Ch2 Riemannian Metric, Connection and Parallel Transport

§2.1 Riemannian metric & connection

Def: Let M be a C^∞ manifold. A Riemannian metric g on M is given by an inner product g_x at each T_xM which depends smoothly on $x \in M$ in the sense that in any nbd. system U with coordinate functions $x^1, \cdots, x^n$,

$$g_{ij}(x) = g_x\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right) \qquad (\forall i, j)$$

is a smooth function on the nbd.

(Caution: same notation, but not the $g_{ij}(x)$ in the def. of vector bundle.)

Notation, most of the time we write $\langle \, , \, \rangle_x$ for g_x (or simply $\langle \, , \, \rangle$ for g).

Note:
- By def. $(g_{ij}(x))$ is a symmetric positive definite $n \times n$ matrix $\forall x \in U$.
- g can be regarded as $(0,2)$-tensor satisfying

$$\begin{cases} g(X,X) \geq 0, \ \forall X \in \Gamma(TM) \\ g_x(X,X) = 0 \Leftrightarrow X(x) = 0. \\ g(X,Y) = g(Y,X), \ \forall X,Y \in \Gamma(TM) \end{cases}$$

Hence $\boxed{g = \sum_{i,j=1}^{n} g_{ij}(x)\, dx^i \otimes dx^j}$ in local coord.

<u>Def</u>: A <u>connection</u> D (∇) on a C^∞ manifold M is a mapping

$$D : \Gamma(TM) \times \Gamma(TM) \to \Gamma(TM)$$
$$(V, X) \mapsto D_V X$$

such that

(C1) $D_{fV+gW} X = f D_V X + g D_W X$

(C2) $D_V(fX) = (Vf) X + f D_V X$

(C3) $D_V(X+Y) = D_V X + D_V Y$

where $X, Y, V, W \in \Gamma(TM)$; $f, g \in C^\infty(M)$.

(and $Vf = D_V f$ is the directional derivative of f in the direction V.)

<u>Note</u>: $D_V X$ is called the <u>covariant derivative</u> of X in the

direction of V (or wrt V)

<u>Fact</u>: If $V, W \in \Gamma(TM)$ are vector fields such that $V(x) = W(x)$, then

$$(D_V X)(x) = (D_W X)(x), \quad \forall X \in \Gamma(TM)$$

(Pf: Ex!)

Using this fact, we have

<u>Def</u>: $\forall v \in T_xM$, one can define

$$D_v X \overset{\text{def}}{=\!=} (D_V X)(x) \quad \text{for any } V \in \Gamma(TM)$$

with $V(x) = v$.

<u>eg</u>: <u>Standard connection on $\mathbb{R}^n$</u>

Recall the directional derivative of function

$$D_v f = \lim_{t \to 0} \frac{f(x+tv) - f(x)}{t}$$

for f = smooth function defined near $x \in \mathbb{R}^n$.

A smooth vector field X on $\mathbb{R}^n$ can be written as

$$\underline{X} = \sum_i \underline{X}^i(x) \frac{\partial}{\partial x^i}$$

where x^i = standard coordinates on $\mathbb{R}^n$

hence $\frac{\partial}{\partial x^i} = e_i = \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix} \leftarrow$ ith-place.

& $\underline{X}^i(x)$ are smooth functions

Then $$D_v \underline{X} \overset{\text{def}}{=\!=} \sum_i (D_v \underline{X}^i)(x) \frac{\partial}{\partial x^i}$$ and

$$(D_V \underline{X})(x) \overset{\text{def}}{=\!=} D_{V(x)} \underline{X}$$

defines a <u>connection</u> on $\mathbb{R}^n$ (Ex: check C1 – C3)

Clearly, for this standard connection on $\mathbb{R}^n$,

$$D_V\left(\frac{\partial}{\partial x^i}\right) = 0 \,, \quad \forall\, i = 1, \cdots, n$$ for the standard basis $\{ \frac{\partial}{\partial x^i} \}$.

