Def: A vector field X along γ is called parallel if

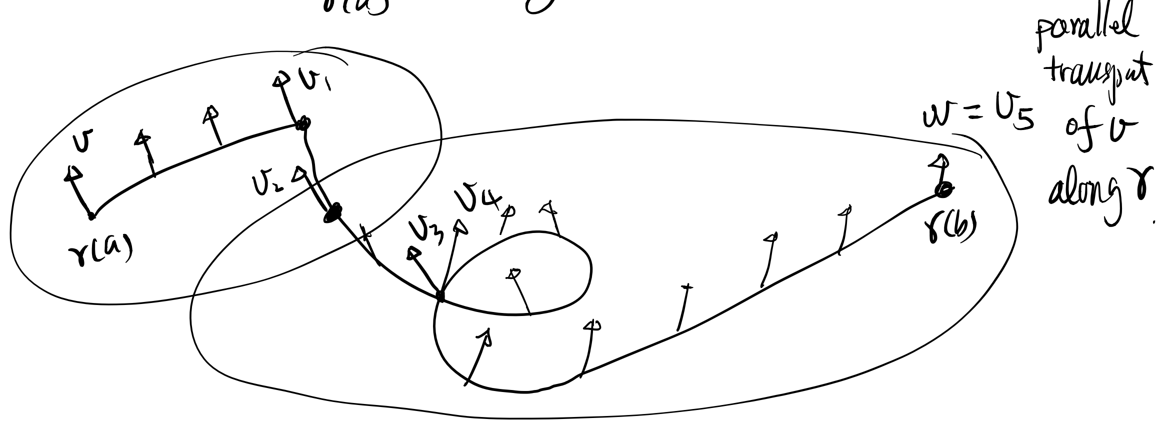
$$D_{\gamma'} X = 0 .$$

Def: A vector $w \in T_{\gamma(b)}M$ is called a parallel transport of a vector $v \in T_{\gamma(a)}M$ along γ if $\exists$ a parallel vector field X along γ such that

$$X(a) = v \ \& \ X(b) = w .$$

Note: (i) parallel transport w of v (along γ) is uniquely determined by v (for fixed γ).

(ii) If γ is not embedded, or not contained in a chart, or γ is only piecewise smooth, we can use subdivision to define parallel transport of a vector $v \in T_{\gamma(a)}M$ along γ.



Hence

<u>Thm</u>: $\forall$ immersed curve $\gamma : [a,b] \to M$ & $v \in T_{\gamma(a)}M$,

$\exists!$ parallel vector field $X(t)$ along γ such that $X(a) = v$.

Hence $\exists!$ $w \in T_{\gamma(b)}M$ such that w is the parallel transport of v along γ.

<u>Remark</u>: This Thm $\Rightarrow$ one can define, $\forall$ immersed curve $\gamma : [a,b] \to M$, a mapping

$$\begin{array}{cccc} P^{\gamma} : & T_{\gamma(a)}M & \longrightarrow & T_{\gamma(b)}M \\ & \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ & v & \longmapsto & w = \text{parallel transport of } v \text{ along } \gamma \end{array}$$

<u>Thm</u>: $P^{\gamma} : T_{\gamma(a)}M \to T_{\gamma(b)}M$ is a vector space isomorphism.

(Pf: Ex.)

<u>Note</u>: (i) P^{γ} is called <u>parallel transport from $\gamma(a)$ to $\gamma(b)$ along γ</u>.

(ii) Furthermore, if D is the Levi-Civita connection of a metric g on M, then $\forall$ two vector fields X & Y along γ (γ embedded),

$$\frac{d}{dt}\langle X, Y\rangle = \gamma'(t)\langle X, Y\rangle$$
$$= \langle D_{\gamma'}X, Y\rangle + \langle X, D_{\gamma'}Y\rangle$$

So if X, Y are parellel along γ, then

$$\frac{d}{dt}\langle X, Y\rangle = 0$$

and $P^{\gamma}: T_{\gamma(a)}M \to T_{\gamma(b)}M$ is in fact an <u>isometry</u> of the inner product spaces (defined by g).

