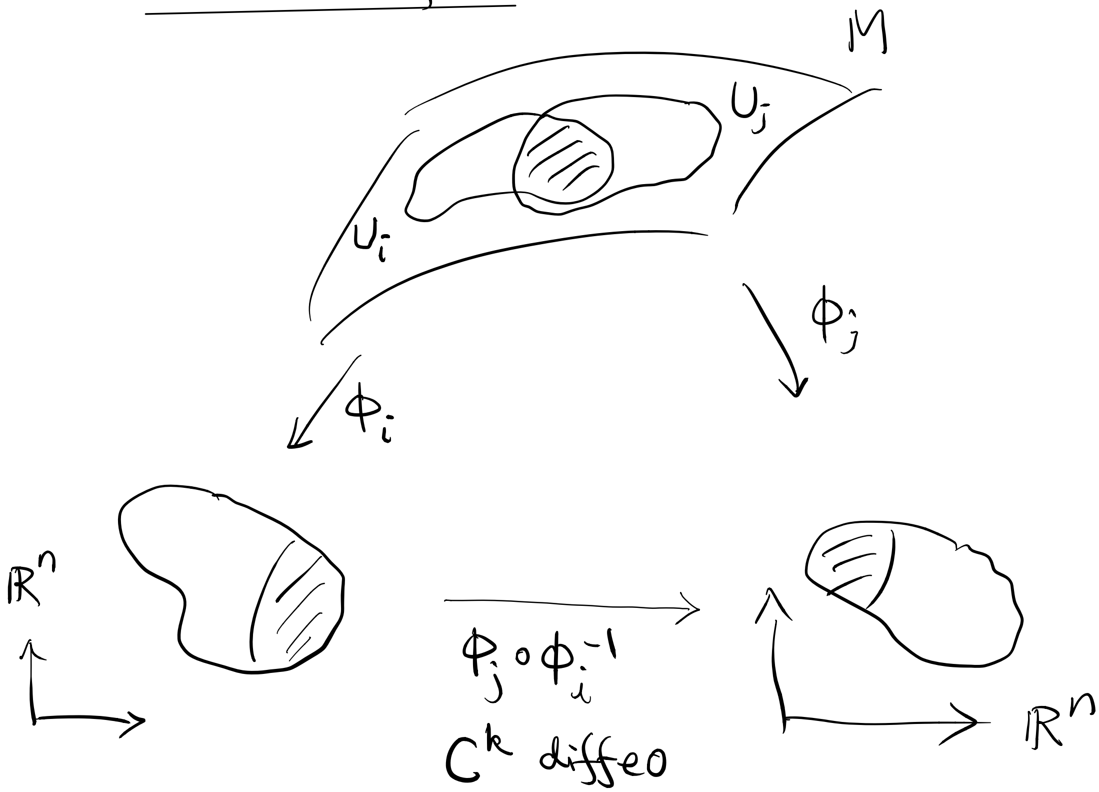
Ch1 Differential Manifolds

1.1 Abstract Manifolds



Def: A C^k atlas on a Hausdroff topological space M is defined by

(i) an open covering U_i, $i \in \Lambda$, of M

(ii) a family of homeomaphisms

$$\phi_i : U_i \to \Omega_i \subset \mathbb{R}^n \quad (\Omega_i \text{ open})$$

such that $\forall\, i, j \in \Lambda$,

$$\phi_j \circ \phi_i^{-1} : \phi_i(U_i \cap U_j) \to \phi_j(U_i \cap U_j)$$

is a C^k-diffeomaphism.

Remark: • $\phi_j \circ \phi_i^{-1}$, $i,j \in \Lambda$ (with $U_i \cap U_j \neq \phi$) are called transition functions.

• (U_i, ϕ_i) is a called a (coordinate) chart.

• $\phi_i^{-1}: \Omega_i \to U_i \subset M$ is a local parametrization.

