0,1] \to \overline{M}$

be the unique minimizing geodesic joining x to y.

Since $x, y \in M$, we have $\varphi(x) = x$ & $\varphi(y) = y$.

$\Rightarrow$ $\varphi \circ \gamma$ is also a minimizing geodesic joining x to y.

Uniqueness $\Rightarrow$ $\varphi \circ \gamma = \gamma$

$\Rightarrow d\varphi(v) = v \qquad (v = \gamma'(0))$

$\Rightarrow v \in \mathcal{F}$

$\therefore \quad y = \exp_x v \in \exp_x(\mathcal{F} \cap B(\delta))$.

(2) $\exp_x(\mathcal{F} \cap B(\delta)) \subset M \cap B_\delta$

<u>Pf</u>: Let $y \in \exp_x(\mathcal{F} \cap B(\delta))$

Then, we have $y \in B_\delta$ and

$\exists\ v \in \mathcal{F} \cap B(\delta)$ s.t. $y = \exp_x v$

Let $\gamma(t) = \exp_x(tv) : [0,1] \to \overline{M}$ be the unique minimizing geodesic joining x to y.

Since $v \in \mathcal{F}$, $d\varphi(\gamma'(0)) = \gamma'(0)$

$\Rightarrow$ $\varphi \circ \gamma$ and γ have the same initial data

Uniqueness $\Rightarrow$ $\varphi \circ \gamma = \gamma$.

$\Rightarrow$ $y = \gamma(1) = \varphi \circ \gamma(1) = \varphi(y)$

$\therefore$ $y \in M \cap B_\delta$ ※

Lemma 5: $S^n \subset \mathbb{R}^{n+1}$ has constant sectional curvature $+1$, $\forall\, n \geq 2$.

Pf: "$n=2$" is proved in undergrad DG (Ex!)

If $n \geq 3$, define

$$\tilde{\varphi} : \mathbb{R}^{n+1} \longrightarrow \mathbb{R}^{n+1}$$

$$(x^1, x^2, x^3, x^4, \cdots, x^{n+1}) \to (x^1, x^2, x^3, -x^4, \cdots, -x^{n+1})$$

Then $|\tilde{\varphi}(x)| = |x|$ (Euclidean norm)

Hence $\tilde{\varphi}$ induces an isometry

$$\varphi : S^n \to S^n$$

The fixed points set

$$M = \{ x \in S^n : \varphi(x) = x \}$$

$$= \{ (x^1, x^2, x^3, 0, \cdots, 0) : (x^1)^2 + (x^2)^2 + (x^3)^2 = 1 \}$$

$= S^2$ is a totally geodesic submanifold.

Hence $K_{S^n}(\pi) = K_{S^2}(\pi) = +1$,

$\forall$ 2-plane $\pi \subset T_x S^2 \subset T_x S^n$ (where $x = (x^1, x^2, x^3, 0 \cdots 0)$)

Repeat the argument for any 3 indices $i, j, k \in \{1, \cdots, n+1\}$ and using the fact that S^n is invariant under rotation, we have proved that $K_{S^n} \equiv +1$.

$\#$

<u>Lemma 6</u> $\left(\mathbb{B}^n, \frac{4}{(1-|x|^2)^2} \sum_{i=1}^n dx^i \otimes dx^i \right)$, where $|x|^2 = \sum_{i=1}^n (x^i)^2$

is a <u>complete</u> Riemannian metric with constant sectional curvature -1. $(n \geq 2)$

Pf : (1) <u>Completeness</u>

<u>Pf</u>: First note that $\forall\ A \in O(n)$

$$A|_{\mathbb{B}^n} : \mathbb{B}^n \to \mathbb{B}^n$$ is an isometry of the hyperbolic geometry

(since A preserves $|X|$ & $\sum_i dx^i \otimes dx^i$)