(iii) Conversely, if D is a connection such that all P^{γ} are isometries of the inner product spaces, then $\forall$ vector fields X, Y, Z, we choose a curve $\gamma: [0,1] \to M$ such that

$$\gamma'(0) = X(x) \quad (\text{at } x = \gamma(0) \in M)$$

Let $\{e_1, \cdots, e_n\}$ be an orthonormal basis of T_xM.

Then parallel transport P^γ along γ defines orthonormal basis $\{e_1(t), \cdots, e_n(t)\}$ $\left(e_i(t) = P^{\gamma|_{[0,t]}} e_i\right)$ of $T_{\gamma(t)}M$, $\forall\, t \in [0,1]$. (by assumption that $P^{\gamma|_{[0,t]}}$ are isometries).

Hence
$$Y(\gamma(t)) = \sum_i Y^i(t)\, e_i(t)$$
$$Z(\gamma(t)) = \sum_i Z^i(t)\, e_i(t)$$

for some $Y^i(t)$ & $Z^i(t)$.

$$\Rightarrow \quad X(x)\langle Y, Z\rangle = \gamma'(0)\langle Y, Z\rangle$$
$$= \frac{d}{dt}\Big|_{t=0} \langle Y, Z\rangle(\gamma(t))$$
$$= \frac{d}{dt}\Big|_{t=0} \sum_i Y^i(t) Z^i(t)$$
$$= \frac{dY^i}{dt}(0) Z^i(0) + Y^i(0) \frac{dZ^i}{dt}(0)$$

We also have
$$D_{\gamma'(0)} Y = D_{\gamma'(0)}\Big(\sum_i Y^i(t) e_i(t)\Big)$$
$$= \sum_i \Big[\big(\gamma'(0)\, Y^i(t)\big) e_i(t) + Y^i(t) \underbrace{\cancel{D_{\gamma'(0)} e_i(t)}}_{0}\Big]$$

$$= \frac{dY^i}{dt}(0)\, e_i(t)$$

Similarly for $D_{\gamma'(0)} Z$. Hence

$$X(x)\langle Y, Z\rangle = \langle D_{\gamma'(0)} Y, Z\rangle + \langle Y, D_{\gamma'(0)} Z\rangle$$

$$= \langle D_X Y, Z\rangle + \langle Y, D_X Z\rangle .$$

Since $x \in M$ is arbitrary, we have shown that

D is compatible with g.

<u>Conclusion</u>: D = compatible with g

$\iff$ P^γ = isometry $\forall \gamma$.

In particular, if D is <u>symmetric</u>, then

D = Levi-Civita $\iff$ P^γ = isometry $\forall \gamma$.

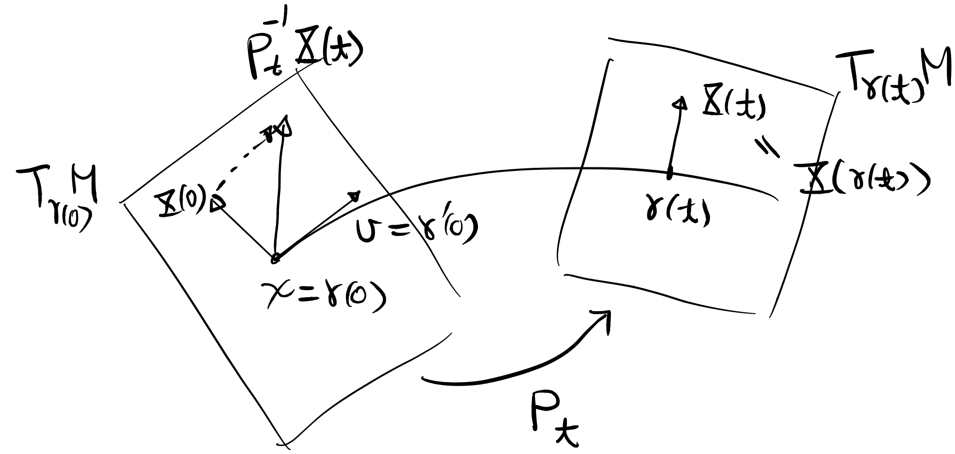
<u>Thm</u>: $\forall\, v \in T_x M$ & $X \in \Gamma(TM)$, D = Levi-Civita