Def: "Two" C^k altas on M, say $(U_i, \phi_i)_{i\in\Lambda_1}$ and $(V_j, \psi_j)_{j\in\Lambda_2}$ are C^k-equivalent if their union is still a C^k-altas, that is, if $\forall i \in \Lambda_1$, $j \in \Lambda_2$ (s.t. $U_i \cap V_j \neq \phi$), then

$$\phi_i \circ \psi_j^{-1} = \psi_j(U_i \cap V_j) \to \phi_i(U_i \cap V_j)$$

are C^k-diffeomaphism.

Def: A differentiable structure of class C^k on M is an equivalence class of C^k altas.

Remark: If M is connected, then the integer n in the definition doesn't depend on the chart and is defined as the dimension of M.

<u>Def</u>: A C^k <u>differentiable manifold</u> of dimension n is a pair $(M, \mathcal{A})$, where M is a Hausdroff top. space and $\mathcal{A} = [(U_i, \phi_i)_{i\in\Lambda}]$ is a differentiable structure given by a C^k altas on M with $\phi_i(U_i) \subset \mathbb{R}^n$.

<u>Remark</u>:
- In this course, we consider only $\underline{C^\infty}$ diff. manifold which is <u>connected</u> and with a further condition such that "partitions of unity" is always possible.
- All <u>compact</u> manifolds satisfy the further condition.
- We'll refer such a manifold as a <u>smooth</u> manifold (or even just <u>manifold</u>)

<u>eg</u>: $M = T^n$, the <u>n-torus</u> $(T^n = \underbrace{S^1 \times \cdots \times S^1}_{n})$ is a manifold.

Pf: Let
$$f : \mathbb{R}^n \to T^n \in \mathbb{C}^n \qquad (f \text{ is onto})$$
$$(x_1, \cdots, x_n) \mapsto (e^{ix_1}, \cdots, e^{ix_n})$$

$\forall\, p \in T^n$, $\exists\, x^p = (x_1^p, \cdots, x_n^p) \in \mathbb{R}^n$ s.t.

$p = f(x^p)$ (we may choose $x_i^p \in [0, 2\pi)$, $i = 1, \cdots, n$)

Consider $\Omega_p = (x_1^p - \pi, x_1^p + \pi) \times \cdots \times (x_n^p - \pi, x_n^p + \pi)$

open in $\mathbb{R}^n$ containing x^p.

Let $U_p = f(\Omega_p) \subset T^n$ ($\Rightarrow U_p$ open & containing p)

$$\phi_p = \left(f|_{\Omega_p}\right)^{-1} : U_p \to \Omega_p \subset \mathbb{R}^n \text{ homeo.}$$

Then $(U_p, \phi_p)_{p \in T^n}$ is an C^∞ altas on T^n:

Infact, if $p, q \in T^n$ s.t. $U_p \cap U_q \neq \phi$, $\qquad \Omega_p \cup$

then $\phi_q \circ \phi_p^{-1}(x_1, \cdots, x_n) \qquad (x_1, \cdots, x_n) \in \phi_p(U_p \cap U_q)$

$$= \phi_q(f(x_1, \cdots, x_n)) = \phi_q(e^{ix_1}, \cdots, e^{ix_n})$$

$$= \left(f|_{\Omega_q}\right)^{-1}(e^{ix_1}, \cdots, e^{ix_n})$$

$$= (x_1 + 2k_1\pi, \cdots, x_n + 2k_n\pi) \text{ for some } k_1, \cdots, k_n$$

such that $x_i + 2k_i\pi \in (x_i^q - \pi, x_i^q + \pi)$

note that k_i are indep. of $(x_1, \cdots, x_n) \in \phi_p(U_p \cap U_q)$ (Ex!)