Now consider the curve

$$\begin{aligned} \zeta(s) : (-\infty, \infty) &\longrightarrow \mathbb{B}^n \\ \psi \qquad\quad & \qquad \psi \\ s &\longmapsto \left(\frac{e^s - 1}{e^s + 1}, 0, \cdots, 0\right) \end{aligned}$$

Then $\zeta'(s) = \left(\dfrac{2e^s}{(e^s+1)^2}, 0, \cdots, 0\right)$

$$\Rightarrow \quad |\zeta'(s)|^2_{hyp} = \frac{4}{(1 - |\zeta(s)|^2)^2} |\zeta'(s)|^2_{Eucl} \overset{(Ex!)}{=\!=} 1.$$

Let $A \in O(n)$ be given by

$$A(x^1, x^2, \cdots, x^n) = (x^1, -x^2, \cdots, -x^n)$$

Then $\zeta((-\infty, \infty)) = \{x \in \mathbb{B}^n : Ax = x\} = \{(x^1, 0, \cdots, 0) : -1 < x^1 < 1\}$

Lemma 4 $\Rightarrow$ ξ = normalized geodesic defined on the whole $(-\infty, \infty)$ with $\xi'(0)$ in the e_1-direction ($\{e_i\}$ = standard basis of $\mathbb{R}^n$)

Applying other $A \in O(n)$, we have geodesic with

$$(A\xi)'(0) = \text{any given direction},$$

and $(A\xi)(s)$ is defined on the whole $(-\infty, \infty)$.

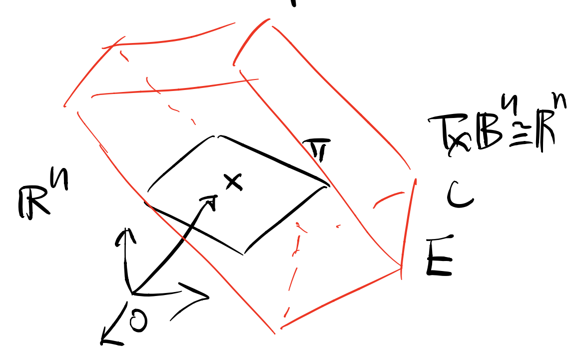
Therefore, $\exp_0$ is defined on the whole $T_0\mathbb{B}^n$.

Hence Hopf-Rinow $\Rightarrow$ $\mathbb{B}^n$ is complete.

(2) Curvature $\equiv -1$

Pf: Let $x \in \mathbb{B}^n$ and $\pi \subset T_x\mathbb{B}^n$ be a 2-plane.

Identify $T_x\mathbb{B}^n \cong \mathbb{R}^n$ and x can be considered as an element in $\mathbb{R}^n$.



Assume $n \geq 3$, take a 3-dim'l subspace $E \subset \mathbb{R}^n$ s.t. $\text{span}\{x, \pi\} \subset E$

(If $X \neq 0$ & $X \notin \Pi$, then E is unique, otherwise not)

Then $\mathbb{R}^n = E \oplus E^\perp$ orthogonal (in Euclidean)

and one can defines a map

$$\phi : (e, e') \mapsto (e, -e'), \quad e \in E, e' \in E^\perp.$$

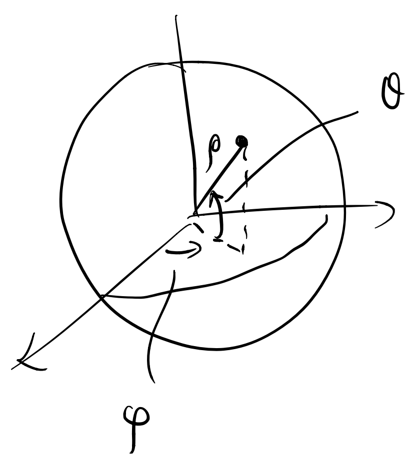
Then $\phi|_{\mathbb{B}^n}$ is an isometry of $\mathbb{B}^n$ with fixed point set $E \cap \mathbb{B}^n$

$\Rightarrow$ $\mathbb{B}^3 = E \cap \mathbb{B}^n$ is a totally geodesic submanifold of $\mathbb{B}^n$.

