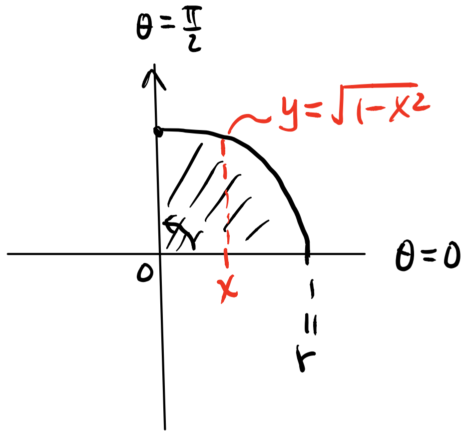
eg13 Convert integrals between Cartesian and Polar coordinates

(a) $\int_0^{\frac{\pi}{2}} \int_0^1 r^3 \sin\theta \cos\theta \, dr \, d\theta$

(b) $\int_1^2 \int_0^{\sqrt{2x-x^2}} y \, dy \, dx$



Soln: (a) $\int_0^{\frac{\pi}{2}} \int_0^1 r^3 \sin\theta\cos\theta \, dr \, d\theta$

$$= \int_0^{\frac{\pi}{2}} \left[\int_0^1 r^3 \sin\theta \cos\theta \, dr \right] d\theta$$

$\therefore$ Region : $0 \leq r \leq 1$, $0 \leq \theta \leq \frac{\pi}{2}$.

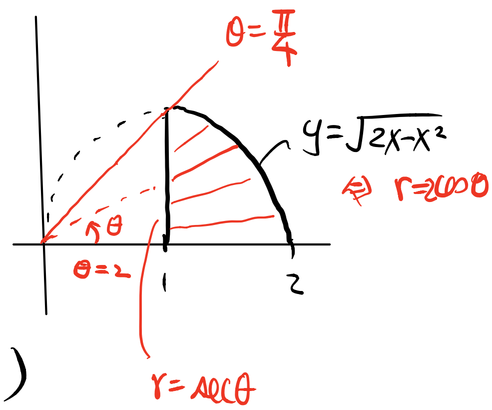
$$\Rightarrow \int_0^{\frac{\pi}{2}} \int_0^1 r^3 \sin\theta\cos\theta \, dr \, d\theta = \int_0^{\frac{\pi}{2}} \int_0^1 r^2 \sin\theta\cos\theta \, \underbrace{r \, dr \, d\theta}$$

$$= \int_0^1 \int_0^{\sqrt{1-x^2}} xy \, \underbrace{dy \, dx}$$

$$\left(\text{or} \; = \int_0^1 \int_0^{\sqrt{1-y^2}} xy \, dx \, dy \right)$$

(b) $$\int_1^2 \int_0^{\sqrt{2x-x^2}} y \, dy \, dx = \int_1^2 \left[\int_0^{\sqrt{2x-x^2}} y \, dy \right] dx$$

Region is $1 \leq x \leq 2$, $0 \leq y \leq \sqrt{2x - x^2}$



The curve $x = 1$

$\Leftrightarrow r\cos\theta = 1$

$\Leftrightarrow r = \frac{1}{\cos\theta} = \sec\theta \quad (0 \leq \theta \leq \frac{\pi}{4})$

The other curve $y=\sqrt{2x-x^2}$

$$\Leftrightarrow \quad r\sin\theta = \sqrt{2r\cos\theta - r^2\cos^2\theta}$$

$$\vdots$$

$$\Leftrightarrow \quad r^2 = 2r\cos\theta \qquad \text{(check!)}$$

$$\Leftrightarrow \quad r = 2\cos\theta \qquad \left(0\le\theta\le\tfrac{\pi}{4}\right)$$