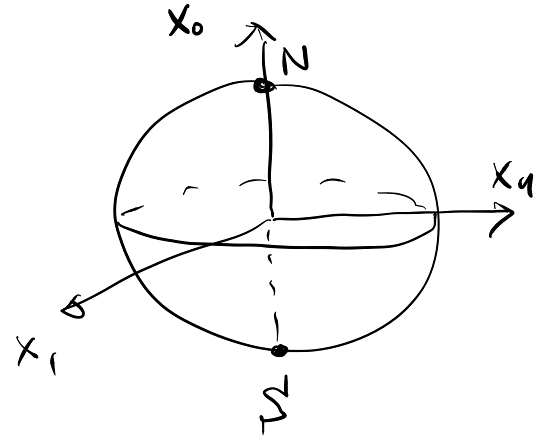
Hence $\phi_q \circ \phi_p^{-1}$ is just a translation in $\mathbb{R}^n$

$\Rightarrow \phi_q \circ \phi_p^{-1}$ is a C^∞ diffeo.

$\therefore (T^n, (U_p, \phi_p)_{p \in T^n})$ is a smooth manifold. #

eg $M = S^n$, the n-sphere $S^n = \{(x_0, x_1, \cdots, x_n) : \sum_{j=0}^{n} x_j^2 = 1\} \subset \mathbb{R}^{n+1}$

is a manifold.



(Sketch)

Let $\begin{cases} N = (1, 0, \cdots, 0) \in S^n \\ S = (-1, 0, \cdots, 0) \in S^n \end{cases}$

$\begin{cases} U_1 = S^n \setminus \{N\} \\ U_2 = S^n \setminus \{S\} \end{cases}$ $\left(\Rightarrow U_1 \cup U_2 = S^n\right)$

Let $\phi_1 : U_1 \to \mathbb{R}^n$ (Stereographic projection)

$$\begin{cases} \phi_1 : U_1 \to \mathbb{R}^n \\ (x_0, x_1, \cdots, x_n) \mapsto \dfrac{1}{1 - x_0}(x_1, \cdots, x_n) \\ \phi_2 : U_2 \to \mathbb{R}^n \\ (x_0, x_1, \cdots, x_n) \mapsto \dfrac{1}{1 + x_0}(x_1, \cdots, x_n) \end{cases}$$

are homeomorphisms.

Note that if $\phi_1(x_0, x_1, \cdots, x_n) = (y_1, \cdots, y_n) \neq 0$

then $y = (y_1, \cdots, y_n) \in \phi_1(U_1 \cap U_2)$

$\left(= \phi_2(U_1 \cap U_2)\right)$

and $\phi_2 \circ \phi_1^{-1}(y) = \dfrac{y}{|y|^2}$ $\left(\forall y \in \mathbb{R}^n \setminus \{0\}\right)$

which is a C^∞-diffeomorphism.

$\Rightarrow$ $\{(U_1,\phi_1),(U_2,\phi_2)\}$ is an C^∞ altas

and hence $(S^n, \mathcal{A}=[\{(U_1,\phi_1),(U_2,\phi_2)\}])$ is a smooth manifold. ※

<u>eg</u> $\mathbb{RP}^n$ the <u>real projective space</u> (in some book: $\mathbb{P}^n(\mathbb{R})$)

- As topological space

  $\mathbb{RP}^n$ = <u>quotient</u> space of $\mathbb{R}^{n+1}\setminus\{0\}$ by the equivalent relation : $x \sim y \iff \exists\, \lambda \neq 0 \in \mathbb{R}$ s.t. $x = \lambda y$

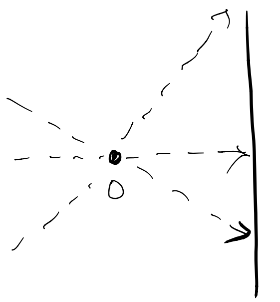
  $= S^n / \{\pm Id\}$ (hence $\mathbb{RP}^n$: Hausdroff, compact, connected.)

- Let $\pi: \mathbb{R}^{n+1}\setminus\{0\} \to \mathbb{RP}^n$ be the canonical projection map, ie $\pi(x)$ = equi. class of x.