$$\boxed{D_v X = \frac{d}{dt}\Big|_{t=0} P_t^{-1} X(\gamma(t))}$$

where $\gamma : [0,1] \to M$ is a curve such that

$\gamma(0) = x$, $\gamma'(0) = v$,

$P_t : T_x M \to T_{\gamma(t)} M$ = parallel transport along $\gamma\,\big|_{[0,t]}$.



Pf: Let $\{e_i\}$ be an orthonormal basis of $T_x M$

Define $e_i(t) = P_t e_i$

Then $\{e_i(t)\}$ is an o.n. basis for $T_{\gamma(t)} M$.

Write $X(\gamma(t))$ in terms of $\{e_i(t)\}$:

$$X(\gamma(t)) = \sum_i X^i(t) e_i(t)$$

for some $X^i(t)$.

$$\Rightarrow \quad D_v X = \sum_i \frac{dX^i}{dt}(0)\, e_i$$

And $P_t^{-1}(X(\gamma(t))) = \sum_i X^i(t)\, P_t^{-1} e_i(t)$

$$= \sum_i \overline{X}^i(t)\, e_i$$

$$\Rightarrow \quad \frac{d}{dt}\Big|_{t=0} P_t^{-1}(\overline{X}(\gamma(t))) = \sum_i \frac{d\overline{X}^i}{dt}(0)\, e_i = D_v \overline{X}$$

#

§2.3 Geodesic

Def: A curve $\gamma : [a,b] \to M$ is called a *geodesic* wrt the connection D if $\gamma'(t)$ is parallel along γ.

In local coordinates $\{x^i\}$, then

$$\gamma(t) = (x^1(t), \cdots, x^n(t))$$

$$\Rightarrow \quad \gamma'(t) = \sum_i \frac{dx^i}{dt}(t) \frac{\partial}{\partial x^i}\Big|_{\gamma(t)}$$

Hence

$$D_{\gamma'(t)} \gamma'(t) = \sum_k \left[\frac{d}{dt}\left(\frac{dx^k}{dt}\right) + \Gamma^k_{ij}(\gamma(t)) \frac{dx^i}{dt}\frac{dx^j}{dt} \right] \frac{\partial}{\partial x^k}$$

$\Rightarrow$

$$\gamma \text{ is a geodesic (wrt } D) \iff D_{\gamma'}\gamma' = 0$$

$$\iff \boxed{\frac{d^2x^k}{dt^2} + \Gamma^k_{ij}(x^1, \cdots, x^n) \frac{dx^i}{dt}\frac{dx^j}{dt} = 0, \quad \forall k = 1, \cdots, n}$$

which is a <u>non-linear</u> <u>2^{nd} order</u> ODE system in $(x^1(t), \cdots, x^n(t))$.

ODE theory $\Rightarrow$

<u>Lemma</u>: $\forall$ connection D on M,

$\forall\ v \in T_x M$

$\Rightarrow \exists!$ geodesic $\gamma(t)$ wrt D on some interval $(-\varepsilon, \varepsilon)$ such that

$$\begin{cases} \gamma(0) = x, \\ \gamma'(0) = v \end{cases}$$

<u>Note</u>: If D is Levi-Civita connection of g. Then $\forall$ geodesic γ of D, we have

$$\frac{d}{dt}\langle \gamma', \gamma' \rangle = \langle D_{\gamma'}\gamma', \gamma' \rangle + \langle \gamma', D_{\gamma'}\gamma' \rangle = 0$$