<u>Lemma</u> The set of connections on M is <u>convex</u>

i.e. If $D^1, \cdots, D^k$ are connections on M,

$f_1, \cdots, f_k$ are functions $\in C^\infty(M)$ with

$$\sum_{i=1}^k f_i = 1,$$

then $D = \sum_{i=1}^{k} f_i D^i$ is a connection on M.

(ie. $D_V X \overset{def}{=} \sum_{i=1}^{k} f_i D^i_V X$)

Pf: C1 & C3 are clear (and don't need $\sum_i f_i = 1$)

For C2, we have

$$D_V(fX) = \sum_i f_i D^i_V(fX)$$
$$= \sum_i f_i [(Vf)X + f D^i_V X]$$
$$= (Vf)X + f D_V X \quad \text{since } \sum_i f_i = 1.$$

⌘

Thm: Let M be a C^∞ manifold. Then $\exists$ a connection on M.

Pf: Let $\{(U_i, \phi_i)\}$ be an atlas on M. Then $\{U_i\}$ is an open cover of M

$\Rightarrow$ $\exists$ partitions of unity $\{\varphi_i\}$ subordinate to U_i

(WLOG, we may assume $\{V_k\}_{k\in N} = \{U_i\}_{i\in\Lambda}$)

On each U_i, the standard connection on $\mathbb{R}^n$ induces a connection D^i. Then $\sum \varphi_i D^i$ is a connection on M by the previous lemma. ※

<u>Remark</u>: Similar argument shows that the existence of Riemannian metric on any C^∞ manifold. (Ex!)

<u>Lemma</u>: Let $v \in T_x M$ and $\gamma : [0, \epsilon) \to M$ be a curve such that $\gamma(0) = x$ & $\gamma'(0) = v$. Suppose $X, Y \in \Gamma(TM)$ be two vector fields such that

$$X(\gamma(t)) = Y(\gamma(t)), \ \forall t.$$

Then $D_v X = D_v Y$.

(i.e. $D_{\gamma'(0)} X$ is determined by $X \circ \gamma$)

(Pf: Ex!)

<u>Thm</u> Let M = manifold

$g = \langle \, , \rangle$ = Riemannian metric on M

Then $\exists !$ connection D such that

(compatible with g) (L1) $X\langle Y, Z\rangle = \langle D_X Y, Z\rangle + \langle Y, D_X Z\rangle$

(torsion free) (L2) $D_X Y - D_Y X - [X, Y] = 0$.

<u>Pf</u>: (<u>Uniqueness</u>)

In coordinates, any vector field can be written as

$$X = \sum X^i \frac{\partial}{\partial x^i}$$

$$\Rightarrow \qquad D_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} = \sum_k \Gamma_{ij}^k \frac{\partial}{\partial x^k}$$

for some functions Γ_{ij}^k .

Now for $X = X^j \frac{\partial}{\partial x^j}$, $V = V^i \frac{\partial}{\partial x^i}$, then

$$D_V X = D_{\left(V^i \frac{\partial}{\partial x^i}\right)} \left(X^j \frac{\partial}{\partial x^j} \right)$$

$$= V^i \left[D_{\frac{\partial}{\partial x^i}} \left(X^j \frac{\partial}{\partial x^j} \right) \right]$$

$$= V^i \left[\frac{\partial X^j}{\partial x^i} \frac{\partial}{\partial x^j} + X^j D_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} \right]$$

$$= V^i \left[\frac{\partial X^{\cancel{j}\,k}}{\partial x^i} \frac{\partial}{\partial x^{\cancel{j}\,k}} + X^j \Gamma_{ij}^k \frac{\partial}{\partial x^k} \right]$$

$$= \left(V^i \frac{\partial X^k}{\partial x^i} + \Gamma_{ij}^k V^i X^j \right) \frac{\partial}{\partial x^k}$$

$\Rightarrow \quad \{\Gamma^k_{ij}\}$ determines $D_V X$.