  Define $V_i = \{x = (x_0, x_1, \cdots, x_n) \in \mathbb{R}^{n+1}, x_i \neq 0\}$

  $$\Phi_i : V_i \longrightarrow \mathbb{R}^n$$
  $$x \longmapsto \left(\frac{x_0}{x_i}, \cdots, \widehat{\frac{x_i}{x_i}}, \cdots, \frac{x_n}{x_i}\right)$$

  (this means the term is deleted)



$\{x_i = 1\} \cong \mathbb{R}^n$

Then $\forall\, x, y \in V_i$, we have

$$\boxed{\begin{array}{l} \Phi_i(x) = \Phi_i(y) \\ \Longleftrightarrow \ \pi(x) = \pi(y) \end{array}} \quad (*)$$

(i.e. $x \sim y$)

This gives

$$\begin{array}{ccc} \mathbb{R}^{n+1}\setminus\{0\} \supset V_i & \xrightarrow{\ \Phi_i\ } & \mathbb{R}^n \\ \pi \big\downarrow & \circlearrowright & \nearrow \phi_i \\ \mathbb{RP}^n \supset U_i = \pi(V_i) & & \end{array} \qquad (\phi_i \circ \pi = \Phi_i)$$

where ϕ_i is defined by

$$\begin{array}{rcl} \phi_i : U_i = \pi(V_i) & \longrightarrow & \mathbb{R}^n \\ \psi & & \psi \\ \text{equi. class of } x & \longmapsto & \Phi_i(x) \end{array}$$

(ϕ_i is well-defined because of $(*)$)

Further $\phi_i : U_i \to \mathbb{R}^n$ is a homeomorphism (check!)

with inverse

$$\phi_i^{-1}(y_0, \cdots, y_{n-1}) = \pi(y_0, \cdots, y_{i-1}, 1, y_i, \cdots, y_{n-1}).$$

Therefore, if $y_j \neq 0$, say for $j < i$, we have

$$(\phi_j \circ \phi_i^{-1})(y_0, \cdots, y_{n-1}) = \phi_j(\pi(y_0, \cdots, y_{i-1}, 1, y_i, \cdots, y_{n-1}))$$

$$= \Phi_j(y_0, \cdots, y_{i-1}, 1, y_i, \cdots, y_{n-1})$$

$$= \left(\frac{y_0}{y_j}, \cdots, \frac{\widehat{y_j}}{y_j}, \cdots \frac{y_{i-1}}{y_j}, \frac{1}{y_j}, \frac{y_i}{y_j}, \cdots, \frac{y_{n-1}}{y_j}\right)$$

$\therefore \phi_j \circ \phi_i^{-1} : \phi_i(U_i \cap U_j) \to \phi_j(U_i \cap U_j)$ is a C^∞ diffeo.