$$\Rightarrow \quad |\gamma'(t)| = \text{constant}.$$

§2.4 Induced connection

Let M = Riemannian manifold

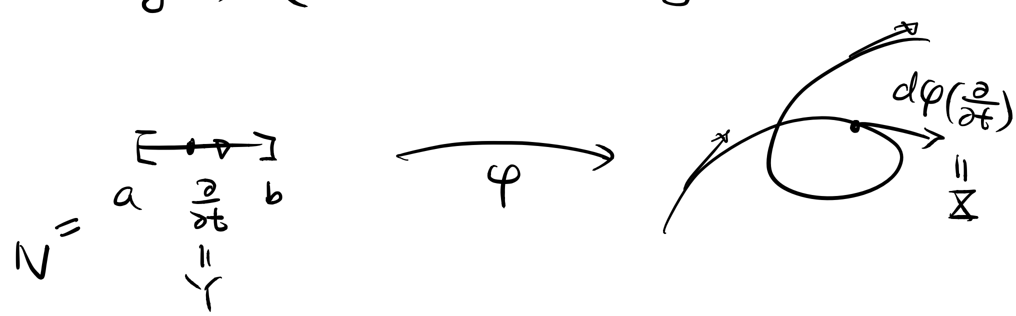
N = differentiable manifold

and $\varphi : N \to M$ C^∞ map

Def: A map $X : N \to TM$ is called a vector field along φ if $\forall x \in N$, $X(x) \in T_{\varphi(x)}M$

$$\begin{tikzcd} & TM \arrow[d, "\pi"] \\ N \arrow[ur, "X"] \arrow[r, "\varphi"'] \arrow[ur, phantom, "\circlearrowleft" near end, shift right=3] & M \end{tikzcd}$$

eg: $\forall Y \in \Gamma(TN)$, $X = d\varphi(Y)$ is a vector field along φ (but not necessary $\in \Gamma(TM)$)



Note: If $v \in T_x N$, and $\{E_i\}_{i=1}^n$ is a "frame field" in a nbd. V of $\varphi(x) \in M$

(i.e. $\{E_i(p)\}$ is a basis for T_pM, $\forall p \in V$ and $E_i(p)$ are smooth in p.)

Then $\forall\ x \in \varphi^{-1}(V) \subset N$,

$$\mathbb{X}(x) = \sum_i \mathbb{X}^i(x)\, E_i(\varphi(x)) \in T_{\varphi(x)}M,$$

for some functions $\mathbb{X}^i$ on N.

Define

$$\boxed{\tilde{D}_v \mathbb{X} = \sum_i \left[v(\mathbb{X}^i)(x)\, E_i(\varphi(x)) + \mathbb{X}^i(x)\, D_{d\varphi(v)} E_i \right]}$$

where D = Levi-Civita connection on M.

Fact: $\tilde{D}_v \mathbb{X}$ is well-defined (indep. of the choice of $\{E_i\}$'s.) (Pf: Ex!)

Def: • $\tilde{D}$ is called the <u>induced connection</u>.

• $\forall\ V \in \Gamma(TN)$, $\mathbb{X}$ = vector field along φ

$$(\tilde{D}_V \mathbb{X})(x) \overset{\text{def}}{=\!=} \tilde{D}_{V(x)} \mathbb{X}$$

Fact: (Ex!) Since D = Levi-Civita on M,

- $\forall\, X, Y \in \Gamma(TN)$

$$\tilde{D}_X \, d\varphi(Y) - \tilde{D}_Y \, d\varphi(X) - d\varphi([X,Y]) = 0$$