Let $\quad g_{ij} = \langle \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \rangle, \quad \forall i, j.$

$$\Rightarrow \quad \frac{\partial g_{jk}}{\partial x^i} = \frac{\partial}{\partial x^i} \langle \frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^k} \rangle$$

(by h1) $$= \langle \nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^k} \rangle + \langle \frac{\partial}{\partial x^j}, \nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^k} \rangle$$

$$= \langle \Gamma^l_{ij} \frac{\partial}{\partial x^l}, \frac{\partial}{\partial x^k} \rangle + \langle \frac{\partial}{\partial x^j}, \Gamma^l_{ik} \frac{\partial}{\partial x^l} \rangle$$

$$= \Gamma^l_{ij} g_{lk} + \Gamma^l_{ik} g_{jl}$$

$\therefore$

$$\begin{cases} \dfrac{\partial g_{jk}}{\partial x^i} = \Gamma^l_{ij} g_{lk} + \Gamma^l_{ik} g_{jl} & \text{--- (1)} \\[2ex] \dfrac{\partial g_{ki}}{\partial x^j} = \Gamma^l_{jk} g_{li} + \Gamma^l_{ji} g_{kl} & \text{--- (2)} \\[2ex] \dfrac{\partial g_{ij}}{\partial x^k} = \Gamma^l_{ki} g_{lj} + \Gamma^l_{kj} g_{il} & \text{--- (3)} \end{cases}$$

By (h2),

$$0 = D_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} - D_{\frac{\partial}{\partial x^j}} \frac{\partial}{\partial x^i} - \cancelto{0}{\left[\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right]}$$

$$= (\Gamma^k_{ij} - \Gamma^k_{ji}) \frac{\partial}{\partial x^k}$$

$$\Rightarrow \quad \Gamma_{ij}^{k} = \Gamma_{ji}^{k}, \quad \forall\, i,j,k$$

Then (1)+(2)−(3) $\Rightarrow$

$$\frac{\partial g_{jk}}{\partial x^i} + \frac{\partial g_{ki}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^k} = 2\, g_{k\ell} \Gamma_{ij}^{\ell}$$

Denote the inverse matrix of (g_{ij}) by (g^{ij}). Then

$$g^{sk} g_{k\ell} = \delta_\ell^s, \quad \forall\, s,\ell$$

$$\Rightarrow \ (\Gamma) \quad \boxed{\Gamma_{ij}^{k} = \frac{1}{2} g^{k\ell} \left[\frac{\partial g_{j\ell}}{\partial x^i} + \frac{\partial g_{\ell i}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^\ell} \right]}$$

$\therefore \{\Gamma_{ij}^k\}$ & hence D satisfying L1 & L2 is uniquely determined by g.

(Existence): Let $\{(U_\beta, \phi_\beta)\}$ = atlas of M

For $X = X^j \frac{\partial}{\partial x^j}$ & $V = V^i \frac{\partial}{\partial x^i}$ on U_β, we define

$$D_V X \overset{\text{def}}{=} V^i \left(\frac{\partial X^k}{\partial x^i} + \Gamma_{ij}^k X^j \right) \frac{\partial}{\partial x^k}$$

with Γ_{ij}^k defined by (Γ) (locally). Then one can check that $D_V X$ doesn't depend on the coordinate chart (U_β, ϕ_β). Hence it defines a connection D on M. The properties L1 & L2 are then easy to check. ※

Note: • The connection given by this Theorem is called the Levi-Civita connection of g (or Riemannian connection of g)

• The coefficients Γ_{ij}^k of D are called the Christoffel symbols if D is Levi-Civita.

• The formula (Γ) is equivalent to

$$\langle D_X Y, Z\rangle = \frac{1}{2}\left\{ \begin{array}{l} X\langle Y,Z\rangle + Y\langle Z,X\rangle - Z\langle X,Y\rangle \\ +\langle Z,[X,Y]\rangle + \langle Y,[Z,X]\rangle - \langle X,[Y,Z]\rangle \end{array} \right\}$$

for $X, Y, Z \in \Gamma(TM)$ (Ex!)

eg Fact: on S^3, $\exists\ \hat{i}, \hat{j}, \hat{k}$ orthonormal vector fields such that $[\hat{i},\hat{j}] = \hat{k}$, $[\hat{j},\hat{k}] = \hat{i}$ & $[\hat{k},\hat{i}] = \hat{j}$.