(Similarly for $i < j$). Hence $\mathbb{RP}^n$ with the altas

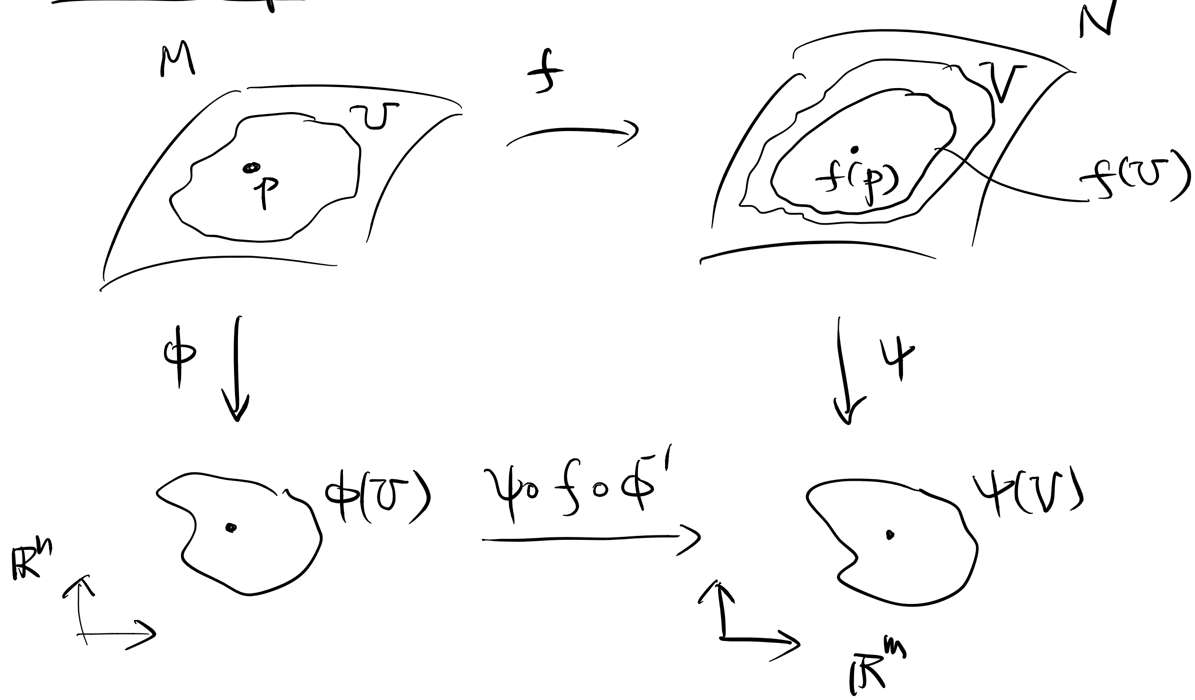
$(U_i, \phi_i)_{i=0,\cdots,n}$ is a smooth manifold.

Note: $\mathbb{RP}^n$ is non-orientable for n even according to the following definition:

Def: A smooth manifold M is said to be orientable if $\exists$ an altas on M s.t. the Jacobian determinant

$$\det\left(J(\phi_j \circ \phi_i^{-1})\right) > 0, \quad \forall i, j.$$

1.2 Smooth Maps



Def: Let M & N be C^k manifolds. A continuous map $f: M \to N$ is C^ℓ map (for $\ell \le k$) if $\forall p \in M$ $\exists$ charts (U, ϕ), (V, ψ) for M and N around p and $f(p)$ respectively with $f(U) \subset V$ s.t.

$$\psi \circ f \circ \phi^{-1} : \phi(U) \to \psi(V) \text{ is } C^\ell$$

(coordinate representation of f) (as a map: $\mathbb{R}^n \to \mathbb{R}^m$)

Note: This definition doesn't depend on the charts since transition functions are C^k $(k \ge \ell)$. (Ex!)

Def: A C^k map $\gamma : (a,b) \to M$ from an open interval to a smooth manifold is called a <u>C^k curve</u> on M

Def: A C^k map $f : M \to \mathbb{R}$ (or $\mathbb{C}$) is called a <u>C^k function</u> on M

Def: A smooth map $g : \mathbb{R}^n \to \mathbb{R}^m$ is a <u>submersion</u> (an <u>immersion</u>, a <u>local diffeomorphism</u>) at $x \in \mathbb{R}^n$ if $D_x g : \mathbb{R}^n \to \mathbb{R}^m$ is <u>surjective</u> (<u>injective</u>, <u>bijective</u>).

Def: Let M & N be smooth manifolds. A smooth map $f : M \to N$ is a <u>submersion</u> (<u>immersion</u>, <u>local diffeo</u>) at $p \in M$, if $\exists$ charts (U, ϕ) for M around p, (V, ψ) for N around $f(p)$, with $f(U) \subset V$ s.t. $\psi \circ f \circ \phi^{-1}$ is a <u>submersion</u> (<u>immersion</u>, <u>local diffeo</u>) at $\phi(p) \in \phi(U) \subset \mathbb{R}^n$.