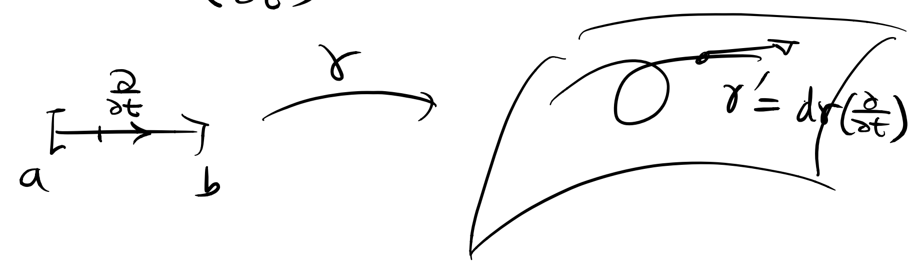
$$\left(d\varphi([X,Y]) = [\, d\varphi(X), d\varphi(Y)] \right)$$

- $\forall\, V, W$ vector fields along φ and $u \in T_x N$,

$$u \langle V, W \rangle = \langle \tilde{D}_u V, W \rangle + \langle V, \tilde{D}_u W \rangle .$$

Note: If $\gamma : [0,1] \to M$ is a smooth curve (not necessary embedded) then

$$\gamma' = d\gamma\left(\frac{\partial}{\partial t}\right) \text{ is a vector field along } \gamma$$



$$\begin{tikzcd} & TM \arrow[d, "\pi"] \\ {[a,b]} \arrow[ur, "d\gamma(\frac{\partial}{\partial t})"] \arrow[r, "\gamma"'] & M \end{tikzcd}$$

We (alternatively can) define $D_{\gamma'} \gamma' \overset{\text{def}}{=} \tilde{D}_{\frac{\partial}{\partial t}} \gamma'$

<u>Check</u>: If γ is embedded, this definition coincides with the previous one.

$\therefore$ geodesic and P^γ (wrt Levi-Civita) connection can be defined for any smooth curve.

Ch3 Covariant derivative, Curvature Tensor

§3.1 Covariant derivative of tensors

Fact: Let $\varphi: V \to W$ be an isomorphism between vector spaces, then φ can be extended to an isomorphism between the tensor algebras:

$$\tilde{\varphi}: \bigoplus_{r,s} T^{r,s}V \longrightarrow \bigoplus_{r,s} T^{r,s}W$$

where $T^{r,s}V = (\underbrace{V\otimes\cdots\otimes V}_{r}) \otimes (\underbrace{V^*\otimes\cdots\otimes V^*}_{s})$

$V^* =$ dual of V.

In fact: we can first define

$$\begin{aligned} \varphi^*: W^* &\to V^* \\ \alpha &\mapsto \varphi^*(\alpha) \end{aligned} \quad \text{by } \boxed{\varphi^*(\alpha)(v) \overset{\text{def}}{=} \alpha(\varphi(v))}$$

Then $\varphi =$ isom $\Rightarrow \varphi^*$ isom

$\Rightarrow \quad (\varphi^*)^{-1}: V^* \to W^*$ exists

Hence we can define

$\forall \ v_1 \otimes \cdots \otimes v_r \otimes \alpha^1 \otimes \cdots \otimes \alpha^s \in T^{r,s}V,$

$$\tilde{\varphi}(v_1 \otimes \cdots \otimes v_r \otimes \alpha^1 \otimes \cdots \otimes \alpha^s)$$
$$= \varphi(v_1) \otimes \cdots \otimes \varphi(v_r) \otimes (\varphi^*)^{-1}(\alpha^1) \otimes \cdots \otimes (\varphi^*)^{-1}(\alpha^s)$$
$$\in T^{r,s}W$$

Finally, extend $\tilde{\varphi}$ to all $\bigoplus_{r,s} T^{r,s}V$ by linearity, and one can check that $\tilde{\varphi}$ is an isomorphism.

