Proof of Thm 10 (3-dim'l case)

Only the "$\Leftarrow$" part remains to be proved:

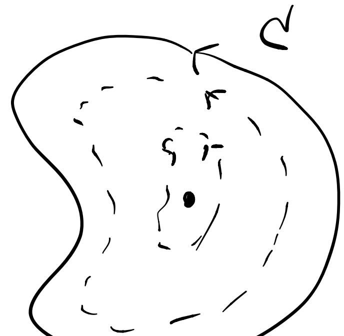
By assumption $\vec{F} = M\hat{i} + N\hat{j} + L\hat{k}$ satisfies the system of eqts in the Cor to the Thm 9, that is

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}, \quad \frac{\partial N}{\partial z} = \frac{\partial L}{\partial y}, \quad \text{and} \quad \frac{\partial L}{\partial x} = \frac{\partial M}{\partial z}$$

Hence $\vec{\nabla} \times \vec{F} = 0$

Let C be a simple closed curve in a simply-connected region D.

Then C can be deformed to a point inside D.



The process of deformation gives an oriented surface $S \subset D$ such that the boundary of $S = C$.

By Stokes' Thm,
$$\oint_C \vec{F} \cdot d\vec{r} = \iint_S (\vec{\nabla} \times \vec{F}) \cdot \hat{n} \, d\sigma$$
$$= 0 \qquad (\text{since } \vec{\nabla} \times \vec{F} = 0)$$

Then Thm 9 $\Rightarrow$ $\vec{F}$ is conservative. #

Summary

$n=2$	$n=3$
Tangential form of Green's Thm	Stokes' Thm
$$\oint_C \vec{F}\cdot d\vec{r} = \iint_R \vec{\nabla}\times\vec{F}\cdot\hat{k}\, dA$$	$$\oint_C \vec{F}\cdot d\vec{r} = \iint_S \vec{\nabla}\times\vec{F}\cdot\hat{n}\, d\sigma$$
Normal form of Green's Thm	Divergence Thm (see below)
$$\oint_C \vec{F}\cdot\hat{n}\, ds = \iint_R \vec{\nabla}\cdot\vec{F}\, dA$$	$$\iint_S \vec{F}\cdot\hat{n}\, d\sigma = \iiint_D \vec{\nabla}\cdot\vec{F}\, dV$$

flux: by definition $\hat{n}$ is the "outward" normal of curve "C" in "plane"

surface normal of S, "outward" pointing which can be defined when S encloses a solid region D.

Thm 13 (Divergence Theorem)

no boundary

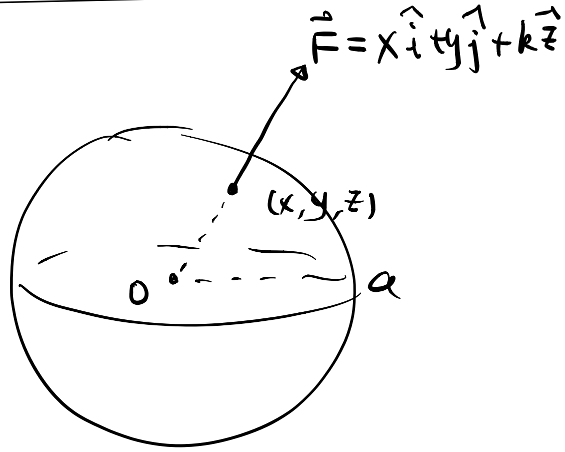
Let $\vec{F}$ be a C^1 vector field on $\Omega \overset{\text{open}}{\subseteq} \mathbb{R}^3$,

S be a piecewise smooth orientable closed surface

enclosing a (solid) region $D \subseteq \Omega$

Let $\hat{n}$ be the outward pointing unit normal vector field on S. Then

$$\boxed{\iint_S \vec{F} \cdot \hat{n} \, d\sigma = \iiint_D \operatorname{div} \vec{F} \, dV = \iiint_D \vec{\nabla} \cdot \vec{F} \, dV}$$



eg 63 Verify Divergence Thm for

$\vec{F} = x\hat{i} + y\hat{j} + z\hat{k}$

$S: x^2 + y^2 + z^2 = a^2, \ a > 0$

(Sphere)

($S = S_a^2$ 2-dim'l surface of radius a)

D = solid sphere (ball) bounded by S.

Soln: At $(x, y, z) \in S$

$$\hat{n} = \frac{x\hat{i} + y\hat{j} + z\hat{k}}{\sqrt{x^2 + y^2 + z^2}} = \frac{1}{a}(x\hat{i} + y\hat{j} + z\hat{k})$$ is the outward pointing unit normal.

$$\iint_S \vec{F} \cdot \hat{n} \, d\sigma = \iint_S (x\hat{i} + y\hat{j} + z\hat{k}) \cdot \frac{1}{a}(x\hat{i} + y\hat{j} + z\hat{k}) \, d\sigma$$

$$= \iint_{S} a \, d\sigma = a \, \text{Area}(S) = 4\pi a^3 \quad (\text{check!})$$