$\Rightarrow$ $K_{\mathbb{B}^n}(\Pi) = K_{\mathbb{B}^3}(\Pi)$.

So we only need to show the case that $n=3$.

Let $\{\rho, \varphi, \theta\}$ be the spherical coordinates on $\mathbb{B}^3$.



$\Rightarrow$ on $\mathbb{B}^3 \setminus \{0\}$, the metric

$\frac{4}{(1-|x|^2)^2} \sum dx^i \otimes dx^i$ can be written as

$$\frac{4}{(1-\rho^2)^2}\left(d\rho^2+\rho^2 d\theta^2+\rho^2\cos^2\theta\, d\varphi^2\right)$$

(where $d\rho^2 = d\rho\otimes d\rho, \cdots$)

Let
$$\begin{cases} e_1 = \dfrac{1-\rho^2}{2}\dfrac{\partial}{\partial\rho} \\ e_2 = \dfrac{1-\rho^2}{2\rho}\dfrac{\partial}{\partial\theta} \\ e_3 = \dfrac{1-\rho^2}{2\rho\cos\theta}\dfrac{\partial}{\partial\varphi} \end{cases}$$

Then $\langle e_i, e_j\rangle = \delta_{ij}$ (Ex!)

$$\Rightarrow \langle D_{e_i}e_j, e_k\rangle = \frac{1}{2}\Big\{\cancel{e_i\langle e_j, e_k\rangle} + \cancel{e_j\langle e_k, e_i\rangle} - \cancel{e_k\langle e_i, e_j\rangle}$$
$$+\langle e_k,[e_i,e_j]\rangle + \langle e_j,[e_k,e_i]\rangle - \langle e_i,[e_j,e_k]\rangle\Big\}$$

$$= \frac{1}{2}\Big\{\langle e_k,[e_i,e_j]\rangle + \langle e_j,[e_k,e_i]\rangle - \langle e_i,[e_j,e_k]\rangle\Big\}$$

Now
$$[e_1,e_2] = \frac{1-\rho^2}{2}\frac{\partial}{\partial\rho}\left(\frac{1-\rho^2}{2\rho}\frac{\partial}{\partial\theta}\right) - \frac{1-\rho^2}{2\rho}\frac{\partial}{\partial\theta}\left(\frac{1-\rho^2}{2}\frac{\partial}{\partial\rho}\right)$$
$$= \frac{1-\rho^2}{2}\left(\frac{1-\rho^2}{2\rho}\right)'\frac{\partial}{\partial\theta} = -\frac{1+\rho^2}{2\rho}e_2 \quad (\text{Ex.})$$

Similarly $\begin{cases} [e_2, e_3] = \dfrac{1-\rho^2}{2\rho} \tan\theta \, e_3 \\ [e_1, e_3] = -\dfrac{1+\rho^2}{2\rho} e_3 \end{cases}$ (Ex!)

Then straight forward calculation (Ex.) $\Rightarrow$

$$\begin{cases} D_{e_1} e_1 = 0, \quad D_{e_2} e_1 = \dfrac{1+\rho^2}{2\rho} e_2, \quad D_{e_3} e_1 = \dfrac{1+\rho^2}{2\rho} e_3 \\ D_{e_1} e_2 = 0, \quad D_{e_2} e_2 = -\dfrac{1+\rho^2}{2\rho} e_1, \quad D_{e_3} e_2 = -\dfrac{1-\rho^2}{2\rho} \tan\theta \, e_3 \\ D_{e_1} e_3 = 0, \quad D_{e_2} e_3 = 0, \quad D_{e_3} e_3 = -\dfrac{1+\rho^2}{2\rho} e_1 + \dfrac{1-\rho^2}{2\rho} \tan\theta \, e_2 \end{cases}$$