<u>Def</u>: Let M = Riemannian manifold, $x \in M$, $v \in T_xM$
γ = curve with $\gamma(0) = x$, $\gamma'(0) = v$

Then $\forall$ tensor field K on M, we define the <u>covariant derivative</u> of K wrt v by

$$\boxed{D_v K = \frac{d}{dt}\Big|_{t=0} \left(\widetilde{P_t^\gamma}\right)^{-1}(K(\gamma(t)))}$$

where $\widetilde{P_t^\gamma} : \bigoplus_{r,s} T^{r,s}(T_xM) \to \bigoplus_{r,s} T^{r,s}(T_{\gamma(t)}M)$

is the extension of the parallel transport

$P_t^\gamma : T_xM \to T_{\gamma(t)}M$ wrt Levi-Civita connection.

Caution: We need to check $D_v K$ doesn't depend on γ.

Properties:

(1) If K is a (r,s)-tensor, then $D_v K$ is also a (r,s)-tensor

(2) D_v is a derivation on the tensor algebra

$$D_v(K_1 \otimes K_2) = (D_v K_1) \otimes K_2 + K_1 \otimes (D_v K_2)$$

(3) D_v commutes with "contractions".

Def (of contraction) The contractions C_{pq}, $p=1,\cdots,r$, $q=1,\cdots,s$

are linear maps

$$C_{pq} : (\otimes^r TM) \otimes (\otimes^s T^*M) \to (\otimes^{r-1} TM) \otimes (\otimes^{s-1} T^*M)$$

defined by

$$C_{pq}(v_1 \otimes \cdots \otimes v_r \otimes \alpha^1 \otimes \cdots \otimes \alpha^s)$$
$$= \alpha^q(v_p)\, v_1 \otimes \cdots \otimes \hat{v}_p \otimes \cdots \otimes v_r \otimes \alpha^1 \otimes \cdots \otimes \hat{\alpha}^q \otimes \cdots \otimes \alpha^s$$

omitted.

eg: For $C_{11} : TM \otimes T^*M \to \mathbb{R} \; (\cong (\otimes^0 TM) \otimes (\otimes^0 T^*M))$

takes $C_{11}(\frac{\partial}{\partial x^i} \otimes dx^j) = dx^j(\frac{\partial}{\partial x^i}) = \delta_i^j \in \mathbb{R}$

- For $C_{11}: TM \otimes (\overset{2}{\otimes} T^*M) \to T^*M$

$$\frac{\partial}{\partial x^i} \otimes (dx^{j_1} \otimes dx^{j_2}) \mapsto dx^{j_1}\left(\frac{\partial}{\partial x^i}\right) dx^{j_2}$$

$$= \delta_i^{j_1} dx^{j_2}$$

Property (3) means if $\mathcal{C} = C_{pq}$ is a contraction, then

$$\boxed{D_v(\mathcal{C}K) = \mathcal{C}(D_v K)}$$

Pf: (1) is clear

(2) We do a special case only. The general case can be proved similarly.

Suppose $K = X \otimes Y \otimes \rho \in (\overset{2}{\otimes} TM) \otimes (T^*M)$

Then X, Y are vector fields & ρ is a 1-form

Hence we need to show

$$D_v K = (D_v X) \otimes Y \otimes \rho + X \otimes D_v Y \otimes \rho + X \otimes Y \otimes D_v \rho$$

Let $\{e_1(t), \cdots, e_n(t)\}$ be <u>parallel</u> vector field along γ s.t. $\{e_i(t)\}$ forms a basis of $T_{\gamma(t)}M$

Then $\forall t$, $\exists$ dual basis $\{\alpha^1(t), \cdots, \alpha^n(t)\}$ for $T^*_{\gamma(t)}M$

i.e. $\alpha^i(t)\big(e_j(t)\big) = \delta^i_j$, $\forall t$.

<u>Claim</u>: $\{\alpha^i(t)\}$ are all parallel.