On the other hand

$$\text{div}\vec{F} = \vec{\nabla}\cdot\vec{F} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\cdot\left(x\hat{i} + y\hat{j} + z\hat{k}\right)$$

$$= \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z} = 3$$

$$\Rightarrow \iiint_{D} \text{div}\vec{F}\, dV = \iiint_{D} 3\, dV = 3\, \text{Vol}(D) = 3 \cdot \frac{4\pi a^3}{3} = 4\pi a^3$$

$$= \iint_{S} \vec{F}\cdot\hat{n}\, d\sigma .$$

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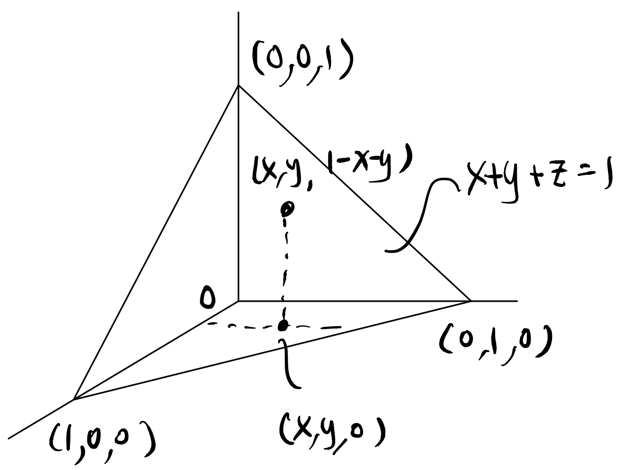
eg 63: $\vec{F} = x\sin y\,\hat{i} + (\cos y + z)\hat{j} + z^2\hat{k}$

Compute outward flux of $\vec{F}$ across boundary ∂T of

$$T = \left\{ (x,y,z) \in \mathbb{R}^3 : \begin{array}{l} x+y+z=1 \\ x,y,z \geq 0 \end{array} \right\}$$

(tetrahedron)

$$\iint_{\partial T} \vec{F}\cdot\hat{n}\, d\sigma$$



Soln:

$$\text{div}\vec{F} = \vec{\nabla}\cdot\vec{F} = \frac{\partial}{\partial x}(x\sin y) + \frac{\partial}{\partial y}(\cos y + z) + \frac{\partial}{\partial z}(z^2)$$

$$= 2z \quad (\text{check!})$$

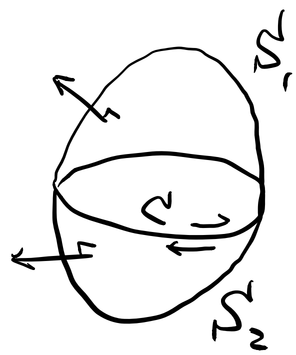
Divergence Thm

$$\Rightarrow \iint_{\partial T} \vec{F}\cdot\hat{n}\, d\sigma = \iiint_{T} \text{div}\vec{F}\, dV$$

$$= \int_0^1 \int_0^{1-x} \int_0^{1-x-y} 2z \, dz \, dy \, dx$$

$$= \frac{1}{12} \quad (\text{check!})$$

eg 64 Let S_1, S_2 be 2 surfaces with common boundary curve C such $S_1 \cup S_2$ forms a close surface enclosing a solid region D (without hole)



Suppose $\hat{n}$ is the outward normal of the solid region D.

Then the orientation of C wrt $(S_1, \hat{n})$ and $(S_2, \hat{n})$ are opposite (since "$\hat{b}$" pointing to opposite side)

Stokes' Thm $\Rightarrow$

$$\iint_{S_1} \vec{\nabla} \times \vec{F} \cdot \hat{n} \, d\sigma = \oint_C \vec{F} \cdot d\vec{r}$$

(+ve oriented wrt $(S_1, \hat{n})$)

$$= - \oint_C \vec{F} \cdot d\vec{r}$$

(+ve oriented wrt $(S_2, \hat{n})$)

$$= - \iint_{S_2} \vec{\nabla} \times \vec{F} \cdot \hat{n} \, d\sigma$$

$$\Rightarrow \iint_{S_1 \cup S_2} \vec{\nabla} \times \vec{F} \cdot \hat{n} \, d\sigma = 0$$

Divergence Thm $\Rightarrow$

$$\iiint_D \operatorname{div}(\vec{\nabla}\times\vec{F})\,dV = \iint_{S_1' \cup S_2'} (\vec{\nabla}\times\vec{F})\cdot\hat{n}\,d\sigma = 0$$

(true for "any" C^2 vector fields $\vec{F}$ defined on "any" D)

It is <u>consistent</u> with $\boxed{\vec{\nabla}\cdot(\vec{\nabla}\times\vec{F}) = 0}$ (Ex!)

$\forall$ C^2 vector field

i.e. $\boxed{\operatorname{div}(\operatorname{curl}\vec{F}) = 0} \leftrightarrow \boxed{\vec{\nabla}\cdot(\vec{\nabla}\times\vec{F}) = 0}$

Compare $\boxed{\operatorname{curl}(\operatorname{grad} f) = 0} \leftrightarrow \boxed{\vec{\nabla}\times(\vec{\nabla} f) = 0}$

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