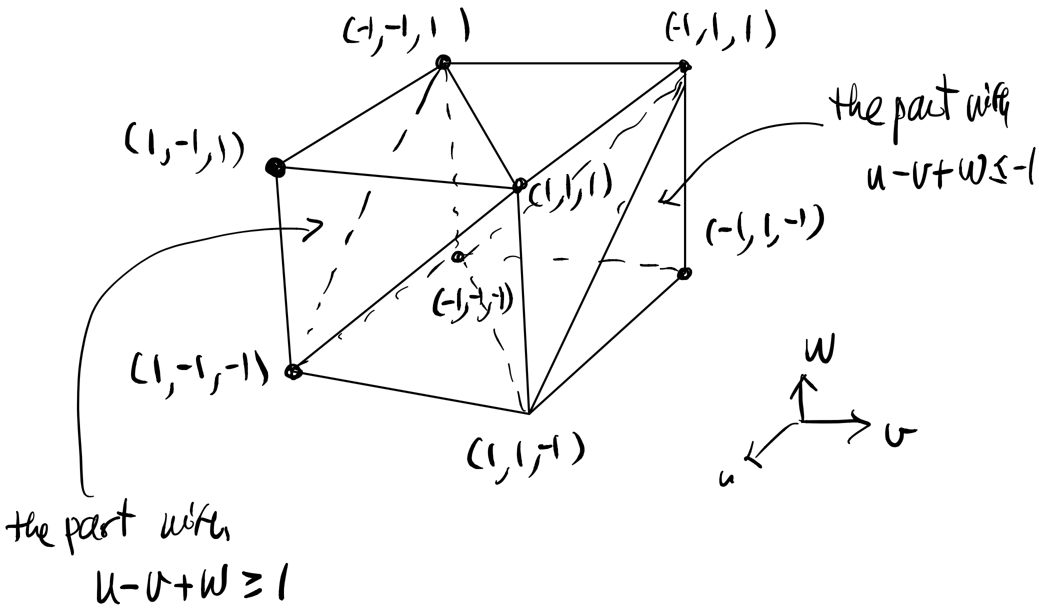
By calculating the values of $u-v+w$ on the vertexes, we see that

$$B = \iiint_{\substack{\text{solid determined}\\ \text{by the 4 vertexes}\\ (1,-1,1),\ (1,1,1)\\ (-1,-1,1),\ (1,-1,-1)}} \frac{u^4}{4}\, dv\, dw\, du$$



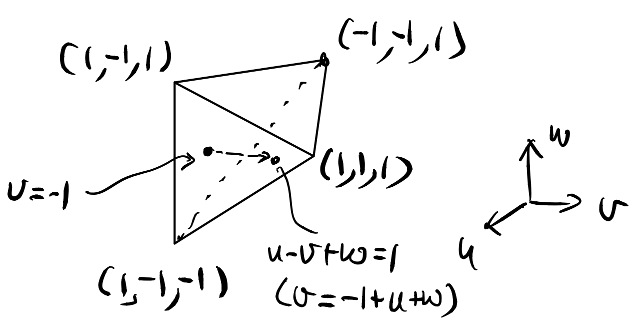
By symmetry, the solid for the integration C is determined by the other 4 vertexes

$(-1,1,-1)$, $(-1,-1,-1)$, $(1,1,-1)$, $(-1,1,1)$

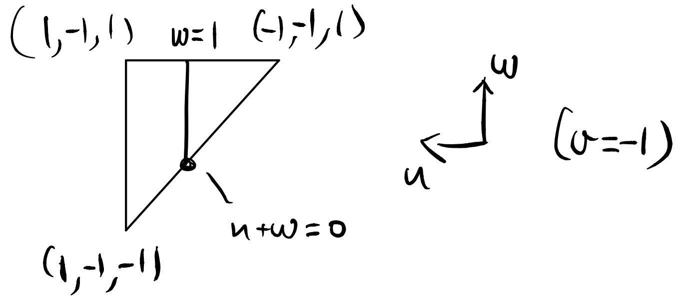
and $C = B$ (by change of variables $(u,v,w) \leftrightarrow (-u,-v,-w)$)

$$\therefore\ B = \int_{-1}^{1} \int_{-u}^{1} \left(\int_{-1}^{-1+u+w} \frac{u^4}{4}\, dv \right) dw\, du$$

$$= \frac{3}{35} \text{ (check!)}$$



Hence $C = \frac{3}{35}$ also



and

$$\iiint_D (x+y+z)^4\,dV = A - B - C$$
$$= \frac{2}{5} - \frac{3}{35} - \frac{3}{35} = \frac{8}{35} \quad \#$$

Pf of Thm 6

Recall:

Thm6: Suppose $\phi: \binom{u}{v} \mapsto \binom{x}{y}$ is a diffeomorphism (1-1, onto, ϕ & $\phi^{-1} \in C^1$) mapping a region G (closed and bounded) in the uv-plane onto a region R (closed and bounded) in xy-plane (except possibly on the boundary). Suppose $f(x,y)$ is continuous on R, then

$$\iint_R f(x,y)\,dx\,dy = \iint_G f\circ\phi(u,v) \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv$$

Step 0: We need better notations and terminology:

In this proof, we'll denote

$$J(\phi) = \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix}$$ the Jacobian matrix

and $\dfrac{\partial(x,y)}{\partial(u,v)} = \det J(\phi)$ the <u>Jacobian determinant</u>.