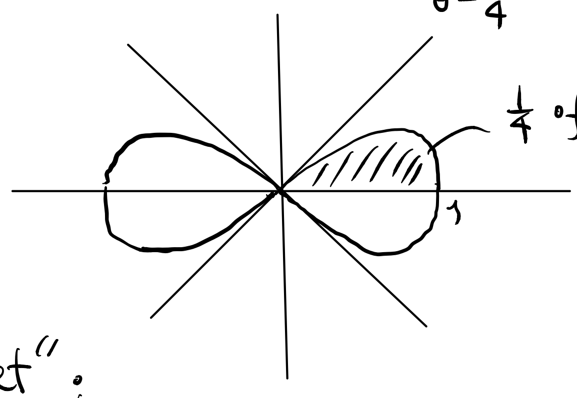
Hence $$\int_1^2\int_0^{\sqrt{2x-x^2}} y\,\underbrace{dy\,dx}$$

$$= \int_0^{\frac{\pi}{4}}\int_{\sec\theta}^{2\cos\theta} r\sin\theta\cdot \underbrace{r\,dr\,d\theta}$$

$$= \int_0^{\frac{\pi}{2}}\int_{\sec\theta}^{2\cos\theta} r^2\sin\theta\,dr\,d\theta .$$

eg14: Find area enclosed by $r^2 = 4\cos 2\theta$

Soln:



Remark: r is "not really" a function of θ, it should be regarded as a "level set".

(i) there is no soln. when $\frac{\pi}{4}<\theta<\frac{3\pi}{4}$, & $\frac{5\pi}{4}<\theta<\frac{7\pi}{4}$

(ii) in terms of (x,y) coordinates

$$F(x,y) = (x^2+y^2)^2 - 4(x^2-y^2) = 0 \qquad \text{(check!)}$$

which has a critical point at $(x,y)=(0,0)$ on the level set (Implicit Function Theorem)

By the symmetry

$$\text{Area} = 4\int_0^{\frac{\pi}{4}}\int_0^{\sqrt{4\cos 2\theta}} 1\cdot \underbrace{r\,dr\,d\theta} = 8\int_0^{\frac{\pi}{4}}\cos 2\theta\, d\theta = 4$$

(check!)

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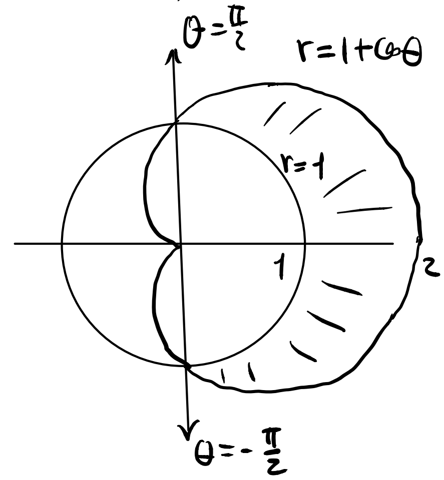
eg15: Integrate $f(x,y) = \dfrac{1}{\sqrt{x^2+y^2}}$ over the region R bounded

between $\begin{cases} r = 1+\cos\theta & \text{(cardioid)} \\ r = 1 \end{cases}$

and outside the circle $r=1$ $(\Rightarrow \cos\theta \geq 0)$



Soln: Intersections: $1+\cos\theta = 1$

$\Leftrightarrow \cos\theta = 0$

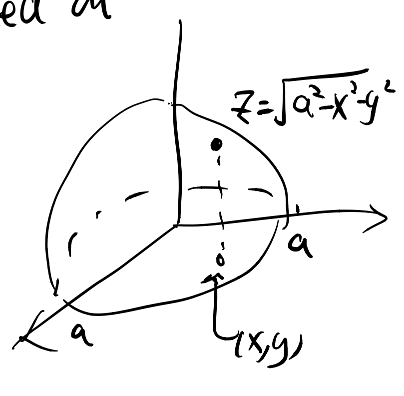
$\Leftrightarrow \theta = \frac{\pi}{2} + k\pi$

$\therefore\ \theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ (choice)

$$\Rightarrow \iint_R f(x,y)\,dA = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_1^{1+\cos\theta}\frac{1}{r}\cdot r\,dr\,d\theta = 2 \text{ (check!)}$$

eg16: Let $z = \sqrt{a^2-x^2-y^2}$ be a function defined on

$$R = \{(x,y) : x^2+y^2 \leq a^2\}$$



The graph of z is the hemisphere of radius a. Find the average height of the hemisphere