Hence

$$\langle D_{\hat{i}}\hat{j}, \hat{k}\rangle = \frac{1}{2}\Big\{ \hat{i}\,\cancel{\langle \hat{j},\hat{k}\rangle}^{\to 0} + \hat{j}\,\cancel{\langle \hat{k},\hat{i}\rangle}^{\to 0} - \hat{k}\,\cancel{\langle \hat{i},\hat{j}\rangle}^{\to 0}$$
$$+ \langle \hat{k}, [\hat{i},\hat{j}]\rangle + \langle \hat{j}, [\hat{k},\hat{i}]\rangle - \langle \hat{i}, [\hat{j},\hat{k}]\rangle \Big\}$$
$$= \frac{1}{2}\left[\langle \hat{k},\hat{k}\rangle + \langle \hat{j},\hat{j}\rangle - \langle \hat{i},\hat{i}\rangle\right] = \frac{1}{2}$$

Similarly $\langle D_{\hat{i}}\hat{j}, \hat{i}\rangle = \langle D_{\hat{i}}\hat{j}, \hat{j}\rangle = 0$ (Ex!)

$$\Rightarrow \quad D_{\hat{i}}\hat{j} = \frac{1}{2}\hat{k}.$$

(Similarly for others (Ex!))

<u>Geometry meaning of Levi-Civita connection</u>

<u>Def</u>: Let N be a (embedded) submanifold of M. Suppose g is a metric on M, then the <u>induce metric</u> $\bar{g}$ of g on N is defined by

$$\bar{g}(X,Y) = g(X,Y), \ \forall X,Y \in TN \subset TM.$$

(e.g. If $N \subset M$ is open subdomain, then $\bar{g} = g|_N$)

<u>Def</u>: Let (M,g) be a Riemannian manifold

D = Levi-Civita connection of g

Suppose $N \subset M$ is a submanifold (embedded), then one can define a connection on N by

$$\bar{D}_X Y = (D_X Y)^{\perp}$$

where $(\ )^{\perp}_x : T_xM \to T_xN$ is the <u>orthogonal projection</u> wrt g_x on T_xM.

<u>Facts</u>: • $\bar{D}$ is well-defined, i.e. $\bar{D}$ satisfies C1–C3

• $\bar{D}$ is the <u>Levi-Civita connection</u> of the <u>induce metric</u> $\bar{g}$ (Ex!)

<u>Note</u>: If $M = \mathbb{R}^n$, g = standard metric (= flat metric) then Levi-Civita connection D = usual directional derivative on $\mathbb{R}^n$. Hence, the facts above give a geometric interpretation of the Levi-Civita connection on submanifolds of $\mathbb{R}^n$.

"<u>Meaning</u>" of (L2): $D_X Y - D_Y X - [X, Y] = 0$

(L2) doesn't use the metric g, and in local coordinates

$$(L2) \Leftrightarrow \Gamma^k_{ij} = \Gamma^k_{ji}$$

Hence connections satisfying (L2) are called <u>symmetric</u>

Moreover, $T(X,Y) = D_X Y - D_Y X - [X,Y]$ defines a $(1,2)$-tensor on M called the <u>torsion tensor</u>.

i.e. $T \in \Gamma(TM \otimes (\otimes^2 T^*M))$. Hence

$$D \text{ is symmetric} \Leftrightarrow T \equiv 0$$
$$\Leftrightarrow D \text{ is torsion free}.$$