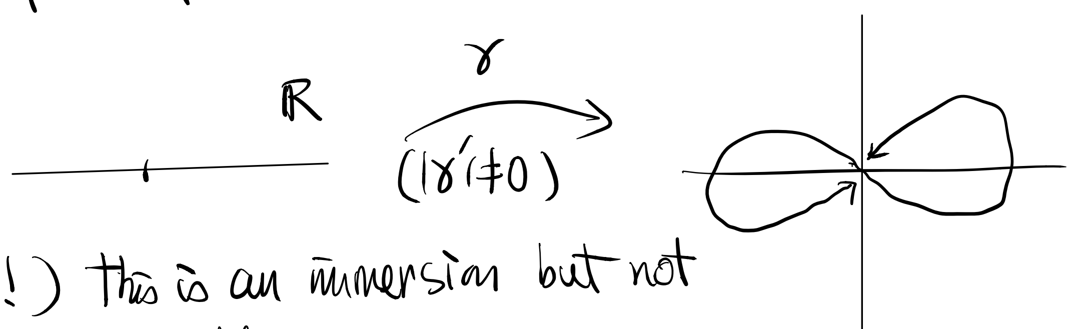
(Check: this is well-defined)

Def: A map $f : M \to N$ is a <u>submersion</u> (<u>immersion</u>, <u>local diffeo</u>) if it has the property at any point of M.

Def: A map $f: M \to N$ is a diffeomorphism if it is a bijection such that both f and f^{-1} are smooth.

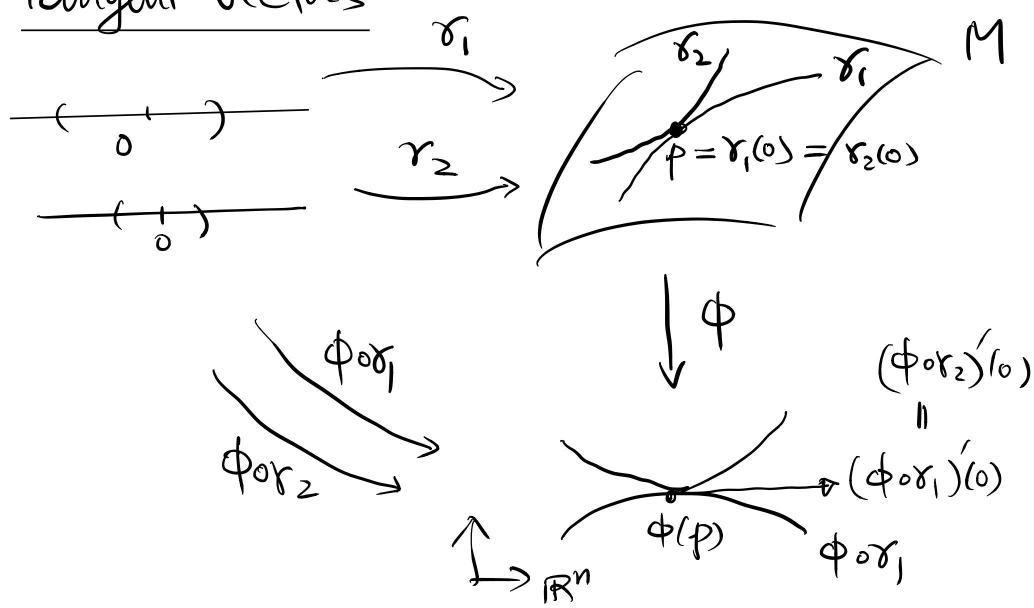
Def: A map $f: M \to N$ is an embedding if it is an immersion and $f: M \to f(M) \subset N$ (with subspace top.) is a homeomorphism.

eg:



(Ex!) This is an immersion but not embedding

1.3 Tangent vectors



Def 1: Let M be a smooth manifold and $p \in M$. A <u>tangent vector</u> to M at p is an <u>equi. class</u> of C^∞ curves $\gamma : I \to M$, where I = interval containing 0, such that $\gamma(0) = p$, for the equi. relation defined by $\gamma_1 \sim \gamma_2$

$$\Leftrightarrow \quad (\phi \circ \gamma_1)'(0) = (\phi \circ \gamma_2)'(0) \quad \text{for a chart } (U, \phi) \text{ around } p.$$

Ex: Check that the equi. relation is well-defined by showing that for any other chart (V, ψ) around p, we have

$$\boxed{(\psi \circ \gamma)'(0) = D_{\phi(p)}(\psi \circ \phi^{-1})\,(\phi \circ \gamma)'(0)}$$

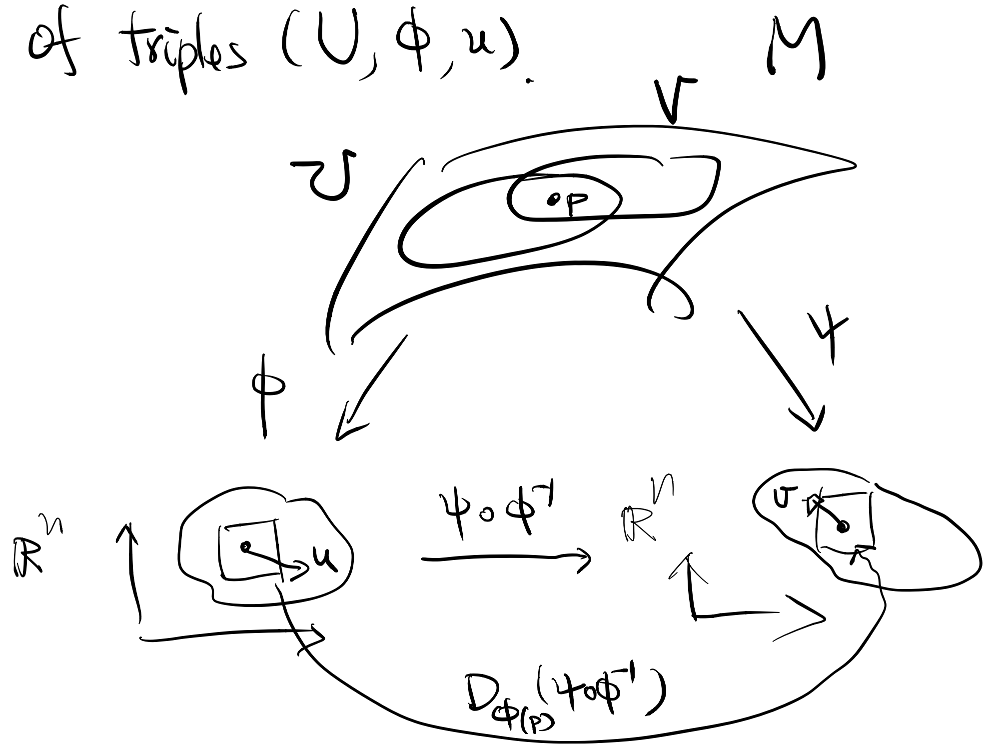
where $D_{\phi(p)}(\psi \circ \phi^{-1})$ is the <u>Jacobi matrix</u> (or <u>differential</u>) of the map $\psi \circ \phi^{-1}$ at $\phi(p)$.

Def 2 (Equivalent definition for tangent vectors)
Let M be a smooth manifold, $p \in M$, (U, ϕ) & (V, ψ) be 2 charts for M around p. Let u, v be

2 vectors in $\mathbb{R}^n$ (considered as tangent vectors to $\mathbb{R}^n$ at $\phi(p)$ and $\psi(p)$ respectively). We say that

$$(U, \phi, u) \sim (V, \psi, v) \iff D_{\phi(p)}(\psi \circ \phi^{-1}) u = v.$$

Then a tangent vector to M at p is an equi. class of triples (U, ϕ, u).