- We also use "index" notations for variables:

$$(x_1, x_2) \text{ or } \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \quad \left(\text{instead of } (x,y), \begin{pmatrix} x \\ y \end{pmatrix}\right)$$

<u>Step 1</u> Let $F: \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mapsto \begin{pmatrix} f_1(x_1,x_2) \\ f_2(x_1,x_2) \end{pmatrix}$ <u>near</u> a point p

with $\dfrac{\partial(f_1,f_2)}{\partial(x_1,x_2)} \neq 0$ at p. Then, near a point p, F

can be decomposed into $F = H \circ K$

with H, K of the forms

$$K: \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mapsto \begin{pmatrix} k(x_1,x_2) \\ x_2 \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

$$\left(\text{or } \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mapsto \begin{pmatrix} k(x_1,x_2) \\ x_1 \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \right)$$

and

$$H: \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \mapsto \begin{pmatrix} y_1 \\ h(y_1,y_2) \end{pmatrix}$$

such that $\det J(K) \neq 0$ and

$\det J(H) \neq 0$.

<u>Pf of Step 1</u>: By assumption $0 \neq \dfrac{\partial(f_1,f_2)}{\partial(x_1,x_2)} = \det \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{pmatrix}$ near p.

Case 1 $\frac{\partial f_1}{\partial x_1}(p) \neq 0$

Define $k(x_1, x_2) = f_1(x_1, x_2)$ near p

Then the transformation

$$K: \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mapsto \begin{pmatrix} y_1 = f_1(x_1, x_2) \\ y_2 = x_2 \end{pmatrix}$$

is of the required form and has Jacobian matrix

$$J(K) = \begin{pmatrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_1}{\partial x_2} \\ \frac{\partial y_2}{\partial x_1} & \frac{\partial y_2}{\partial x_2} \end{pmatrix} = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ 0 & 1 \end{pmatrix}$$

$$\Rightarrow \quad \det J(K)(p) = \frac{\partial f_1}{\partial x_1}(p) \neq 0 .$$

By Inverse Function Theorem, K is invertible near p and

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = K^{-1}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} g(y_1, y_2) \\ y_2 \end{pmatrix}$$ is differentiable at $K(p)$

(since $x_2 = y_2$)

with

$$J(K^{-1})_{K(p)} \cdot J(K)_p = Id$$

ie

$$\begin{pmatrix} \frac{\partial g}{\partial y_1} & \frac{\partial g}{\partial y_2} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\Leftrightarrow \quad \frac{\partial g}{\partial y_1} \cdot \frac{\partial f_1}{\partial x_1} = 1 \quad \& \quad \frac{\partial g}{\partial y_1} \cdot \frac{\partial f_1}{\partial x_2} + \frac{\partial g}{\partial y_2} = 0 .$$

In particular $\quad \det J(K^{-1})_{K(p)} = \dfrac{1}{\det J(K)_p} \neq 0$

Now, define

$$h(y_1, y_2) = f_2(x_1, x_2) = f_2 \circ K^{-1}(y_1, y_2)$$
$$= f_2(g(y_1, y_2), y_2)$$

and $\quad H : \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \mapsto \begin{pmatrix} z_1 = y_1 \\ z_2 = h(y_1, y_2) \end{pmatrix}$

(is of the required form)

Moreover $J(H) \neq 0$

$\vdots$

(to be cont'd)

Preview for next time:

Step 2 Let $\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = K\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} k(x_1, x_2) \\ x_2 \end{pmatrix}$ be a diffeomorphism from region R_1 to $R_2 = K(R_1)$. Then for any function $f(y_1, y_2)$ on R_2,

$$\iint_{R_2} f(y_1, y_2)\, dy_1 dy_2 = \iint_{R_1} f \circ K(x_1, x_2) \left| \det J(K) \right| dx_1 dx_2$$

$$= \iint_{R_1} f(k(x_1, x_2), x_2) \left| \frac{\partial(y_1, y_2)}{\partial(x_1, x_2)} \right| dx_1 dx_2$$

Step 3: If the change of variables formula holds for F & G, then it holds for $F \circ G$

Pf: Easily by $J(F \circ G) = J(F) J(G)$ (Chain Rule)

$\Rightarrow \quad |\det J(F \circ G)| = |\det J(F)| \, |\det J(G)|$

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Final step: Combining steps 1-3, and using additivity property of integration, we've proved the Thm 6 for general change of variables formula

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