Hence

$$R(e_1, e_2, e_1, e_2) = \langle R_{e_1 e_2} e_1, e_2 \rangle$$

$$= \langle D_{[e_1, e_2]} e_1 - [D_{e_1}, D_{e_2}] e_1, e_2 \rangle$$

$$= -\frac{1+\rho^2}{2\rho} \langle D_{e_2} e_1, e_2 \rangle - \langle D_{e_1}(D_{e_2} e_1) - D_{e_2}(\cancel{D_{e_1} e_1}), e_2 \rangle$$

$$= -\left(\frac{1+\rho^2}{2\rho}\right)^2 - \langle D_{e_1}\left(\frac{1+\rho^2}{2\rho} e_2\right), e_2 \rangle$$

$$= -\left(\frac{1+\rho^2}{2\rho}\right)^2 - e_1\left(\frac{1+\rho^2}{2\rho}\right)$$

$$= -\left(\frac{1+\rho^2}{2\rho}\right)^2 - \frac{1-\rho^2}{2}\left(\frac{1+\rho^2}{2\rho}\right)'$$

$$= -1 \quad (\text{Ex.})$$

Similarly $R(e_1,e_3,e_1,e_3) = R(e_2,e_3,e_2,e_3) = -1$ (Ex!)

To complete the proof, we need to show that all other (Ex!)

$$R(e_i,e_j,e_k,e_l) = 0$$

Since $n=3$, the indices have to be repeated.

It is clear that if $i=j=k=l$ or 3 of the indices are equal, then $R(e_i,e_j,e_k,e_l)=0$

Therefore, we only need to consider

$$R(e_i,e_j,e_i,e_k) \quad \text{with} \quad j<k \quad (i,j,k \text{ distinct.})$$

other cases are clearly zero or can be reduced to this case. (If $j=k$, this is the previous situation)

For $i=3$

$$R(e_3,e_1,e_3,e_2) = \langle R_{e_3 e_1} e_3, e_2 \rangle$$

$$= \langle D_{[e_3,e_1]} e_3, e_2 \rangle - \langle D_{e_3} \cancel{D_{e_1}} e_3, e_2 \rangle + \langle D_{e_1} D_{e_3} e_3, e_2 \rangle$$

$$= \frac{1+\rho^2}{2\rho}\langle D_{e_3}e_3, e_2\rangle + \langle D_{e_1}(D_{e_3}e_3), e_2\rangle \qquad \left(\text{using } D_{e_1}e_1 = 0 \text{ \& } \langle e_1, e_2\rangle = 0\right)$$

$$= \frac{1+\rho^2}{2\rho}\cdot\frac{1-\rho^2}{2\rho}\tan\theta + \left\langle D_{e_1}\left(\frac{1-\rho^2}{2\rho}\tan\theta\, e_2\right), e_2\right\rangle$$

$$= \frac{1+\rho^2}{2\rho}\cdot\frac{1-\rho^2}{2\rho}\tan\theta + e_1\left(\frac{1-\rho^2}{2\rho}\tan\theta\right)$$

$$= \frac{1-\rho^4}{4\rho^2}\tan\theta + \frac{1-\rho^2}{2}\left(\frac{1-\rho^2}{2\rho}\right)'\tan\theta$$

$$= 0 \quad (\text{Ex.})$$

Similarly $R(e_1,e_2,e_1,e_3) = R(e_2,e_1,e_2,e_3) = 0$ (Ex!)

Hence $\mathbb{B}^3$ has sectional curvature $\equiv -1$. ✕✕

<u>Existence of Thm 1</u>: By Lemma 5 and Lemma 6, we have complete simply-connected Riemannian manifolds of any dimension ≥ 2 with constant sectional curvature $= \pm 1$. By scaling, we have $K_{\frac{1}{c}g} = c K_g$ ($\forall$ metric g (Ex!)) $= \pm c$

Together with $\mathbb{R}^n$, we've proved the existence part of Thm 1. ✕✕