Notes: • In def 1, a tangent vector is represented by a curve γ. We usually write $\gamma'(0)$ for the tangent vector $[\gamma]$ for simplicity (indep. of chart).

• In def 2, the "same" tangent vector will be represented in a chart (U, ϕ) by a vector $u \in \mathbb{R}^n$

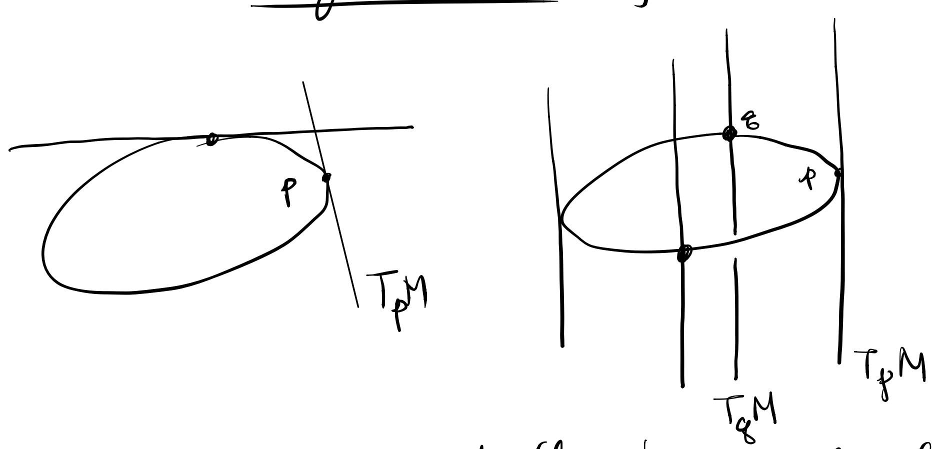
- Def 1 $\Leftrightarrow$ Def 2 by taking $\boxed{u = (\phi \circ \gamma)'(0)}$.

Notation: The set of tangent vectors to M at p is denoted by T_pM (Tangent space to M at $p \in M$)

Note: If a chart (U, ϕ) is given, then we have an "isomorphism"

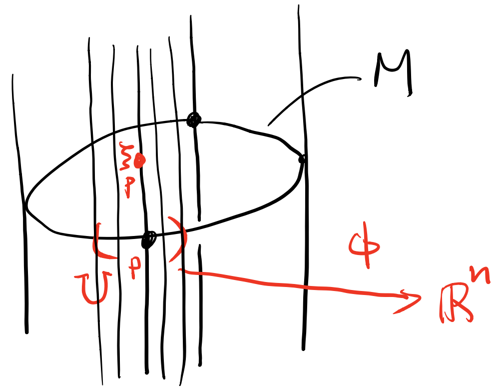
$$\theta_{U,\phi,p} : \mathbb{R}^n \longrightarrow T_pM$$
$$u \longmapsto [(U, \phi, u)]$$

Def: The disjoint union TM of T_pM, $\forall p \in M$, is called the tangent bundle of M.



Thm: Let M be an n-dim'l C^k manifold ($k \geq 1$)
Then TM can be equipped with a 2n-dim'l C^{k-1} abstract manifold structure.

(Sketch of Pf)



For each chart (U, ϕ) of M define a "chart"

$$\left(\coprod_{p\in U} T_pM, \Phi\right) \text{ for } TM \text{ by}$$

$$\Phi(\xi_p) = \left(\phi(p), \ \theta^{-1}_{U,\phi,p}(\xi_p)\right) \in \mathbb{R}^n \times \mathbb{R}^n$$

$\forall\ \xi_p \in T_pM$ & $p \in U$.

The one can see all these "$\coprod_{p\in U} T_pM$" give a topology on TM s.t. Φ are homeomorphisms. And one can check that TM is Hausdroff and $\left\{\left(\coprod_{p\in U} T_pM, \Phi\right)\right\}_{(U,\phi)}$ forms an C^{k-1} altas of TM.

(We've differentiated once in the equi. relation for tangent vectors.)

※