<u>Pf</u>: In fact, by def of $\widetilde{P}_t$, we see that

$$\widetilde{P}_t(\alpha^i(0)) \overset{def}{=} \left(P_t^*\right)^{-1}(\alpha^i(0))$$

$$\Leftrightarrow \quad P_t^*\left(\widetilde{P}_t(\alpha^i(0))\right) = \alpha^i(0)$$

$$\Leftrightarrow \quad P_t^*\left(\widetilde{P}_t(\alpha^i(0))\right)(e_j(0)) = \alpha^i(0)\big(e_j(0)\big) = \delta^i_j, \ \forall j$$

$$\Leftrightarrow \quad \widetilde{P}_t(\alpha^i(0))\Big(P_t(e_j(0))\Big) = \delta^i_j, \ \forall j$$

$$\Leftrightarrow \quad \widetilde{P}_t(\alpha^i(0))\big(e_j(t)\big) = \delta^i_j, \ \forall j$$

$$\Leftrightarrow \quad \widetilde{P}_t(\alpha^i(0)) = \alpha^i(t). \ \divideontimes$$

Now, write

$$X(t) = X(\gamma(t)) = \sum X^i(t)\, e_i(t)$$

$$Y(t) = Y(\gamma(t)) = \sum Y^j(t)\, e_j(t)$$

$$\rho(t) = \rho(\gamma(t)) = \sum \rho_\ell(t)\, \alpha^\ell(t)$$

Then $K(t) = \sum_{i,j,l} X^i(t) Y^j(t) \rho_l(t)\; e_i(t) \otimes e_j(t) \otimes \alpha^l(t)$

$$\Rightarrow (\tilde{P}_t)^{-1} K(t) = \sum_{i,j,l} X^i(t) Y^j(t) \rho_l(t)\; e_i(0) \otimes e_j(0) \otimes \alpha^l(0)$$

$$\Rightarrow D_v K = \frac{d}{dt}\Big|_{t=0} (\tilde{P}_t)^{-1} K(t)$$

$$= \sum_{i,j,l} \left(\frac{dX^i}{dt} Y^j \rho_l + X^i \frac{dY^j}{dt} \rho_l + X^i Y^j \frac{d\rho_l}{dt} \right) e_i \otimes e_j \otimes \alpha^l$$

(at $t=0$)

Compare with
$$\begin{cases} D_v X = \sum \frac{dX^i}{dt} e_i \\ D_v Y = \sum \frac{dY^j}{dt} e_j \\ D_v \rho = \sum \frac{d\rho_l}{dt} \alpha^l \end{cases} \quad (\text{at } t=0)$$

$$\Rightarrow D_v K = (D_v X) \otimes Y \otimes \rho + X \otimes D_v Y \otimes \rho + X \otimes Y \otimes D_v \rho.$$

\#

Pf of (3) We do only special case that

$$K = X \otimes \rho \in TM \otimes T^*M \quad \text{and}$$

$$\begin{array}{rcl} C : TM \otimes T^*M & \longrightarrow & \mathbb{R} \\ \Psi & & \Psi \\ X \otimes \rho & \longmapsto & \rho(X) \end{array}$$

In this case $\quad CK = C(X\otimes\rho) = \rho(X)$

$$D_v(CK) = v(\rho(X))$$

$$C(D_vK) = C(D_v(X\otimes\rho))$$

$$= C(D_vX\otimes\rho + X\otimes D_v\rho)$$

$$= \rho(D_vX) + (D_v\rho)(X).$$

Note that

$$\rho(X) = (\rho_\ell\,\alpha^\ell(t))(X^i e_i(t))$$

$$= \rho_i X^i$$

$$\rho(D_vX) = \rho_i \frac{dX^i}{dt}$$

$$(D_v\rho)(X) = \frac{d\rho_i}{dt} X^i$$

$$\therefore \quad v(\rho(X)) = v(\rho_i X^i) = \rho_i \frac{dX^i}{dt} + \frac{d\rho_i}{dt} X^i$$

$$(v = \gamma'(0),\ \gamma(0) = x)$$

$$= \rho(D_vX) + (D_v\rho)(X).$$

<u>Note</u>: This can be used to define $D_v\rho$:

$$(D_v\rho)(X) = v(\rho(X)) - \rho(D_vX), \quad \forall X\in\Gamma(M).$$