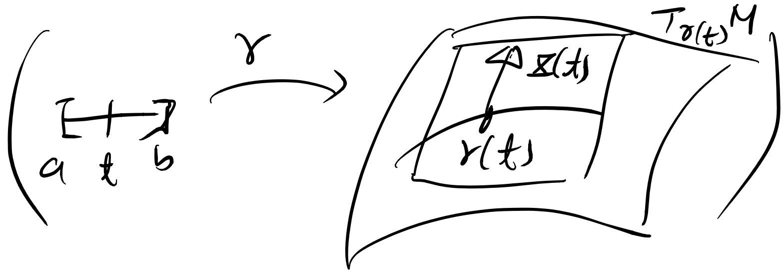
§2.2 Parallel Transport

Let D be a connection (not necessary Levi-Civita) on M, $\gamma : [a,b] \to M$ be an <u>embedded</u> curve such that $\gamma([a,b])$ is contained in a coordinate nbd. with coordinate functions $\{x^i\}$.

Suppose X is a <u>vector field along γ</u>,

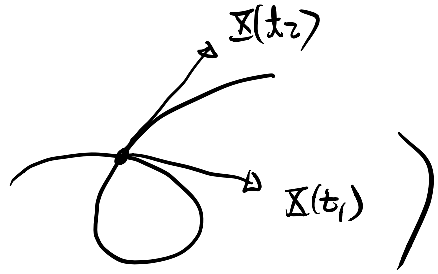
i.e. X depends smoothly on t and $X(t) \in T_{\gamma(t)}M$, $\forall t \in [a,b]$

$$
\begin{tikzcd}
 & TM \arrow[d, "\pi"] \\
{[a,b]} \arrow[ur, "X"] \arrow[r, "\gamma"'] & M
\end{tikzcd}
$$



Since γ is embedded, X can be extended to a smooth vector field $\tilde{X}$ on M. (Ex!)

(Not true for <u>immersed curve</u> :



)

Now for any 2 extensions $\tilde{X}$ & $\tilde{Y}$, we must have

$$\tilde{X}(\gamma(t)) = X(t) = \tilde{Y}(\gamma(t)), \quad \forall t \in [a,b]$$

$$\Rightarrow \qquad D_{\gamma'(t)}\tilde{X} = D_{\gamma'(t)}\tilde{Y}$$

$\therefore$ $\underline{D_{\gamma'(t)}X \text{ is well-defined}}$ (for vector field $X(t)$ along $\gamma(t)$)

In local coordinates

$$\begin{cases} \gamma'(t) = (\gamma')^i(t) \left.\dfrac{\partial}{\partial x^i}\right|_{\gamma(t)} \\ X(t) = X^i(t) \left.\dfrac{\partial}{\partial x^i}\right|_{\gamma(t)} \end{cases}$$

for some functions $(\gamma')^i(t)$ & $X^i(t)$.

Then
$$\boxed{D_{\gamma'(t)} X = \left(\frac{dX^k}{dt} + \Gamma^k_{ij} (\gamma')^i X^j \right) \frac{\partial}{\partial x^k} \Big|_{\gamma(t)}}$$

(where Γ^k_{ij} are given by $D_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} = \Gamma^k_{ij} \frac{\partial}{\partial x^k}$)

Pf: $D_{\gamma'(t)} X = D_{\gamma'(t)} \left(X^j \frac{\partial}{\partial x^j} \right)$

$$= \left(D_{\gamma'(t)} X^j \right) \frac{\partial}{\partial x^j} + X^j D_{\gamma'(t)} \frac{\partial}{\partial x^j}$$

$$= \left(\frac{dX^k}{dt} + \Gamma^k_{ij} (\gamma')^i X^j \right) \frac{\partial}{\partial x^k}$$

$\#$

Observation:

$$D_{\gamma'(t)} X = 0 \iff \frac{dX^k}{dt} + \Gamma^k_{ij} (\gamma')^i X^j = 0, \quad \forall k = 1, \cdots, n$$

in local coordinates

which is a linear ODE system

in $X^1, \cdots, X^n$.

Linear ODE theory $\Rightarrow$

$\forall\, v \in T_{\gamma(t)}M$, $\exists!$ soln. $X(t)$ to the IVP

$$\begin{cases} D_{\gamma'(t)} X = 0, & \forall\, t \in [a,b] \\ X(a) = v \end{cases}$$