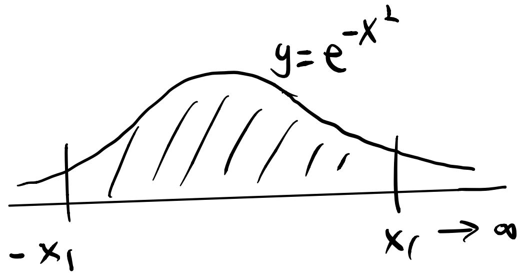
Soln: Average height $= \dfrac{1}{\text{Area}(R)} \displaystyle\iint_R z \, dA$

$$= \frac{1}{\pi a^2} \int_0^{2\pi} \int_0^a \sqrt{a^2-r^2} \, \underbrace{r\,dr\,d\theta}$$

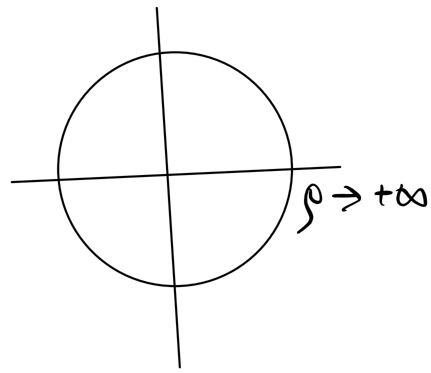
$$= \frac{2a}{3} \quad \text{(Check!)} \;\#$$

eg 17 (Improper integral)

Find $\displaystyle\int_{-\infty}^{\infty} e^{-x^2} dx$.



Soln: Consider $\displaystyle\iint_{\mathbb{R}^2} e^{-x^2-y^2} dA$

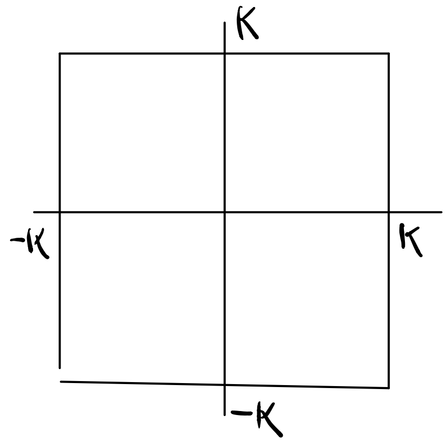


$$= \lim_{\rho \to +\infty} \iint_{\{x^2+y^2 \le \rho^2\}} e^{-(x^2+y^2)} dA$$

$$= \lim_{\rho \to +\infty} \int_0^{2\pi} \int_0^{\rho} e^{-r^2} r\,dr\,d\theta$$

$$= \lim_{\rho \to +\infty} \pi(1-e^{-\rho^2}) = \pi$$

On the other hand



$$\iint_{\mathbb{R}^2} e^{-(x^2+y^2)} dA$$

$$= \lim_{K \to +\infty} \int_{-K}^{K} \int_{-K}^{K} e^{-x^2-y^2} dx\,dy$$

$$= \lim_{K\to+\infty} \int_{-K}^{K}\int_{-K}^{K} e^{-x^2}e^{-y^2}\,dx\,dy$$

$$= \lim_{K\to+\infty} \left(\int_{-K}^{K} e^{-x^2}dx\right)\left(\int_{-K}^{K} e^{-y^2}dy\right)$$

$$= \lim_{K\to+\infty} \left(\int_{-K}^{K} e^{-x^2}dx\right)^2$$

$$= \left(\int_{-\infty}^{\infty} e^{-x^2}dx\right)^2$$

$$\therefore \quad \int_{-\infty}^{\infty} e^{-x^2}dx = \sqrt{\pi}$$

[Caution: we are calculating $\iint_{\mathbb{R}^2} e^{-x^2-y^2}dA$ using <u>two</u> different limiting processes. Why are they equal?]

Answer: $e^{-x^2} > 0$ and :